

## VERTEX-DISTINGUISHING IE-TOTAL COLORINGS OF COMPLETE BIPARTITE GRAPHS $K_{m,n}(m < n)$ <sup>1</sup>

XIANG'EN CHEN, YUPING GAO AND BING YAO

*College of Mathematics and Information Science*  
*Northwest Normal University, Lanzhou 730070, P. R. China*

**e-mail:** chenxe@nwnu.edu.cn

### Abstract

Let  $G$  be a simple graph. An IE-total coloring  $f$  of  $G$  is a coloring of the vertices and edges of  $G$  so that no two adjacent vertices receive the same color. Let  $C(u)$  be the set of colors of vertex  $u$  and edges incident to  $u$  under  $f$ . For an IE-total coloring  $f$  of  $G$  using  $k$  colors, if  $C(u) \neq C(v)$  for any two different vertices  $u$  and  $v$  of  $G$ , then  $f$  is called a  $k$ -vertex-distinguishing IE-total-coloring of  $G$ , or a  $k$ -VDIET coloring of  $G$  for short. The minimum number of colors required for a VDIET coloring of  $G$  is denoted by  $\chi_{vt}^{ie}(G)$ , and is called vertex-distinguishing IE-total chromatic number or the VDIET chromatic number of  $G$  for short. VDIET colorings of complete bipartite graphs  $K_{m,n}(m < n)$  are discussed in this paper. Particularly, the VDIET chromatic numbers of  $K_{m,n}(1 \leq m \leq 7, m < n)$  as well as complete graphs  $K_n$  are obtained.

**Keywords:** complete bipartite graphs, IE-total coloring, vertex-distinguishing IE-total coloring, vertex-distinguishing IE-total chromatic number.

**2010 Mathematics Subject Classification:** 05C15.

For an edge coloring (proper or not) of a graph  $G$  and a vertex  $v$  of  $G$ , denote by  $S(v)$  the set of colors used to color the edges incident to  $v$ .

A proper edge coloring of a graph  $G$  is said to be *vertex-distinguishing* if for any  $u, v \in V(G), u \neq v, S(u) \neq S(v)$ . In other words,  $S(u) \neq S(v)$  whenever  $u \neq v$ . A graph  $G$  has a vertex-distinguishing proper edge coloring if and only if it has no more than one isolated vertex and no isolated edges. Such a graph will be referred to as a *vdec-graph*. The minimum number of colors required

---

<sup>1</sup>This research is supported by NNSFC (No.61163037, 61163054), the Scientific Research Project of Northwest Normal University (No. nwnu-kjcxgc-03-61), the Natural Foudation Project of Ningxia (No. NZ1154) and the Scientific Research Foundation Project of Ningxia University ((E):ndzr10-7)

for a vertex-distinguishing proper edge coloring of a *vdec*-graph  $G$  is denoted by  $\chi'_s(G)$ . Vertex-distinguishing proper edge coloring has been considered in several papers [1-5, 8-9].

A general edge coloring (not necessarily proper) of a graph  $G$  is said to be *vertex-distinguishing* if  $S(u) \neq S(v)$  is required for any two distinct vertices  $u, v$ . The *point-distinguishing chromatic index* of a *vdec*-graph  $G$ , denoted by  $\chi_0(G)$ , refers to the minimum number of colors required for a vertex-distinguishing general edge coloring of  $G$ . This parameter was introduced by Harary and Plantholt in [7]. Although the structure of complete bipartite graph is simple, it seems that the problem of determining  $\chi_0(K_{m,n})$  is not easy, especially in the case  $m = n$ , as documented by papers of Horňák and Soták [10, 11], Zagaglia Salvi [13, 14] and Horňák and Zagaglia Salvi [12].

A *total coloring* of a graph  $G$  is an assignment of some colors to the vertices and edges of  $G$ . It is *proper* if the following three conditions are satisfied:

**Condition (v):** No two adjacent vertices receive the same color;

**Condition (e):** No two adjacent edges receive the same color;

**Condition (i):** No edge receives the same color as any one of its incident vertices.

For a total coloring (proper or not)  $f$  of  $G$  and a vertex  $v$  of  $G$ , denote by  $C_f(v)$ , or simply  $C(v)$  if no confusion arise, the set of colors used to color the vertex  $v$  as well as the edges incident to  $v$ . Let  $\overline{C}(v)$  be the complementary set of  $C(v)$  in the set of all colors we used. Obviously  $|C(v)| \leq d_G(v) + 1$  and the equality holds if the total coloring is proper.

For a proper total coloring, if  $C(u) \neq C(v)$  for any two distinct vertices  $u$  and  $v$ , then the coloring is called a *vertex-distinguishing proper total coloring* and the minimum number of colors required for a vertex-distinguishing proper total coloring is denoted by  $\chi_{vt}(G)$ . This concept was considered in [6, 15]. In [15], the following conjecture was given.

**Conjecture 1.** *Suppose  $G$  is a simple graph and  $n_d$  is the number of vertices of degree  $d$ ,  $\delta \leq d \leq \Delta$ . Let  $k$  be the minimum positive integer such that  $\binom{k}{d+1} \geq n_d$  for all  $d$  such that  $\delta \leq d \leq \Delta$ . Then  $\chi_{vt}(G) = k$  or  $k + 1$ .*

From [15] we know that the above conjecture is valid for complete graphs, complete bipartite graphs, paths and cycles, etc.

In this paper we propose a kind of vertex-distinguishing general total coloring called IE-total coloring. The relationship between this coloring and vertex-distinguishing proper total coloring is similar to the relationship between vertex-distinguishing general edge coloring and vertex-distinguishing proper edge coloring.

An *IE-total coloring* of a graph  $G$  is a total coloring of  $G$  such that the Condition (v) is satisfied. If  $f$  is an IE-total coloring of graph  $G$  using  $k$  colors

and for any  $u, v \in V(G)$ ,  $u \neq v$ , we have  $C(u) \neq C(v)$ , then  $f$  is called a  $k$ -vertex-distinguishing IE-total coloring, or a  $k$ -VDIET coloring. The number

$$\min\{k : G \text{ has a } k\text{-VDIET coloring}\}$$

is called the *vertex-distinguishing IE-total chromatic number* of a graph  $G$  and is denoted by  $\chi_{vt}^{ie}(G)$ .

The following proposition is obviously true.

**Proposition 2.**  $\chi_{vt}^{ie}(G) \leq \chi_{vt}(G)$ .

For a graph  $G$ , let  $n_i$  denote the number of vertices of degree  $i$ ,  $\delta \leq i \leq \Delta$ . Let  $\xi(G) = \min \left\{ k \mid \binom{k}{1} + \binom{k}{2} + \binom{k}{3} + \dots + \binom{k}{s+1} \geq n_\delta + n_{\delta+1} + \dots + n_s, \delta \leq s \leq \Delta \right\}$ . Obviously,  $\chi_{vt}^{ie}(G) \geq \xi(G)$ .

In the following we will consider the VDIET colorings of complete bipartite graphs  $K_{m,n}$  ( $1 \leq m < n$ ) and complete graphs  $K_n$ , then we will give three conjectures.

For a complete bipartite graph  $K_{m,n}$  ( $1 \leq m < n$ ),  $\xi(K_{m,n})$  is the minimum positive integer  $l$  such that

$$(1) \quad \binom{l}{1} + \binom{l}{2} + \binom{l}{3} + \dots + \binom{l}{m+1} \geq n,$$

$$(2) \quad \binom{l}{1} + \binom{l}{2} + \binom{l}{3} + \dots + \binom{l}{n+1} \geq n + m.$$

**Proposition 3.** (i) If  $n = \sum_{i=1}^{m+1} \binom{m+2}{i} - m + 1$ , then  $\xi(K_{m,n}) = m + 2$ ;

(ii) If  $\sum_{i=1}^{m+1} \binom{m+2}{i} - m + 2 \leq n \leq \sum_{i=1}^{m+1} \binom{m+3}{i} - m$ , then  $\xi(K_{m,n}) = m + 3$ .

**Proof.** (i) When  $l = m + 1$ , (1) is not valid, because

$$\binom{m+1}{1} + \binom{m+1}{2} + \dots + \binom{m+1}{m+1} = 2^{m+1} - 1,$$

$$n = 2^{m+2} - 2 - m + 1 = 2^{m+2} - m - 1 > 2^{m+1} - 1.$$

Therefore  $\xi(K_{m,n}) \geq m + 2$ . Since

$$\binom{m+2}{1} + \binom{m+2}{2} + \dots + \binom{m+2}{m+1} = 2^{m+2} - 2 \geq 2^{m+2} - m - 1 = n,$$

$$\binom{m+2}{1} + \binom{m+2}{2} + \dots + \binom{m+2}{n+1} = 2^{m+2} - 1 = m + n,$$

so we have  $\xi(K_{m,n}) = m + 2$ .

