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Dedicated to Mietek Borowiecki on the occasion of his 70th birthday

A SURVEY OF THE PATH PARTITION CONJECTURE[1]

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Abstract

The Path Partition Conjecture (PPC) states that if G is any graph and (λ_1, λ_2) any pair of positive integers such that G has no path with more than $\lambda_1 + \lambda_2$ vertices, then there exists a partition (V_1, V_2) of the vertex set of G such that V_i has no path with more than λ_i vertices, $i = 1, 2$. We present a brief history of the PPC, discuss its relation to other conjectures and survey results on the PPC that have appeared in the literature since its first formulation in 1981.

Keywords: Path Partition Conjecture, Path Kernel Conjecture, generalized colourings, additive hereditary properties.

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1. Introduction

The Path Partition Conjecture (PPC) is an intriguing conjecture that is easy to state but difficult to settle. It is more than thirty years old and has generated a host of other interesting problems and conjectures. The PPC has been attacked by several authors and in this process two stronger conjectures have been created, explored and proved false. However, the PPC has survived and a variety of results supporting the conjecture have been proved. In addition, variations of the PPC have brought to light new questions for investigation.

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The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If $U \subseteq V(G)$, the subgraph of G induced by U is denoted by $G[U]$. The number of vertices in a longest path of a graph G is denoted by $\lambda(G)$. A simple way of stating the Path Partition Conjecture is as follows.

Conjecture 1.1 (PPC)**.** *If G is any graph and (λ_1, λ_2) any pair of positive integers such that $\lambda_1 + \lambda_2 = \lambda(G)$, then there exists a partition (V_1, V_2) of $V(G)$ such that $\lambda(G[V_i]) \leq \lambda_i,\ i = 1, 2$.*

As indicated in [27], the author's interest in the PPC was sparked in 1993 by a problem concerning certain generalized colourings introduced by Chartrand, Geller and Hedetniemi [23]. During a visit to Peter Mihók in 1995, the author was surprised to learn that the PPC was already an existing conjecture. In fact, it was orally formulated during a discussion between Mihók and Lovász in Szeged in 1981 and thereafter addressed in theses of Vronka and Hajnal [51, 36], supervised by Mihók and Lovász, respectively.

Laborde, Payan and Xuong were the fist authors to publish a paper on the PPC. Their paper [42], which appeared in 1983, dealt mostly with digraphs, but in the paper they formulated the PPC for undirected graphs and stated only the case $\lambda_1 = 1$ of the analogous conjecture for digraphs.

In 1995 Bondy [13] stated a conjecture that is close to the digraph analogue of the PPC, but the PPC for undirected graphs was not mentioned in the literature again until 1997, when it was addressed in Problem 1 of [15] and also in [20]. Since then, several authors have contributed results towards proving the PPC (see [17, 18, 21, 22, 25, 26, 29, 30, 33, 34, 45, 46, 47]) and its digraph analogue, the DPPC (see [1, 2, 3, 4, 5, 6, 8, 9, 28, 31, 32, 37]). The PPC and DPPC are also discussed in the text books [35] and [10], respectively.

The lattice of additive hereditary properties of graphs, treated in the paper [15] by Borowiecki *et al.*, provides a convenient framework for investigating partition problems that involve additive hereditary properties. The purpose of this survey is to bring together the results on the PPC that have appeared to date and discuss them within this framework. Section 2 provides the necessary notation and terminology for this purpose.

In Section 3 we discuss two refuted conjectures, either of which would have implied the truth of the PPC. In the place of each fallen conjecture we formulate a weaker conjecture that still implies the PPC but is as yet unrefuted. Moreover, each new conjecture has led to further progress towards settling the PPC in the affirmative.

In Section 4 we provide some useful results relating the cycle structure of a graph to the partitions required by the PPC. In Section 5 it is briefly explained how these results on cycle structure, together with the Ryjáček closure operation, led to a proof that the PPC holds for claw-free graphs. Results on the PPC for other special classes of graphs, such as the class of planar graphs, are also presented.

A graph G of order n is *t-deficient* if $t = n - \lambda(G)$. The main result in Section 6 implies that the PPC holds asymptotically for t-deficient graphs, for each $t \geq 0$.

The generalized colouring problem that had originally motivated the author to investigate the PPC is discussed in Section 7.

In Section 8 the directed and oriented analogues of the PPC are briefly discussed.

2. The PPC and the Lattice of Additive Hereditary Properties

The notation provided in this section is taken from the survey [15] by Borowiecki, Broere, Frick, Mihók and Semanišin.