(ii) When  $\sum_{i=1}^{m+1} \binom{m+2}{i} - m + 2 \leq n \leq \sum_{i=1}^{m+1} \binom{m+3}{i} - m$ , i.e.,  $2^{m+2} - m \leq n \leq 2^{m+3} - (m+3) - 2 - m = 2^{m+3} - 2m - 5$ , we have

$$\binom{m+2}{1} + \binom{m+2}{2} + \cdots + \binom{m+2}{n+1} = 2^{m+2} - 1 \leq m + n - 1.$$

Therefore, (2) is not valid if  $l = m + 2$ . So,  $\xi(K_{m,n}) \geq m + 3$ . When  $l = m + 3$ , (1) and (2) are right, so  $\xi(K_{m,n}) = m + 3$ . ■

**Proposition 4.** (i) If  $\sum_{i=1}^{m+1} \binom{k-1}{i} - m < n \leq \sum_{i=1}^{m+1} \binom{k-1}{i}$  and  $k \geq m + 4$ , then

$$\xi(K_{m,n}) = k - 1;$$

(ii) If  $\sum_{i=1}^{m+1} \binom{k-1}{i} < n \leq \sum_{i=1}^{m+1} \binom{k}{i} - m$  and  $k \geq m + 4$ , then  $\xi(K_{m,n}) = k$ .

**Proof.** (i) As

$$\begin{aligned} \binom{k-2}{1} + \binom{k-2}{2} + \cdots + \binom{k-2}{m+1} &\leq \left[ \binom{k-2}{0} + \binom{k-2}{1} \right] + \left[ \binom{k-2}{1} + \binom{k-2}{2} \right] \\ &+ \cdots + \left[ \binom{k-2}{m} + \binom{k-2}{m+1} \right] - m = \binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} - m < n, \end{aligned}$$

(1) is not valid if  $l = k - 2$ . Therefore,  $\xi(K_{m,n}) \geq k - 1$ . Because

$$\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{m+1} \geq n,$$

$$\binom{k-1}{1} + \binom{k-1}{2} + \cdots + \binom{k-1}{n+1} \geq n + \binom{k-1}{m+2} + \binom{k-1}{m+3} + \cdots + \binom{k-1}{n+1} > m + n,$$

so (1) and (2) are valid if  $l = k - 1$ . We have  $\xi(K_{m,n}) = k - 1$ .

(ii) When  $\sum_{i=1}^{m+1} \binom{k-1}{i} < n \leq \sum_{i=1}^{m+1} \binom{k}{i} - m$ , (1) is not valid if  $l = k - 1$ , whereas (1) and (2) are valid if  $l = k$ . Therefore  $\xi(K_{m,n}) = k$ . ■

**Theorem 5.** Let  $m \geq 1$ ,  $n > 2^{m+2} - m - 2$ . Then  $\chi_{vt}^{ie}(K_{m,n}) = k$  when  $\sum_{i=1}^{m+1} \binom{k-1}{i} - m < n \leq \sum_{i=1}^{m+1} \binom{k}{i} - m$ .

**Proof.** As  $n > 2^{m+2} - m - 2$ , we have  $k > m + 2$  (otherwise, if  $k \leq m + 2$ , then  $n \leq \sum_{i=1}^{m+1} \binom{k}{i} - m \leq \sum_{i=1}^{m+1} \binom{m+2}{i} - m = 2^{m+2} - 2 - m$ , a contradiction).

We prove that  $K_{m,n}$  does not have a  $(k - 1)$ -VDIET coloring. If not, suppose  $g$  is a VDIET coloring of  $K_{m,n}$  using colors  $1, 2, \dots, k - 1$ . Let  $B_0 = \{g(u_1), g(u_2), \dots, g(u_m)\}$ ,  $B_i = \{1, 2, \dots, k - 1\} \setminus C_g(u_i), i = 1, 2, \dots, m$ . Note that none of  $B_0, B_1, B_2, \dots, B_m$  is the color set of any vertex  $v_j$ . Let  $T = \{j : |C_g(v_j)| = 1, j = 1, 2, \dots, n\}$  and  $t = |T|$ . Then  $B_0 \cap \{g(v_j) | j \in T\} = \emptyset$ . Without loss of generality, we assume that  $C_g(v_j) = \{j\}, j = 1, 2, \dots, t$ , then we have  $|C_g(v_j)| \geq 2, j = t + 1, \dots, n$  and  $C_g(u_i) \supseteq \{1, 2, \dots, t, g(u_i)\}, i = 1, 2, \dots, m$ .

*Case 1.*  $t \geq k - m - 3$ . For each  $i \in \{1, 2, \dots, m\}$ , we have  $|C_g(u_i)| \geq t + 1$  and  $|B_i| \leq (k - 1) - (t + 1) \leq (k - 1) - (k - m - 3 + 1) = m + 1$ . Note that  $|B_0| \leq m + 1$  and none of  $B_0, B_1, B_2, \dots, B_m$  is the color set of any vertex  $v_j$ . Therefore there are at most  $\binom{k-1}{1} + \binom{k-1}{2} + \dots + \binom{k-1}{m+1} - m$  subsets of  $\{1, 2, \dots, k - 1\}$  with cardinality between 1 and  $m + 1$  which may become the color sets of vertices  $v_1, v_2, \dots, v_n$ . This is a contradiction.

*Case 2.*  $t \leq k - m - 4$ . In this case, there are at least  $(k - 1) - (k - m - 4) = m + 3$  subsets of  $\{1, 2, \dots, k - 1\}$  with cardinality 1 which are not the color sets of vertices  $v_1, v_2, \dots, v_n$ . This is also a contradiction because  $\binom{k-1}{1} + \binom{k-1}{2} + \dots + \binom{k-1}{m+1} - (m + 3) < \binom{k-1}{1} + \binom{k-1}{2} + \dots + \binom{k-1}{m+1} - m < n$ , and at most  $\binom{k-1}{1} + \binom{k-1}{2} + \dots + \binom{k-1}{m+1} - (m + 3)$  subsets of  $\{1, 2, \dots, k - 1\}$  with cardinality between 1 and  $m + 1$  cannot distinguish  $n$  vertices.

In the following we prove that  $K_{m,n}$  has a  $k$ -VDIET coloring. Let  $V(K_{m,n}) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$  and  $E(K_{m,n}) = \{u_i v_j : i = 1, 2, \dots, m, j = 1, 2, \dots, n\}$ .

Put  $D(u_i) = \{1, 2, \dots, k\} \setminus \{i\}, i = 1, 2, \dots, m - 1, D(u_m) = \{1, 2, \dots, k\}, D(v_j) = \{j, k\}, j = 1, 2, \dots, m - 1, D(v_j) = \{j\}, j = m, m + 1, \dots, k - 1$ .

Now distribute other subsets of  $\{1, 2, \dots, k\}$  with cardinality between 2 and  $m + 1$  to vertices  $v_k, v_{k+1}, \dots, v_n$ . These  $n - k + 1$  subsets are denoted by  $D(v_k), D(v_{k+1}), \dots, D(v_n)$ , respectively.

Construct a mapping  $f$  from  $V(K_{m,n}) \cup E(K_{m,n})$  to  $\{1, 2, \dots, k\}$  as follows:

Put  $f(u_i) = k, i = 1, 2, \dots, m, f(v_j) = \min D(v_j), j = 1, 2, \dots, n,$

$f(u_i v_i) = k$  for  $i = 1, 2, \dots, m - 1, f(u_m v_m) = m,$

$f(u_i v_j) = j, i = 1, 2, \dots, m, j = 1, 2, \dots, k - 1, i \neq j.$

For each  $j = k, k + 1, \dots, n$ , we recursively let  $f(u_1 v_j) = \min (D(u_1) \cap (D(v_j) \setminus \{f(v_j)\}))$  or  $f(u_1 v_j) \in D(u_1) \cap D(v_j)$  when  $D(u_1) \cap (D(v_j) \setminus \{f(v_j)\}) = \emptyset$ .

When  $2 \leq i \leq m, f(u_i v_j) = \min (D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j), \dots, f(u_{i-1} v_j)\}))$  or  $f(u_i v_j) \in D(u_i) \cap D(v_j)$  when  $D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j), \dots, f(u_{i-1} v_j)\}) = \emptyset$ .

It is not hard to see that  $C_f(u_i) = D(u_i), i = 1, 2, \dots, m; C_f(v_j) = D(v_j), j = 1, 2, \dots, n$  and moreover  $f(u_i) > f(v_j)$ , therefore our coloring  $f$  is a vertex distinguishing IE-total coloring and then  $\chi_{vt}^{ie}(K_{m,n}) \leq k$ . ■

**Theorem 6.** *Let  $m \geq 2$ ,  $\binom{m+2}{1} + \binom{m+2}{2} + \dots + \binom{m+2}{m+1} - 2m + 1 < n \leq \binom{m+2}{1} + \binom{m+2}{2} + \dots + \binom{m+2}{m+1} - m$ , i.e.,  $2^{m+2} - 2m - 1 < n \leq 2^{m+2} - m - 2$ . Then  $\chi_{vt}^{ie}(K_{m,n}) = m + 3$ .*

**Proof.** When  $2^{m+2} - 2m - 1 < n \leq 2^{m+2} - m - 2$ , we have  $\chi_{vt}^{ie}(K_{m,n}) \geq \xi(K_{m,n}) = m + 2$ . We first prove that  $K_{m,n}$  does not have a  $(m + 2)$ -VDIET coloring. Otherwise, suppose  $g$  is a VDIET coloring of  $K_{m,n}$  using colors  $1, 2, \dots, m + 2$ .

Let  $B_0 = \{g(u_1), g(u_2), \dots, g(u_m)\}$ ,  $B_i = \{1, 2, \dots, m + 2\} \setminus C_g(u_i), i = 1, 2, \dots, m$ . Note that  $B_0, B_1, B_2, \dots, B_m$  are distinct and at most one of them is an empty set.  $B_0, B_1, B_2, \dots, B_m$  are not the color sets of vertices  $v_1, v_2, \dots, v_n$ . We give a fact as follows.

**Observation 7.**  $|C_g(u_i)| \geq 2, i = 1, 2, \dots, m$ . Furthermore, there exists a vertex  $v \in \{v_1, v_2, \dots, v_n\}$  with  $|C_g(v)| = 1$ .

**Proof.** Suppose that there exists a vertex  $u_i \in \{u_1, u_2, \dots, u_m\}$  with  $C_g(u_i) = \{\alpha\}, \alpha \in \{1, 2, \dots, m + 2\}$ . Then  $\alpha \in C_g(v_j), j = 1, 2, \dots, n$ . However,  $2^{m+1} - 1 < 2^{m+2} - 2m - 1 < n$ , i.e., the subsets of  $\{1, 2, \dots, m + 2\}$  which contain  $\alpha$  cannot distinguish  $n$  vertices, this is a contradiction. Therefore,  $|C_g(u_i)| \geq 2, i = 1, 2, \dots, m$ .

Suppose  $|C_g(v_j)| \geq 2, j = 1, 2, \dots, n$ , i.e., all 1-subsets of  $\{1, 2, \dots, m + 2\}$  are not the color sets of vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ . Therefore, there are at most  $2^{m+2} - 1 - (m + 2) < 2^{m+2} - 1 - m < m + n$  nonempty subsets of  $\{1, 2, \dots, m + 2\}$  which may become the color sets of vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ . This is a contradiction. □

Using the above observation, without loss of generality, we assume  $C_g(v_1) = \{1\}$ , Then  $1 \in C_g(u_i), i = 1, 2, \dots, m, g(u_i) \neq 1, i = 1, 2, \dots, m$ .

It is obvious that  $B_0, B_1, B_2, \dots, B_m$  are not the color sets of any vertex  $u_i, i = 1, 2, \dots, m$ . Therefore, there are at most  $2^{m+2} - 1 - m < m + n$  nonempty subsets of  $\{1, 2, \dots, m + 2\}$  which may become the color sets of vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ . This is a contradiction.

So,  $\chi_{vt}^{ie}(K_{m,n}) \geq m + 3$ .

In the following we prove that  $K_{m,n}$  has a  $(m + 3)$ -VDIET coloring when  $2^{m+2} - 2m - 1 < n \leq 2^{m+2} - m - 2$ .

By Theorem 5, we can give  $K_{m,t}$  a  $(m + 3)$ -VDIET coloring  $f$  using colors  $1, 2, \dots, m + 3$ , where  $2^{m+2} - 2 - m < t \leq 2^{m+3} - 2m - 5$ . Now delete the vertices  $v_{n+1}, v_{n+2}, \dots, v_t$  and their colors, delete the edges  $u_i v_j, i = 1, 2, \dots, m, j = n + 1, n + 2, \dots, t$  and their colors. It is not hard to see that under the resulting coloring the color sets of  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$  do not change, so we get a  $(m + 3)$ -VDIET coloring  $g$  of  $K_{m,n}$  using colors  $1, 2, \dots, m + 3$ . ■

**Theorem 8.** *Let  $s$  be the minimum positive integer such that  $2^s - 1 \geq 3m$ . When  $2^r - 2m - 1 < n \leq 2^{r+1} - 2m - 1$ , we have  $\chi_{vt}^{ie}(K_{m,n}) = r + 1$ , where  $r = m + 1, m, m - 1$  and  $r \geq s$ .*

**Proof.**  $\xi(K_{m,n}) = \begin{cases} r, & \text{when } 2^r - 2m - 1 < n \leq 2^r - m - 1; \\ r + 1, & \text{when } 2^r - m - 1 < n \leq 2^{r+1} - 2m - 1. \end{cases}$

When  $2^r - 2m - 1 < n \leq 2^r - m - 1$ , it is obvious that  $\chi_{vt}^{ie}(K_{m,n}) \geq r$ . We prove that  $K_{m,n}$  does not have an  $r$ -VDIET coloring when  $r = m + 1, m, m - 1$ . If not, let  $g$  be an  $r$ -VDIET coloring of  $K_{m,n}$  using colors  $1, 2, \dots, r$ . First we give four claims as follows.

**Claim 9.**  $|C(v_j)| \geq 2, j = 1, 2, \dots, n$ .

**Proof.** Suppose the claim is not true, without loss of generality, we assume  $C(v_1) = \{1\}$ . Then  $1 \in C(u_i), i = 1, 2, \dots, m$ . Let  $B_0 = \{g(u_1), g(u_2), \dots, g(u_m)\}$ ,  $B_i = \{1, 2, \dots, r\} \setminus C(u_i), i = 1, 2, \dots, m$ . Note that  $1 \notin B_0, 1 \notin B_i, i = 1, 2, \dots, m$ , we have  $B_0, B_1, B_2, \dots, B_m$  are distinct and not the color sets of vertices  $u_1, u_2, \dots, u_m$ . Moreover, none of  $B_0, B_1, B_2, \dots, B_m$  is the color set of any vertex  $v_j, j = 1, 2, \dots, n$ , (because  $C(u_i) \cap C(v_j) = \emptyset, i = 1, 2, \dots, m, j = 1, 2, \dots, n$ , and two adjacent vertices must have different colors). At most one of  $B_0, B_1, B_2, \dots, B_m$  is an empty set, so there are at most  $2^r - 1 - m$  nonempty subsets of  $\{1, 2, \dots, r\}$  which are available for the vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ . However,  $2^r - 1 - m < m + n$ , i.e., these subsets cannot distinguish  $m + n$  vertices, this is a contradiction. □

**Claim 10.**  $|C(u_i)| \geq 2, i = 1, 2, \dots, m$ .

**Proof.** Suppose the claim is not true. Without loss of generality we assume  $C(u_1) = \{1\}$ . Then  $1 \in C(v_j), j = 1, 2, \dots, n$ . Thus,  $\overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  are not available for vertices  $v_1, v_2, \dots, v_n$ . Moreover,  $\overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  cannot be the color sets of vertices  $u_1, u_2, \dots, u_m$  because  $C(u_i) \cap C(v_j) \neq \emptyset$ . At most one of  $\overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  is an empty set, so there are at most  $2^r - 1 - (n - 1)$  nonempty subsets of  $\{1, 2, \dots, r\}$  which can be the color sets of vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ . However,  $2^r - 1 - (n - 1) \leq 2^r - 1 - m < m + n$ , these subsets cannot distinguish  $m + n$  vertices, this is a contradiction. □

**Claim 11.**  $C(u_1) \cap C(u_2) \cap \dots \cap C(u_m) = \emptyset$ .

**Proof.** Suppose  $1 \in C(u_i), i = 1, 2, \dots, m$ . Then the  $m + 1$  distinct subsets  $\{1\}, \overline{C}(u_1), \overline{C}(u_2), \dots, \overline{C}(u_m)$  are not available for any vertex, and at most one of them is an empty set. Then there are at most  $2^r - 1 - m$  subsets of  $\{1, 2, \dots, r\}$  which can be the color sets of vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ . However,  $2^r - 1 - m < m + n$ , so these subsets cannot distinguish  $m + n$  vertices, this is a contradiction. □

**Claim 12.**  $C(v_1) \cap C(v_2) \cap \dots \cap C(v_n) = \emptyset$ .

*Proof.* Suppose  $1 \in C(v_j), j = 1, 2, \dots, n$ . Then the  $n + 1$  distinct subsets  $\{1\}, \overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  are not available for any vertex, and at most one of them is an empty set. The remaining  $2^r - 1 - n$  subsets of  $\{1, 2, \dots, r\}$  cannot distinguish  $m + n$  vertices because  $2^r - 1 - n \leq 2^r - 1 - m < m + n$ , this is a contradiction. □

Now we consider two cases.