The class of all finite simple graphs is denoted by $\mathcal{I}$. A *graph property* is a nonempty isomorphism-closed subclass of $\mathcal{I}$. A property $\mathcal{P}$ is *hereditary* if it is closed under taking subgraphs and *additive* if it is closed under taking disjoint unions.

Some well-known additive hereditary properties are given in the list below.

$\mathcal{O} = \{G \in \mathcal{I} :\ G$ is edgeless, i.e., $E(G) = \emptyset\}$.

$\mathcal{O}_k = \{G \in \mathcal{I} :\ $ each component of G has at most $k+1$ vertices$\}$

$\mathcal{S}_k = \{G \in \mathcal{I} :\ \Delta(G) \leq k\}$

$\mathcal{D}_k = \{G \in \mathcal{I} :\ G$ is k-degenerate$\}$

$\mathcal{W}_k = \{G \in \mathcal{I} :\ G$ has no path with more than k edges, i.e., $\lambda(G) \leq k+1\}$.

Note that $\mathcal{O} = \mathcal{O}_o = \mathcal{S}_o = \mathcal{D}_0 = \mathcal{W}_0$.

For given hereditary properties $\mathcal{P}_1, \ldots, \mathcal{P}_m$, a $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition of G is a partition $V_1, \ldots, V_m$ of $V(G)$ such that $G[V_i] \in P_i$ for $i = 1, \ldots, m$. The product $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$ is defined as the class of all graphs that have a $(\mathcal{P}_1, \ldots, \mathcal{P}_m)$-partition. If $\mathcal{P}_i = \mathcal{P}$ for $i = 1, \ldots, m$, we write $\mathcal{P}^n = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_m$. Thus, for example, $\mathcal{O}^m$ is the class of all m-colourable graphs. Throughout the paper p and q will denote non-negative integers.

In [15] the Path Partition Conjecture is formulated as follows.

Conjecture 2.1 (PPC)**.** *$\mathcal{W}_{p+q+1} \subseteq \mathcal{W}_p \circ \mathcal{W}_q$ for all p, q.*

The above formulation underlines the similarity with the well-known Lovász Partition Theorem, which may be stated as follows

Theorem 2.2 [44]**.** *$\mathcal{S}_{p+q+1} \subseteq \mathcal{S}_p \circ \mathcal{S}_q$ for all p, q.*

Borodin [14] showed that an analogous result holds for the property $\mathcal{D}_{p+q+1}$.

Theorem 2.3 [14]**.** *$\mathcal{D}_{p+q+1} \subseteq \mathcal{D}_p \circ \mathcal{D}_q$ for all p, q.*

An analogous result also holds for the property $\mathcal{O}_{p+q+1}$, as shown by Jensen and Toft [38].

Theorem 2.4 [38]. *$\mathcal{O}_{p+q+1} \subseteq \mathcal{O}_p \circ \mathcal{O}_q$ for all p, q.*

A tree partition problem which provides a natural generalization of Theorem 2.2 and the PPC was studied by Katrenič and Semanišin [39]. The cycle analogue of the PPC, called the Cycle Partition Conjecture, was introduced by Nielsen [49] and also studied by Yang and Vumar [52].

Broere, Dunbar, Dorfling and Frick [17] showed that the PPC holds for graphs with maximum degree at most 3. This may be stated as follows.

Proposition 2.5 [17]. *$\mathcal{S}_3 \cap \mathcal{W}_{p+q+1} \subseteq \mathcal{W}_p \circ \mathcal{W}_q$ for all p, q.*

Bullock and Frick [22] showed that the PPC holds for the class of 2-degenerate graphs. This result may be stated as follows.

Proposition 2.6 [22]. *$\mathcal{D}_2 \cap \mathcal{W}_{p+q+1} \subseteq \mathcal{W}_p \circ \mathcal{W}_q$ for all p, q.*

Theorem 3.1 of [17] may be stated as follows.

Theorem 2.7 [17]. *If $p \leq q$, then $\mathcal{W}_{\lceil \frac{p}{2} \rceil + q + 1} \subseteq \mathcal{W}_p \circ \mathcal{W}_q$.*

For $p \geq 15$, Broere, Dorfling and Jonck [18] presented the following improvement of Theorem 2.7.