*Case 1.*  $r = m, m + 1$ . By Claims 9 and 10, all 1-subsets of  $\{1, 2, \dots, r\}$  cannot be the color sets of any vertex. So there are at most  $2^r - 1 - r \leq 2^r - m - 1 < m + n$  subsets of  $\{1, 2, \dots, r\}$  which are available for vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ . This is a contradiction.

*Case 2.*  $r = m - 1$ . By Claims 9 and 10, all the 1-subsets  $\{1\}, \{2\}, \dots, \{m - 1\}$  cannot be the color sets of any vertex. The remaining  $2^{m-1} - 1 - (m - 1) = 2^{m-1} - m$  subsets of  $\{1, 2, \dots, m - 1\}$  cannot distinguish  $m + n$  vertices when  $2^{m-1} - 2m < n \leq 2^{m-1} - m - 1$ , this is a contradiction, so  $K_{m,n}$  does not have an  $(m - 1)$ -VDIET coloring when  $2^{m-1} - 2m < n \leq 2^{m-1} - m - 1$ .

Now we consider the case  $n = 2^{m-1} - 2m$ . Let  $t = |\{g(u_1), g(u_2), \dots, g(u_m)\}|$ . Without loss of generality we assume  $\{g(u_1), g(u_2), \dots, g(u_m)\} = \{1, 2, \dots, t\}$ . By Claims 11 and 12 we know that  $2 \leq t \leq r - 2$ , thus if  $r \leq 3$ , this is a contradiction. So  $r \geq 4$ . None of 2-subsets of  $\{1, 2, \dots, t\}$  is available for  $v_1, v_2, \dots, v_n$ .

If  $\{1, 2\} \notin \{C(u_1), C(u_2), \dots, C(u_m)\}$ , then at most  $2^{m-1} - 1 - m < m + n$  subsets of  $\{1, 2, \dots, m - 1\}$  are available for vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ , this is a contradiction.

Therefore,  $\{1, 2\} \in \{C(u_1), C(u_2), \dots, C(u_m)\}$ . Without loss of generality, assume  $C(u_1) = \{1, 2\}$ . By Claim 12, there are at least two colors among  $v_1, v_2, \dots, v_n$ , say  $t + 1, t + 2$ . Then  $\{t + 1, t + 2\} \notin \{C(u_1), C(u_2), \dots, C(u_m)\}$ . If  $\{t + 1, t + 2\} \notin \{C(v_1), C(v_2), \dots, C(v_n)\}$ , then at most  $2^{m-1} - 1 - m < m + n$  subsets of  $\{1, 2, \dots, m - 1\}$  are available for vertices  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ , this is a contradiction. Thus  $\{t + 1, t + 2\} \in \{C(v_1), C(v_2), \dots, C(v_n)\}$ . Then  $t + 1 \in C(u_i)$  or  $t + 2 \in C(u_i), i = 1, 2, \dots, m$ . However,  $C(u_1) = \{1, 2\}$ , this is a contradiction.

So,  $K_{m,n}$  does not have an  $r$ -VDIET coloring when  $2^{m-1} - 2m \leq n \leq 2^{m-1} - m - 1$  and  $r = m + 1, m, m - 1$ .

In the following we give an  $(r + 1)$ -VDIET coloring of  $K_{m,n}$  using colors  $1, 2, \dots, r, r + 1$ , where  $r = m - 1, m, m + 1$ .

Let  $V(K_{m,n}) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$  and  $E(K_{m,n}) = \{u_i v_j : i = 1, 2, \dots, m; j = 1, 2, \dots, n\}$ .

Put  $D(u_i) = \{1, 2, \dots, r + 1\} \setminus \{i\}, i = 1, 2, \dots, m - 1, D(u_m) = \{1, 2, \dots, r + 1\}; D(v_j) = \{j, r + 1\}, j = 1, 2, \dots, m - 1$ .



When  $r = m + 1$ , put  $D(v_{r-1}) = \{r - 1\}, D(v_r) = \{r\}$ . When  $r = m$ , put  $D(v_r) = \{r\}$ .

Now distribute other subsets of  $\{1, 2, \dots, r + 1\}$  with cardinality between 2 and  $r$  to vertices  $v_{r+1}, v_{r+2}, \dots, v_n$ . These  $n - r$  subsets are denoted by  $D(v_{r+1}), D(v_{r+2}), \dots, D(v_n)$ , respectively.

Construct a mapping  $f$  from  $V(K_{m,n}) \cup E(K_{m,n})$  to  $\{1, 2, \dots, r + 1\}$  as follows: Put  $f(u_i) = r + 1, i = 1, 2, \dots, m, f(v_j) = \min D(v_j), j = 1, 2, \dots, n, f(u_i v_i) = r + 1$  for  $i = 1, 2, \dots, m - 1, f(u_i v_j) = j, i = 1, 2, \dots, m, j = 1, 2, \dots, m - 1, i \neq j, f(u_i v_j) = j, i = 1, 2, \dots, m, j = m, \dots, r$  (if  $r = m$  or  $m + 1$ ).

For each  $j = r + 1, r + 2, \dots, n$ , we recursively let  $f(u_1 v_j) = \min (D(u_1) \cap (D(v_j) \setminus \{f(v_j)\}))$  or  $f(u_1 v_j) \in D(u_1) \cap D(v_j)$  when  $D(u_1) \cap (D(v_j) \setminus \{f(v_j)\}) = \emptyset$ .

When  $2 \leq i \leq m, f(u_i v_j) = \min (D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j), \dots, f(u_{i-1} v_j)\}))$  or  $f(u_i v_j) \in D(u_i) \cap D(v_j)$  when  $D(u_i) \cap (D(v_j) \setminus \{f(v_j), f(u_1 v_j), f(u_2 v_j), \dots, f(u_{i-1} v_j)\}) = \emptyset$ .

It is not hard to see that  $C_f(u_i) = D(u_i), i = 1, 2, \dots, m; C_f(v_j) = D(v_j), j = 1, 2, \dots, n$  and moreover  $f(u_i) > f(v_j)$ , therefore our coloring  $f$  is a vertex distinguishing IE-total coloring and then  $\chi_{vt}^{ie}(K_{m,n}) \leq r + 1, r = m - 1, m, m + 1$ .

So  $\chi_{vt}^{ie}(K_{m,n}) = r + 1, r = m - 1, m, m + 1$ . ■

**Theorem 13.**  $\chi_{vt}^{ie}(K_{1,n}) = \begin{cases} 2, & \text{when } n = 1; \\ 3, & \text{when } n = 2; \\ k, & \text{when } \binom{k-1}{1} + \binom{k-1}{2} - 1 < n \leq \binom{k}{1} + \binom{k}{2} - 1, \\ & k \geq 3. \end{cases}$

**Proof.** It is easy to prove the theorem in the case  $n = 1, 2$ . By Theorem 5, this theorem is valid when  $\binom{k-1}{1} + \binom{k-1}{2} - 1 < n \leq \binom{k}{1} + \binom{k}{2} - 1, k \geq 3$ . ■

**Theorem 14.**  $\chi_{vt}^{ie}(K_{2,n}) = \begin{cases} 3, & \text{when } n = 2, 3; \\ 4, & \text{when } n = 4, 5, \dots, 11; \\ 5, & \text{when } n = 12; \\ k, & \text{when } \binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3} - 2 < n \\ & \leq \binom{k}{1} + \binom{k}{2} + \binom{k}{3} - 2, k \geq 5. \end{cases}$

**Proof.** By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when  $n \geq 4$ . Now we consider the case  $n = 2, 3$ . It is obvious that  $\chi_{vt}^{ie}(K_{2,n}) \geq \xi(K_{2,n}) = 3$  when  $n = 2, 3$ . Let  $V(K_{2,n}) = \{u_1, u_2, v_1, v_2, \dots, v_n\}$  and  $E(K_{2,n}) = \{u_i v_j : 1 \leq i \leq 2, 1 \leq j \leq n\}$ . We give a 3-VDIET coloring of  $K_{2,n}$  using colors 1, 2, 3 when  $n = 2, 3$ .

Let  $u_1, u_2$  receive color 1,  $v_1$  and its incident edges receive color 2. We assign color 3, 3, 1 to  $v_2, u_1 v_2, u_2 v_2$ , respectively. And when  $n = 3$ , we assign color 2, 3, 2 to  $v_3, u_1 v_3, u_2 v_3$ , respectively.