Theorem 2.8 [18]. *$\mathcal{W}_{\lceil \frac{2p}{3} \rceil + q + 1} \subseteq \mathcal{W}_p \circ \mathcal{W}_q$ for all $p \geq 15$ and $q \geq 1$.*

A hereditary property $\mathcal{P}$ is uniquely determined by the set $\mathbf{M}(\mathcal{P})$ of *$\mathcal{P}$-maximal graphs*, defined by

$$\mathbf{M}(\mathcal{P}) = \{G \in \mathcal{P} : G + e \notin \mathcal{P} \text{ for each } e \in E(\overline{G})\}.$$

Since $\mathcal{W}_k$-maximal graphs generally have more structure than graphs in $\mathcal{W}_k$, it is convenient to note that the PPC is equivalent to the following conjecture.

Conjecture 2.9 (PPC). *$\boldsymbol{M}(\mathcal{W}_{p+q+1}) \subseteq \mathcal{W}_p \circ \mathcal{W}_q$ for all p, q.*

$\mathcal{W}_k$-maximal graphs are studied in [11, 12, 19, 24, 41].

3. $\mathcal{W}_p$-maximal Sets and P_{p+2}-kernels

Let P_n denote the path with n vertices (and $n-1$ edges). Then a graph G belongs to $\mathcal{W}_p$ if and only if G does not contain a P_{p+2}.

A set M of vertices in a graph G is called a *maximal $\mathcal{W}_p$-set* of G if $G[M] \in \mathcal{W}_p$ and $G[M \cup \{x\}] \notin \mathcal{W}_p$ for every vertex $x \in V(G - M)$. Our next lemma follows directly from this definition.

Lemma 3.1. *M is a maximal $\mathcal{W}_p$-set of a graph G if and only if each of the following two conditions is satisfied.*

1. *$G[M] \in \mathcal{W}_p$, i.e., $G[M]$ does not contain a P_{p+2}.*
2. *For every $x \in V(G-M)$ at least one of the following hold.*
   (a) *x is adjacent to an end-vertex of a P_{p+1} in $G[M]$.*
   (b) *There exist two vertex disjoint paths P_q and P_r in $G[M]$ such that $q+r = p+1$ and x is adjacent to an end-vertex of each of these two paths.*

A $\mathcal{P}$-maximal set of maximum cardinality in a graph G is called a *$\mathcal{P}$-maximum* set.

Let $G \in \mathcal{W}_k$ and let p, q be any two non-negative integers such that $k = p+q+1$. Now, if we can find a set $M \subseteq V(G)$ such that $M \in \mathcal{W}_p$ and $G-M \in \mathcal{W}_{k-(p+1)}$, then we can conclude that $G \in \mathcal{W}_p \circ \mathcal{W}_q$. It was conjectured in [21] that any $\mathcal{W}_p$-maximum set M has the desired property, but Aldred and Thomassen [7] pointed out that any hypotraceable graph of order $k+2$ provides a counterexample to the case $p = k-1$ of that conjecture. Indeed, if xy is an edge in a hypotraceable graph G of order $k+2$ and $M = G - \{x, y\}$, then $G \in \mathcal{W}_k$ and M is a $\mathcal{W}_{k-1}$-maximal set in G, but $G - M \notin \mathcal{W}_0$. However, as pointed out in [7], in order to prove the PPC it would suffice to prove the following conjecture, for which there is as yet no known counterexample.

Conjecture 3.2 (Revised Maximal $\mathcal{W}_p$-set Conjecture)**.** *If p and k are any integers such that $0 \leq p \leq \frac{k-1}{2}$ and G is any graph in $\mathcal{W}_k$ and M any $\mathcal{W}_p$-maximum set in G, then $G - M \in \mathcal{W}_{k-(p+1)}$.*

In 1968 Chartrand, Geller and Hedetniemi showed (in the proof of [23], Theorem 2) that if M is a $\mathcal{W}_p$-maximum set in a graph $G \in \mathcal{W}_k$ $(p < k)$, then $G - M \in \mathcal{W}_{k-1}$. In 2004, Bullock, Dunbar and Frick [21] presented the following improvement of this result.

Theorem 3.3 [21]**.** *If M is any $\mathcal{W}_p$-maximal set in a graph $G \in \mathcal{W}_k$, $(p < k)$, then $G - M \in \mathcal{W}_{k-(\lfloor \frac{2p+1}{3} \rfloor + 1)}$.*

If M is a $\mathcal{W}_p$-maximal set in a graph $G \in \mathcal{W}_k$ such that every $x \in V(G-M)$ satisfies 2(a) of Lemma 3.1, then it is easily seen that $G - M \in \mathcal{W}_{k-(p+1)}$. This motivated the following definition.

Definition. A subset K of a graph G is a P_{p+2}-*kernel* of G if each of the following conditions is satisfied.

1. $G[K]$ does not contain a P_{p+2}.
2. Every vertex in $G - K$ is adjacent to an end-vertex of a P_{p+1} in $G[K]$.

The connection between path kernels and path partitions is as follows.

Proposition 3.4. *If $G \in \mathcal{W}_{p+q+1}$ and G has a P_{p+2}-kernel, then $G \in \mathcal{W}_p \circ \mathcal{W}_q$.*

Mihók introduced the concept of a P_n-kernel in 1985 in [48], where he mentioned that every graph has a P_n-kernel for every $n \leq 6$. In 1999, Dunbar and Frick [25] proved, by means of an algorithmic approach, that every graph has a P_7-kernel. By expanding this algorithm, Mel'nikov and Petrenko [45, 47] proved (in 2002 and 2005, respectively) that every graph has a P_8-kernel and a P_9-kernel. These results are summarized in the following theorem.

Theorem 3.5 [25, 45, 47]**.** *Every graph G has a P_n-kernel for every $n \leq 9$.*

Corollary 3.6. *If $p \leq 7$, then $\mathcal{W}_{p+q+1} \subseteq \mathcal{W}_p \circ \mathcal{W}_q$ for every $q \geq 1$.*

Corollary 3.7. *The PPC holds for the class $\mathcal{W}_{16}$.*

It was originally conjectured that every graph has a P_k kernel for every $k \geq 2$ (see [20]) but in 2004 Aldred and Thomassen [7] presented a graph in $\mathcal{W}_{363}$ that has no P_{364}-kernel. In 2008, Katrenič and Semanisin [40] presented a graph in $\mathcal{W}_{154}$ that has no P_{155}-kernel. They showed, moreover, that for every $r \geq 0$ there exists a graph G such that G has no $P_{\lambda(G)-r}$-kernel. However, they pointed out that in each of their examples $\lambda(G) - r$ is still greater than $\lambda(G)/2$. In order to prove the PPC it will suffice to prove the following conjecture, for which no counterexample has yet been found.

Conjecture 3.8 (Revised Path Kernel Conjecture)**.** *For every positive integer k, every graph in $\mathcal{W}_k$ has a P_{p+2}-kernel for every non-negative integer $p \leq (k-1)/2$.*

Thus, results on the existence of path kernels in graphs could still lead to further progress towards proving the PPC.

The *girth* of a graph G, denote by $g(G)$ is the length of a shortest cycle in G. The length of a longest cycle in G is called the *circumference* of G and denoted by $c(G)$. In 1999 Dunbar and Frick [25] proved the following.

Proposition 3.9 [25]**.** *If G is a graph with $g(G) \geq p$, then G has a P_{p+2}-kernel.*

In 2010 He and Wang [34] presented a substantial improvement of Proposition 3.9.

Theorem 3.10 [34]**.** *If G is a graph with $g(G) > \frac{2}{3}(p+2)$, then G has a P_{p+2}-kernel.*

Theorem 3.10 immediately implies the following.

Corollary 3.11 [34]**.** *If $G \in \mathcal{W}_{p+q+1}$, $p \leq q$, and $g(G) > \frac{2}{3}(p+1)$, then $G \in \mathcal{W}_p \circ \mathcal{W}_q$.*

Mel'nikov and Petrenko [47] presented results on the existence of path kernels in graphs with small circumference.

In the next section we provide further results that show how knowledge of the cycle structure of a graph can help us find the partitions required by the PPC.

4. The PPC and Cycle Structure

The following result of Broere, Dorfling, Dunbar and Frick [17] has turned out to be surprisingly useful.

Theorem 4.1 [17]. *If G is a connected graph in $\mathcal{W}_{p+q+1}$, $p \leq q$, and G contains a cycle of length $q+1$, then $G \in \mathcal{W}_p \circ \mathcal{W}_q$.*

We call a vertex v an *attachment vertex* of a set S in a graph G if v is adjacent to a vertex in $G - S$. In the proof of Theorem 4.1 in [17] the necessary partition is constructed by considering the distance sets of a $(q+1)$-cycle. Dunbar and Frick [26] observed that this technique also yields the following result.