Then under the above coloring, we have  $C(u_1) = \{1, 2, 3\}$ ,  $C(u_2) = \{1, 2\}$ ,  $C(v_1) = \{2\}$ ,  $C(v_2) = \{1, 3\}$  and  $C(v_3) = \{2, 3\}$  (when  $n = 3$ ). Thus the above coloring is a VDIET coloring of  $K_{2,n}(n = 2, 3)$  using 3 colors. ■

**Theorem 15.**  $\chi_{vt}^{ie}(K_{3,n}) = \begin{cases} 4, & \text{when } 3 \leq n \leq 9; \\ 5, & \text{when } 10 \leq n \leq 25; \\ 6, & \text{when } n = 26, 27; \\ k, & \text{when } \binom{k-1}{1} + \dots + \binom{k-1}{4} - 3 < n \\ & \leq \binom{k}{1} + \dots + \binom{k}{4} - 3, k \geq 6. \end{cases}$

*Proof.* By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when  $n \geq 10$ . Now we consider the case  $3 \leq n \leq 9$ .

$$\xi(K_{3,n}) = \begin{cases} 3, & \text{when } n = 3, 4; \\ 4, & \text{when } 5 \leq n \leq 9. \end{cases}$$

Let  $V(K_{3,n}) = \{u_1, u_2, u_3, v_1, v_2, \dots, v_n\}$  and  $E(K_{3,n}) = \{u_i v_j : 1 \leq i \leq 3, 1 \leq j \leq n\}$ . We prove  $K_{3,n}$  does not have a 3-VDIET coloring when  $n = 3, 4$ . If not, let  $g$  be a 3-VDIET coloring of  $K_{3,n}$  using colors 1, 2, 3. Then  $|C(u_i)| \geq 2, i = 1, 2, 3$ . (Otherwise we assume  $C(u_1) = \{1\}$ . Then  $1 \in C(v_j), j = 1, 2, \dots, n$ . Thus  $\overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  are not available for any vertex and at most one of them is an empty set. Therefore there are at most  $2^3 - 1 - 2 = 5$  nonempty subsets of  $\{1, 2, 3\}$  which can be the color sets of vertices  $u_1, u_2, u_3, v_1, v_2, \dots, v_n$ . Five subsets cannot distinguish  $n + 3$  vertices when  $n = 3, 4$ , this is a contradiction.) Furthermore,  $|C(v_j)| \geq 2, j = 1, 2, \dots, n$ . (Otherwise we assume  $C(v_1) = \{1\}$ . Then  $1 \in C(u_i), i = 1, 2, 3$ . Thus  $\overline{C}(u_1), \overline{C}(u_2), \overline{C}(u_3)$  are not available for any vertex and at most one of them is an empty set. Therefore there are at most  $2^3 - 1 - 2 = 5$  nonempty subsets of  $\{1, 2, 3\}$  which can be the color sets of vertices  $u_1, u_2, u_3, v_1, v_2, \dots, v_n$ . Five subsets cannot distinguish  $n + 3$  vertices when  $n = 3, 4$ , this is a contradiction.) So three 1-subsets of  $\{1, 2, 3\}$  are not available for any vertex, the remaining 4 nonempty subsets of  $\{1, 2, 3\}$  cannot distinguish  $n + 3$  vertices when  $n = 3, 4$ , this is a contradiction. Therefore,  $\chi_{vt}^{ie}(K_{3,n}) \geq 4$  when  $n = 3, 4$ .

In the following we give a 4-VDIET coloring of  $K_{3,n}$  using colors 1, 2, 3, 4 when  $3 \leq n \leq 9$ .

Let  $u_1, u_2, u_3$  receive color 4. Suppose  $\mathcal{S}_1 = (\{3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\})$  and let  $D(v_i)$  be the  $i$ -th term of  $\mathcal{S}_1, i = 1, 2, \dots, n$ . Let  $v_1$  and its incident edges receive color 3, let  $v_2, u_3 v_2$  receive color 1 and  $u_1 v_2, u_2 v_2$  receive color 2.

For  $D(v_j) = \{a, b\}, 3 \leq j \leq n, a < b$ , we assign  $a$  to  $u_1 v_j$  and  $v_j$ , assign  $b$  to  $u_2 v_j$  and  $u_3 v_j$ .

For  $D(v_j) = \{a, b, c\}, a < b < c$ , we assign  $a, b, c$  to  $u_1 v_j, u_2 v_j, u_3 v_j$  respectively and assign  $b$  to  $v_j$ .

Then  $C(u_1) = \{1, 2, 3, 4\}$ ,  $C(u_2) = \{2, 3, 4\}$ ,  $C(u_3) = \{1, 3, 4\}$  and  $C(v_j) = D(v_j), j = 1, 2, \dots, n$  with respect to the above coloring. Thus the above coloring is a VDIET coloring of  $K_{3,n}(3 \leq n \leq 9)$  using 4 colors. ■

**Theorem 16.**  $\chi_{vt}^{ie}(K_{4,n}) = \begin{cases} 4, & \text{when } 4 \leq n \leq 7; \\ 5, & \text{when } 8 \leq n \leq 23; \\ 6, & \text{when } 24 \leq n \leq 55; \\ 7, & \text{when } 56 \leq n \leq 58; \\ k, & \text{when } \binom{k-1}{1} + \dots + \binom{k-1}{5} - 4 < n \\ & \leq \binom{k}{1} + \dots + \binom{k}{5} - 4, k \geq 7. \end{cases}$

**Proof.** It is easy to verify the theorem is valid in each case when  $n \geq 8$  by Theorem 5, 6, 8 respectively. Now we consider the case  $4 \leq n \leq 7$ .

It is obvious  $\chi_{vt}^{ie}(K_{4,n}) \geq \xi(K_{4,n}) = 4$ , when  $4 \leq n \leq 7$ .

In the following we give a 4-VDIET coloring of  $K_{4,n}$  using colors 1, 2, 3, 4 when  $4 \leq n \leq 7$ . Let  $V(K_{4,n}) = \{u_1, u_2, u_3, u_4, v_1, v_2, \dots, v_n\}$  and  $E(K_{4,n}) = \{u_i v_j : i = 1, 2, 3, 4; j = 1, 2, \dots, n\}$ .

Let  $u_1, u_2, u_3, u_4$  receive color 4. Suppose  $\mathcal{S}_2 = (\{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\})$  and let  $D(v_i)$  be the  $i$ -th term of  $\mathcal{S}_2, i = 1, 2, \dots, n$ . Let  $v_i$  receive the minimum number of  $D(v_i), i = 1, 2, \dots, n$ .

For  $D(v_j) = \{j, 4\}, j = 1, 2, 3$ , we assign color 4 to  $u_j v_j$  and color  $j$  to  $u_i v_j, i = 1, 2, 3, 4, i \neq j$ .

For  $D(v_j) = \{a, b\}, 4 \leq j \leq n, a < b$ , we assign color  $b$  to all edges  $u_i v_j$  if  $i \neq b$  and color  $a$  to its remaining incident edge  $u_b v_j$ .

For  $D(v_j) = \{1, 2, 3\}$ , we assign color 2 to  $u_i v_j$  if  $i \neq 2$  and assign color 3 to  $u_2 v_j$ .

Then  $C(u_i) = \{1, 2, 3, 4\} \setminus \{i\}, i = 1, 2, 3, C(u_4) = \{1, 2, 3, 4\}$  and  $C(v_j) = D(v_j), j = 1, 2, \dots, n$  with respect to the above coloring. Thus the above coloring is a 4-VDIET coloring of  $K_{4,n}, 4 \leq n \leq 7$ . ■

**Theorem 17.**  $\chi_{vt}^{ie}(K_{5,n}) = \begin{cases} 5, & \text{when } 6 \leq n \leq 21; \\ 6, & \text{when } 22 \leq n \leq 53; \\ 7, & \text{when } 54 \leq n \leq 117; \\ 8, & \text{when } 118 \leq n \leq 121; \\ k, & \text{when } \binom{k-1}{1} + \dots + \binom{k-1}{6} - 5 < n \\ & \leq \binom{k}{1} + \dots + \binom{k}{6} - 5, k \geq 8. \end{cases}$

**Proof.** By Theorem 5, 6, 8 respectively we know the theorem is valid in each case. ■

**Theorem 18.**  $\chi_{vt}^{ie}(K_{6,n}) = \begin{cases} 5, & \text{when } 6 \leq n \leq 19; \\ 6, & \text{when } 20 \leq n \leq 51; \\ 7, & \text{when } 52 \leq n \leq 115; \\ 8, & \text{when } 116 \leq n \leq 243; \\ 9, & \text{when } 244 \leq n \leq 248; \\ k, & \text{when } \binom{k-1}{1} + \dots + \binom{k-1}{7} - 6 < n \\ & \leq \binom{k}{1} + \dots + \binom{k}{7} - 6, k \geq 9. \end{cases}$

*Proof.* By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when  $n \geq 20$ . Now we consider the case  $6 \leq n \leq 19$ .

$$\xi(K_{6,n}) = \begin{cases} 4, & \text{when } 6 \leq n \leq 9; \\ 5, & \text{when } 10 \leq n \leq 19. \end{cases}$$

Let  $V(K_{6,n}) = \{u_1, u_2, \dots, u_6, v_1, v_2, \dots, v_n\}$  and  $E(K_{6,n}) = \{u_i v_j : 1 \leq i \leq 6, 1 \leq j \leq n\}$ . We prove  $K_{6,n}$  does not have a 4-VDIET coloring when  $6 \leq n \leq 9$ . If not, suppose  $g$  is a 4-VDIET coloring of  $K_{6,n}$  ( $6 \leq n \leq 9$ ) using colors 1, 2, 3, 4. Then  $|C(u_i)| \geq 2, i = 1, 2, \dots, 6$ . (Otherwise we assume  $C(u_1) = \{1\}$ . Then  $1 \in C(v_j), j = 1, 2, \dots, n$ . Thus  $\overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  are not available for any vertex and at most one of them is an empty set. Therefore there are at most  $2^4 - 1 - 5 = 10$  nonempty subsets of  $\{1, 2, 3, 4\}$  which can be the color sets of vertices  $u_1, u_2, \dots, u_6, v_1, v_2, \dots, v_n$ . These subsets cannot distinguish  $n + 6$  vertices when  $6 \leq n \leq 9$ , this is a contradiction.)

Furthermore,  $|C(v_j)| \geq 2, j = 1, 2, \dots, n$ . (Otherwise we assume  $C(v_1) = \{1\}$ , then  $1 \in C(u_i), i = 1, 2, \dots, 6$ . Thus  $\overline{C}(u_1), \overline{C}(u_2), \dots, \overline{C}(u_6)$  are not available for any vertex and at most one of them is an empty set. Therefore there are at most  $2^4 - 1 - 5 = 10$  nonempty subsets of  $\{1, 2, 3, 4\}$  which can be the color sets of vertices  $u_1, u_2, \dots, u_6, v_1, v_2, \dots, v_n$ . These subsets cannot distinguish  $n + 6$  vertices when  $6 \leq n \leq 9$ , this is a contradiction.) So four 1-subsets of  $\{1, 2, 3, 4\}$  are not available for any vertex, the remaining 11 nonempty subsets of  $\{1, 2, 3, 4\}$  cannot distinguish  $n + 6$  vertices when  $6 \leq n \leq 9$ , this is a contradiction. Therefore,  $\chi_{vt}^{ie}(K_{6,n}) \geq 5$  when  $6 \leq n \leq 9$ .

In the following we give a 5-VDIET coloring of  $K_{6,n}$  using colors 1, 2, 3, 4, 5 when  $6 \leq n \leq 19$ .

Let  $u_1, u_2, \dots, u_6$  receive color 5. Suppose  $\mathcal{S}_3 = (\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\})$  and let  $D(v_i)$  be the  $i$ -th term of  $\mathcal{S}_3, i = 1, 2, \dots, n$ . Let  $D(u_i) = \{1, 2, 3, 4, 5\} \setminus \{i\}, i = 1, 2, 3, 4, D(u_5) = \{1, 2, 3, 4, 5\}$  and  $D(u_6) = \{1, 2, 5\}$ .

Let  $u_i v_i (i = 1, 2, 3, 4), u_6 v_3$  and  $u_6 v_4$  receive color 5. Let  $v_j$  and the other incident edges of  $v_j$  receive color  $j, j = 1, 2, 3, 4$ .

For  $D(v_j) = \{a, b\}, 5 \leq j \leq n, a < b$ , we assign  $b$  to  $u_i v_j$  if  $b \in D(u_i)$ , assign  $a$  to  $v_j$  and its remaining incident edges.

For  $D(v_j) = \{a, b, c\}, \{b, c\} \neq \{3, 4\}, a < b < c$ , we assign  $b$  to  $u_i v_j$  if  $b \in D(u_i)$ , assign  $c$  to  $u_i v_j$  if  $b \notin D(u_i)$ , and assign  $a$  to  $v_j$ .

For  $D(v_j) = \{a, 3, 4\}, a = 1, 2$ , we assign  $a$  to  $u_i v_j$  if  $a \in D(u_i)$ , assign 3 to  $u_i v_j$  if  $a \notin D(u_i)$ , and assign 4 to  $v_j$ .

For  $D(v_j) = \{1, 2, 3, 4\}$ , we assign 3 to  $u_i v_j$  if  $3 \in D(u_i)$ , assign 4 to  $u_3 v_j$ , assign 2 to  $u_6 v_j$ , and assign 1 to  $v_j$ .

Then  $C(u_i) = D(u_i), 1 \leq i \leq 6$  and  $C(v_j) = D(v_j), 1 \leq j \leq n$  with respect to the above coloring. Thus the above coloring is a 5-VDIET coloring of  $K_{6,n}, 6 \leq n \leq 19$ . ■

$$\text{Theorem 19. } \chi_{vt}^{ie}(K_{7,n}) = \begin{cases} 5, & \text{when } 7 \leq n \leq 17; \\ 6, & \text{when } 18 \leq n \leq 49; \\ 7, & \text{when } 50 \leq n \leq 113; \\ 8, & \text{when } 114 \leq n \leq 241; \\ 9, & \text{when } 242 \leq n \leq 497; \\ 10, & \text{when } 498 \leq n \leq 503; \\ k, & \text{when } \binom{k-1}{1} + \dots + \binom{k-1}{8} - 7 < n \\ & \leq \binom{k}{1} + \dots + \binom{k}{8} - 7, k \geq 10. \end{cases}$$

**Proof.** By Theorem 5, 6, 8 respectively we know the theorem is valid in each case when  $n \geq 50$ . Now we consider the case  $n \leq 49$ .

$$\xi(K_{7,n}) = \begin{cases} 4, & \text{when } n = 7, 8; \\ 5, & \text{when } 9 \leq n \leq 24; \\ 6, & \text{when } 25 \leq n \leq 49. \end{cases}$$

Let  $V(K_{7,n}) = \{u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n\}$  and  $E(K_{7,n}) = \{u_i v_j : 1 \leq i \leq 7, 1 \leq j \leq n\}$ .

We prove  $K_{7,n}$  does not have a 4-VDIET coloring when  $n = 7, 8$ . If not, suppose  $g$  is a 4-VDIET coloring of  $K_{7,n}(n = 7, 8)$  using colors 1, 2, 3, 4. Then  $|C(u_i)| \geq 2, i = 1, 2, \dots, 7$ . Otherwise we assume  $C(u_1) = \{1\}$ . Then  $1 \in C(v_j), j = 1, 2, \dots, n, n = 7, 8$ . Thus  $\overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  are not available for any vertex and at most one of them is an empty set. Therefore there are at most  $2^4 - 1 - 6 = 9$  nonempty subsets of  $\{1, 2, 3, 4\}$  which can be the color sets of vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . These subsets cannot distinguish 14 or 15 vertices, this is a contradiction.

Furthermore,  $|C(v_j)| \geq 2, j = 1, 2, \dots, n, n = 7, 8$ . Otherwise we assume  $C(v_1) = \{1\}$ . Then  $1 \in C(u_i), i = 1, 2, \dots, 7$ . Thus  $\overline{C}(u_1), \overline{C}(u_2), \dots, \overline{C}(u_7)$  are not available for any vertex and at most one of them is an empty set. Therefore there are at most  $2^4 - 1 - 6 = 9$  nonempty subsets of  $\{1, 2, 3, 4\}$  which can be the color sets of vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . These subsets cannot distinguish 14 or 15 vertices, this is also a contradiction.) So four 1-subsets of  $\{1, 2, 3, 4\}$  are not available for any vertex, the remaining 11 nonempty subsets of

$\{1, 2, 3, 4\}$  cannot distinguish 14 or 15 vertices, this is a contradiction. Therefore,  $\chi_{vt}^{ie}(K_{7,n}) \geq 5$  when  $n = 7, 8$ .

In the following we give a 5-VDIET coloring of  $K_{7,n}$  using colors 1, 2, 3, 4, 5 when  $7 \leq n \leq 17$ .

Let  $u_1, u_2, \dots, u_7$  receive color 5. Suppose  $\mathcal{S}_4 = (\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\})$  and let  $D(v_i)$  be the  $i$ -th term of  $\mathcal{S}_4, i = 1, 2, \dots, n$ . Let  $D(u_i) = \{1, 2, 3, 4, 5\} \setminus \{i\}, i = 1, 2, 3, 4, D(u_5) = \{1, 3, 5\}, D(u_6) = \{2, 4, 5\}$  and  $D(u_7) = \{1, 2, 3, 4, 5\}$ .

Let  $u_1v_1$  and  $u_6v_1$  receive color 5,  $v_1$  and its other incident edges receive color 1. Let  $u_2v_2$  and  $u_5v_2$  receive color 5,  $v_2$  and its other incident edges receive color 2. Let  $u_3v_3$  and  $u_6v_3$  receive color 5,  $v_3$  and its other incident edges receive color 3. Let  $u_4v_4$  and  $u_5v_4$  receive color 5,  $v_4$  and its other incident edges receive color 4.