Theorem 4.2 [26]. *If G is a connected graph in $\mathcal{W}_{p+q+1}$, $p \leq q$, and G contains a cycle of length greater than $q+1$ with at most $q+1$ attachment vertices, then $G \in \mathcal{W}_p \circ \mathcal{W}_q$.*

By considering $\mathcal{W}_p$-maximal sets in a graph $G \in \mathcal{W}_{p+q+1}$, Bullock, Dunbar and Frick [21] proved the following.

Theorem 4.3 [21]. *If $G \in \mathcal{W}_{p+q+1}$, $p \leq q$, and $c(G) \leq q+3$, then $G \in \mathcal{W}_p \circ \mathcal{W}_q$.*

A graph G is called *weakly pancyclic* if G has a cycle of every length between $g(G)$ and $c(G)$. By combining Proposition 3.9, Theorem 4.1 and Theorem 4.3, the following result was obtained.

Corollary 4.4 [21, 26]. *The PPC holds for connected weakly pancyclic graphs.*

If G has a cycle of every length from $\max\{3, \lceil\lambda(G)/2\rceil\}$ up to $c(G)$ or $c(G) < \lceil\lambda(G))/2\rceil$, we say G is *semi-pancyclic*. Dunbar and Frick [26] observed that Theorems 4.1 and 4.3 imply the following.

Corollary 4.5 [26]. *The PPC holds for connected semi-pancyclic graphs.*

Another result which followed from Theorem 4.3 is the following.

Theorem 4.6 [26]. *If the PPC holds for 2-connected graphs, then it holds for all graphs.*

A longest cycle in a graph G is called a *circumference cycle* of G. Theorems 4.2 and 4.6 together with Corollaries 4.4 and 4.5 imply that in order to prove the PPC it will suffice to prove the following conjecture.

Conjecture 4.7. *For each $k \geq 2$, every 2-connected $\mathcal{W}_k$-maximal graph G in which every circumference cycle has at least $(k+1)/2$ attachment vertices is weakly pancyclic or semi-pancyclic.*

As noted in [11], there exist $\mathcal{W}_k$-maximal graphs that are neither weakly pancyclic nor semi-pancyclic, but the known examples are not 2-connected and their circumference cycles have few attachment vertices.

Theorem 4.1 also has the following corollary.

Corollary 4.8 [17]**.** *Let $p \leq q$ and suppose $G \in \mathcal{W}_{p+q+1}$. Then $G \in \mathcal{W}_p \circ \mathcal{W}_q$ if one of the following hold.*

1. $\Delta(G) \geq |V(G)| - p - 1$.
2. $G - N(v) \in \mathcal{W}_p$ *for some* $v \in V(G)$.

5. The PPC Restricted to Special Classes of Graphs

We have already seen that the PPC holds for the classes $\mathcal{D}_2$ and S_3 as well as for the classes of weakly pancyclic graphs and semi-pancyclic graphs.

A graph G is *chordal* if every cycle of length greater than 3 in G has a chord. The author has received several queries as to whether the PPC has been proved to hold for chordal graphs. The answer is yes, since every chordal graph is pancyclic.

A graph G is a *join of two graphs G_1 and G_2* (written $G = G_1 + G_2$) if G is obtained from the disjoint union of G_1 and G_2 by joining every vertex in G_1 to every vertex in G_2 with an edge. If a $\mathcal{W}_k$-maximal graph G is a join of two graphs, then $\Delta(G) \geq |V(G)| - 1$, as shown by Broere, Frick and Semanišin [19]. Thus it follows from Corollary 4.8 that Conjecture 2.9, and hence the PPC, holds for joins of graphs.

A graph is *claw-free* if it has no $K_{1,3}$ as induced subgraph. A vertex x in a claw-free graph G is called *eligible* if the graph induced by its neighbourhood is connected and noncomplete. The operation of joining every pair of nonadjacent vertices in the neighbourhood of an eligible vertex x is called the *local completion* of G at x. Ryjáček [50] defined the *closure* $cl(G)$ of a claw-free graph G to be the graph obtained from G by recursively performing the local completion operation to eligible vertices of G until no eligible vertices remain. A claw-free graph G is *closed* if $cl(G) = G$.

Brandt, Favaron and Ryjáček [16] proved that if G is a claw-free graph, then $cl(G)$ is well-defined and claw-free and $\lambda(cl(G)) = \lambda(G)$. Thus, in order to prove the PPC for claw-free graphs, it is sufficient to prove it for closed claw-free graphs.