For  $D(v_j) = \{a, b\}, 5 \leq j \leq n, a < b$ , we assign  $b$  to  $u_iv_j$  if  $b \in D(u_i)$ , assign  $a$  to  $v_j$  and its remaining incident edges.

For  $D(v_j) = \{a, b, c\}, \{a, b, c\} \neq \{1, 2, 4\}, a < b < c$ , we assign  $b$  to  $u_iv_j$  if  $b \in D(u_i)$ , assign  $c$  to  $u_iv_j$  if  $b \notin D(u_i)$ , and assign  $a$  to  $v_j$ .

For  $D(v_j) = \{1, 2, 4\}$ , we assign 1 to  $u_iv_j$  if  $1 \in D(u_i)$ , assign 2 to  $u_iv_j$  if  $1 \notin D(u_i)$ , and assign 4 to  $v_j$ .

For  $D(v_j) = \{1, 2, 3, 4\}$ , we assign 2 to  $u_iv_j$  if  $2 \in D(u_i)$ , assign 4, 3, 1 to  $u_2v_j, u_5v_j$  and  $v_j$  respectively.

Then  $C(u_i) = D(u_i), 1 \leq i \leq 7$  and  $C(v_j) = D(v_j), j = 1, 2, \dots, n$  with respect to the above coloring. Thus the above coloring is a 5-VDIET coloring of  $K_{7,n}, 7 \leq n \leq 17$ .

We prove  $K_{7,n}$  does not have a 5-VDIET coloring when  $18 \leq n \leq 24$ . If not, suppose  $g$  is a 5-VDIET coloring of  $K_{7,n}(18 \leq n \leq 24)$  using colors 1, 2, 3, 4, 5. First we give four claims as follows.

**Claim 20.**  $|C(u_i)| \geq 2, i = 1, 2, \dots, 7$ .

*Proof.* Suppose the claim is not true, without loss of generality we assume  $C(u_1) = \{1\}$ . Then  $1 \in C(v_j), j = 1, 2, \dots, n, 18 \leq n \leq 24$ . Thus  $\overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  are not available for any vertex and at most one of them is an empty set. Therefore there are at most  $2^5 - 1 - 17 = 14$  nonempty subsets of  $\{1, 2, 3, 4, 5\}$  which can be the color sets of vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . These subsets cannot distinguish  $n + 7$  vertices when  $18 \leq n \leq 24$ , this is a contradiction.  $\square$

**Claim 21.**  $|C(v_j)| \geq 2, j = 1, 2, \dots, n, 18 \leq n \leq 24$ .

*Proof.* Suppose the claim is not true, without loss of generality we assume  $C(v_1) = \{1\}$ . Then  $1 \in C(u_i), i = 1, 2, \dots, 7$ . Thus  $\overline{C}(u_1), \overline{C}(u_2), \dots, \overline{C}(u_7), \{g(u_1), g(u_2), \dots, g(u_7)\}$  are not available for any vertex and at most one of them

is an empty set. Therefore there are at most  $2^5 - 1 - 7 = 24$  nonempty subsets of  $\{1, 2, 3, 4, 5\}$  which can be the color sets of vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . These subsets cannot distinguish  $n + 7$  vertices when  $18 \leq n \leq 24$ , this is also a contradiction.  $\square$

**Claim 22.**  $C(u_1) \cap C(u_2) \cap \dots \cap C(u_7) = \emptyset$ .

**Claim 23.**  $C(v_1) \cap C(v_2) \cap \dots \cap C(v_n) = \emptyset, 18 \leq n \leq 24$ .

The proofs of Claim 22 and Claim 23 are analogous to the proofs of Claim 11 and Claim 12 in Theorem 8, respectively.

By Claims 20 and 21, five 1-subsets of  $\{1, 2, 3, 4, 5\}$  are not available for any vertex. The remaining 26 nonempty subsets of  $\{1, 2, 3, 4, 5\}$  cannot distinguish  $n + 7$  vertices when  $20 \leq n \leq 24$ , this is a contradiction. So we assume  $n = 18, 19$  in the following.

Let  $t = |\{g(u_1), g(u_2), \dots, g(u_7)\}|$ , and  $\{g(u_1), g(u_2), \dots, g(u_7)\} = \{1, 2, \dots, t\}$ , by Claim 22 and Claim 23, we know that  $t = 2$  or  $t = 3$ .

*Case 1.*  $t = 2, \{f(u_1), f(u_2), \dots, f(u_7)\} = \{1, 2\}$ . Of course  $\{1, 2\} \notin \{C(v_1), C(v_2), \dots, C(v_n)\}$ . If  $\{1, 2\} \in \{C(u_1), C(u_2), \dots, C(u_7)\}$ , then  $1 \in C(v_j)$  or  $2 \in C(v_j), j = 1, 2, \dots, n$ . Thus  $\{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}$  cannot be the color sets of any vertices. Moreover, five 1-subsets are not available for any vertex. Then at most  $2^5 - 1 - 5 - 4 = 22$  nonempty subsets of  $\{1, 2, 3, 4, 5\}$  are available for the vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . This is a contradiction because 22 subsets cannot distinguish 25 (when  $n = 18$ ) or 26 (when  $n = 19$ ) vertices. So  $\{1, 2\}$  is not available for any vertex.

If  $|C(u_i)| \geq 3, i = 1, 2, \dots, 7$ , then  $\overline{C}(u_1), \overline{C}(u_2), \dots, \overline{C}(u_7)$  cannot be the color sets of any vertices because there are 5 colors in all. At most one of  $\overline{C}(u_1), \overline{C}(u_2), \dots, \overline{C}(u_7)$  is an empty set, so there are at most  $2^5 - 1 - 6 - 1 = 24$  nonempty subsets of  $\{1, 2, 3, 4, 5\}$  are available for the vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . This is a contradiction because 24 subsets cannot distinguish 25 (when  $n = 18$ ) or 26 (when  $n = 19$ ) vertices.

Therefore, there exists a vertex  $u_{i_0}$  with  $|C(u_{i_0})| = 2$ . Since  $\{1, 2\}$  is not available for any vertex, so without loss of generality, we assume  $C(u_{i_0}) = \{1, 3\}$ , then  $1 \in C(v_j)$  or  $3 \in C(v_j), j = 1, 2, \dots, n$ . Thus  $\{4, 5\}$  is not available for any vertex. Furthermore,  $\{1, 2\}$  and five 1-subsets are not available for any vertex. There are at most  $2^5 - 1 - 5 - 2 = 24$  nonempty subsets of  $\{1, 2, 3, 4, 5\}$  are available for the vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . This is a contradiction because 24 subsets cannot distinguish 25 (when  $n = 18$ ) or 26 (when  $n = 19$ ) vertices.

So  $K_{7,n}(n = 18, 19)$  does not have a 5-VDIET coloring in this case.

*Case 2.*  $t = 3, \{f(u_1), f(u_2), \dots, f(u_7)\} = \{1, 2, 3\}$ . By Claim 23,  $|\{f(v_1), f(v_2), \dots, f(v_n)\}| \geq 2$ , so  $\{f(v_1), f(v_2), \dots, f(v_n)\} = \{4, 5\}$ . Then  $\{4, 5\}$  is not

the color set of any vertex  $u_i, i = 1, 2, \dots, 7$ . If  $\{4, 5\} \in \{C(v_1), C(v_2), \dots, C(v_n)\}$ , then  $4 \in C(u_i)$  or  $5 \in C(u_i), i = 1, 2, \dots, 7$ . Thus  $\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$  cannot be the color sets of any vertex. Moreover, five 1-subsets are not available for any vertex. Then at most  $2^5 - 1 - 5 - 4 = 22$  nonempty subsets of  $\{1, 2, 3, 4, 5\}$  are available for the vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . This is a contradiction because 22 subsets cannot distinguish 25 (when  $n = 18$ ) or 26 (when  $n = 19$ ) vertices. So  $\{4, 5\}$  is not available for any vertex.

If  $|C(v_j)| \geq 3, j = 1, 2, \dots, n$ , then  $\overline{C}(v_1), \overline{C}(v_2), \dots, \overline{C}(v_n)$  cannot be the color sets of any vertex because there are 5 colors in all. At most one of them is an empty set, so at most  $2^5 - 1 - (n - 1) \leq 14$  nonempty subsets of  $\{1, 2, 3, 4, 5\}$  are available for the vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . This is a contradiction because these subsets cannot distinguish 25 (when  $n = 18$ ) or 26 (when  $n = 19$ ) vertices.

Therefore, there exists a vertex  $v_{j_0}$  with  $|C(v_{j_0})| = 2$ . Since  $\{4, 5\}$  is not available for any vertex, so without loss of generality, we assume  $C(v_{j_0}) = \{1, 4\}$ . Then  $1 \in C(u_i)$  or  $4 \in C(u_i), i = 1, 2, \dots, 7$ . Thus  $\{2, 3\}$  is not available for any vertex. Moreover,  $\{4, 5\}$  and five 1-subsets are not available for any vertex. There are at most  $2^5 - 1 - 5 - 2 = 24$  nonempty subsets are available for the vertices  $u_1, u_2, \dots, u_7, v_1, v_2, \dots, v_n$ . This is a contradiction because 24 subsets cannot distinguish 25 (when  $n = 18$ ) or 26 (when  $n = 19$ ) vertices.

So  $K_{7,n}$  ( $n = 18, 19$ ) does not have a 5-VDIET coloring.

Therefore,  $\chi_{vit}^{ie}(K_{7,n}) \geq 6$  when  $18 \leq n \leq 49$ .

In the following we give a 6-VDIET coloring of  $K_{7,n}$  using colors 1, 2, 3, 4, 5, 6 when  $18 \leq n \leq 49$ .

Arrange all 49 subsets of  $\{1, 2, 3, 4, 5, 6\}$  except for  $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{1, 2, 4, 5, 6\}, \{1, 2, 3, 5, 6\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 6\}$  into a sequence  $\mathcal{S}_5$  such that the first 5 terms are  $\{1, 6\}, \{2, 6\}, \{3, 6\}, \{4, 6\}, \{5, 6\}$  respectively. Let  $D(v_j)$  be the  $j$ -th term of  $\mathcal{S}_5, j = 1, 2, \dots, n$ . Let  $D(u_i) = \{1, 2, 3, 4, 5, 6\} \setminus \{i\}, i = 1, 2, 3, 4, 5, D(u_6) = \{1, 2, 3, 4, 5, 6\}, D(u_7) = \{1, 2, 3, 6\}$ .

Let  $u_1, u_2, \dots, u_7$  receive color 6. Let  $v_j$  receive color  $j, j = 1, 2, \dots, 5$ . Let  $u_i v_i$  receive color 6,  $i = 1, 2, \dots, 5$ . Let  $u_i v_j$  receive color  $j, i = 1, 2, \dots, 6, j = 1, 2, \dots, 5, i \neq j$ . Let  $u_7 v_1, u_7 v_2, u_7 v_3, u_7 v_4$  and  $u_7 v_5$  receive colors 1, 2, 3, 6 and 6 respectively.

For  $D(v_j) = \{a, b\}, 6 \leq j \leq n, a < b$ , we assign  $b$  to  $u_i v_j$  if  $b \in D(u_i)$ , assign  $a$  to  $v_j$  and its remaining incident edges.

For  $D(v_j) = \{a, 4, 5\}, 1 \leq a \leq 3$ , we assign 5 to  $v_j, a$  to  $u_i v_j$  if  $a \in D(u_i)$ , assign 4 to  $u_i v_j$  otherwise.

For  $D(v_j) = \{a, b, c\}, a < b < c, \{b, c\} \neq \{4, 5\}$ , we assign  $a$  to  $v_j, b$  to  $u_i v_j$  if  $b \in D(u_i)$ , assign  $c$  to  $u_i v_j$  otherwise.

For  $D(v_j) = \{a, b, c, d\}, a < b < c < d$ , we assign  $a$  to  $v_j, b$  to  $u_i v_j$  if



$b \in D(u_i), i \neq 6$ , assign  $c$  to  $u_i v_j$  if  $b \notin D(u_i), c \in D(u_i), i \neq 6$ , and assign  $d$  to the remaining incident edges of  $v_j$ .

For  $D(v_j) = \{1, 2, 3, 4, 5\}$ , we assign 1 to  $v_j$ , assign 2, 3, 4, 5 to  $u_3 v_j, u_4 v_j, u_5 v_j, u_6 v_j$  respectively and assign 3 to the remaining incident edges of  $v_j$ .

Then  $C(u_i) = D(u_i), 1 \leq i \leq 7$  and  $C(v_j) = D(v_j), j = 1, 2, \dots, n$  with respect to the above coloring. Thus the above coloring is a 6-VDIET coloring of  $K_{7,n}, 24 \leq n \leq 49$ . ■

**Theorem 24.** *Let  $K_n$  be the complete graph of order  $n(n \geq 3)$ . Then  $\chi_{vt}^{ie}(K_n) = n$ .*

**Proof.** As any two vertices in  $K_n$  must receive different colors under an arbitrary VDIET coloring, therefore  $\chi_{vt}^{ie}(K_n) \geq n$ . Of course we may be able to show that  $\chi_{vt}^{ie}(K_n) = n$  by giving a VDIET coloring of  $K_n$  using  $n$  colors  $1, 2, \dots, n$  as follows. Assign colors  $1, 2, \dots, n$  to vertices  $v_1, v_2, \dots, v_n$  of  $K_n$  respectively and then let all edges receive the same color 1. ■

From the results obtained in this paper, we know that for any graph  $G$  discussed in this paper except  $K_n(n \geq 6)$ , we have  $\chi_{vt}^{ie}(G) = \xi(G)$  or  $\xi(G) + 1$ . So we propose the following conjectures.

**Conjecture 25.** *For a simple graph  $G$ , if its (proper vertex coloring) chromatic number  $\chi(G) \leq 4$ , then we have  $\chi_{vt}^{ie}(G) = \xi(G)$  or  $\xi(G) + 1$ .*

**Conjecture 26.** *For a simple graph  $G$ , we have  $\chi_{vt}^{ie}(G) \leq \max\{\xi(G) + 1, \chi(G)\}$ .*

**Conjecture 27.** *Let  $s$  be the minimum positive integer such that  $2^s - 1 \geq 3m$ . When  $2^r - 2m - 1 < n \leq 2^{r+1} - 2m - 1$ , we have  $\chi_{vt}^{ie}(K_{m,n}) = r + 1$ , where  $r = s, s + 1, \dots, m - 2, s \leq m - 2$ .*

**Acknowledgement**

The authors would like to thank the referees for their valuable comments and helpful suggestions.

REFERENCES

[1] P.N. Balister, B. Bollobás and R.H. Shelp, *Vertex distinguishing colorings of graphs with  $\Delta(G) = 2$* , Discrete Math. **252** (2002) 17–29.  
doi:10.1016/S0012-365X(01)00287-4

[2] P.N. Balister, O.M. Riordan and R.H. Schelp, *Vertex distinguishing edge colorings of graphs*, J. Graph Theory **42** (2003) 95–109.  
doi:10.1002/jgt.10076

- [3] C. Bazgan, A. Harkat-Benhamdine, H. Li and M. Woźniak, *On the vertex-distinguishing proper edge-colorings of graphs*, J. Combin. Theory (B) **75** (1999) 288–301.  
doi:10.1006/jctb.1998.1884
- [4] A.C. Burriss and R.H. Schelp, *Vertex-distinguishing proper edge-colorings*, J. Graph Theory **26** (1997) 73–82.  
doi:10.1002/(SICI)1097-0118(199710)26:2<73::AID-JGT2>3.0.CO;2-C
- [5] J. Černý, M. Horňák and R. Soták, *Observability of a graph*, Math. Slovaca **46** (1996) 21–31.
- [6] X. Chen, *Asymptotic behaviour of the vertex-distinguishing total chromatic numbers of  $n$ -cube*, J. Northwest Univ. **41(5)** (2005) 1–3.
- [7] F. Harary and M. Plantholt, *The point-distinguishing chromatic index*, in: F. Harary, J.S. Maybee (Eds.), Graphs and Application, New York (1985) 147–162.
- [8] M. Horňák and R. Soták, *Observability of complete multipartite graphs with equipotent parts*, Ars Combin. **41** (1995) 289–301.
- [9] M. Horňák and R. Soták, *Asymptotic behaviour of the observability of  $Q_n$* , Discrete Math. **176** (1997) 139–148.  
doi:10.1016/S0012-365X(96)00292-0
- [10] M. Horňák and R. Soták, *The fifth jump of the point-distinguishing chromatic index of  $K_{n,n}$* , Ars Combin. **42** (1996) 233–242.
- [11] M. Horňák and R. Soták, *Localization jumps of the point-distinguishing chromatic index of  $K_{n,n}$* , Discuss. Math. Graph Theory **17** (1997) 243–251.  
doi:10.7151/dmgt.1051
- [12] M. Horňák and N. Zagaglia Salvi, *On the point-distinguishing chromatic index of complete bipartite graphs*, Ars Combin. **80** (2006) 75–85.
- [13] N. Zagaglia Salvi, *On the point-distinguishing chromatic index of  $K_{n,n}$* , Ars Combin. **25B** (1988) 93–104.
- [14] N. Zagaglia Salvi, *On the value of the point-distinguishing chromatic index of  $K_{n,n}$* , Ars Combin. **29B** (1990) 235–244.
- [15] Z. Zhang, P. Qiu, B. Xu, J. Li, X.Chen and B.Yao, *Vertex-distinguishing total colorings of graphs*, Ars Combin. **87** (2008) 33–45.

Received 11 October 2010

Revised 11 July 2011

Accepted 5 March 2012