Using results on the structure of closed claw-free graphs proved in [16], Dunbar and Frick [26] showed that if C is any circumference cycle in a closed claw-free graph, then C either has at most $\lceil(\lambda(G)/2)\rceil$ attachment vertices or G is semi-pancyclic. This, together with Theorem 4.2 and Corollary 4.5 imply the following.

Theorem 5.1 [26]**.** *The PPC holds for claw-free graphs.*

Glebov and Zambalaeva [33] considered the PPC for planar graphs and reported the following.

Theorem 5.2 [33]**.** *The PPC holds for planar graphs with girths* $5, 8, 9$ *and* 16*. Moreover, if* G *is a planar graph, the following hold.*

1. *If* $g(G) = 8$*, then* $G \in \mathcal{W}_1 \circ \mathcal{W}_2$*.*
2. *If* $g(G) = 9$*, then* $G \in \mathcal{W}_1 \circ \mathcal{W}_1$*.*
3. *If* $g(G) = 16$*, then* $G \in \mathcal{W}_0 \circ \mathcal{W}_1$*.*

It remains an interesting open problem to prove the PPC for the class of planar graphs.

6. THE PPC FOR t-DEFICIENT GRAPHS

Recall that a graph G of order n is *t-deficient* if $t = n - \lambda(G)$. The PPC obviously holds for 0-deficient graphs and this fact is a best possible result, since there are 0-deficient graphs in $\mathcal{W}_{p+q+2}$ that are not in $\mathcal{W}_p \circ \mathcal{W}_q$, for example K_{p+q+3}. However, Frick and Whitehead [30] showed that for 1-deficient as well as for 2-deficient graphs a stronger result than the PPC holds.

Theorem 6.1 [30]**.** *If* $G \in \mathcal{W}_{p+q+2}$ *and* G *is* 1*-deficient or* 2*-deficient, then* $G \in \mathcal{W}_p \circ \mathcal{W}_q$*.*

Theorem 6.1 is best possible, as shown in [30].

Frick and Schiermeyer [29] proved the following two results.

Theorem 6.2 [29]**.** *If* $G \in \mathcal{W}_{p+q+1}$ *and* G *is* 3*-deficient, then* $G \in \mathcal{W}_p \circ \mathcal{W}_q$*.*

Theorem 6.3 [29]**.** *Let* $t \geq 4$ *and suppose* G *is a* t*-deficient graph of order* n *in* $\mathcal{W}_{p+q+1}$ *Then* $G \in \mathcal{W}_p \circ \mathcal{W}_q$ *if either of the following hold.*

1. $p \geq t - 1$ *and* $n \geq 10t^2 - 3t$*.*
2. $p < t - 1$ *and* $n \geq 4t^2 - 6t - 4$*.*

We conclude that the PPC for t-deficient graphs is true for $t \leq 3$ and asymptotically true for $t \geq 4$.

7. Generalized Chromatic Numbers

Let $\mathcal{P}$ and $\mathcal{Q}$ be additive hereditary properties of graphs. Broere, Dorfling and Jonck [21] defined the *generalized chromatic number* $\chi_{\mathcal{Q}}(\mathcal{P})$ as follows:

$$\chi_{\mathcal{Q}}(\mathcal{P}) = m \text{ if } \mathcal{P} \in \mathcal{Q}^m \text{ and } \mathcal{P} \notin \mathcal{Q}^{m-1}.$$

The chromatic number $\chi_{\mathcal{W}_m}$ was introduced by Chartrand, Geller and Hedetniemi [23] in 1968.

From Theorems 2.2, 2.3 and 2.4, the following bounds for the corresponding generalized chromatic numbers are easily derived. (The bound for $\chi_{\mathcal{D}_m}(\mathcal{D}_k)$ was also given by Lick and White [43].)

$$\chi_{\mathcal{S}_m}(\mathcal{S}_k) \le \left\lceil \frac{k+1}{m+1} \right\rceil, \quad \chi_{\mathcal{D}_m}(\mathcal{D}_k) \le \left\lceil \frac{k+1}{m+1} \right\rceil, \quad \chi_{\mathcal{O}_m}(\mathcal{O}_k) \le \left\lceil \frac{k+1}{m+1} \right\rceil.$$

It is therefore not unreasonable to expect the following conjecture to be true.

Conjecture 7.1. *If $0 \le m \le k$, then $\chi_{\mathcal{W}_m}(\mathcal{W}_k) \le \left\lceil \frac{k+1}{m+1} \right\rceil$.*

If the PPC is true, then Conjecture 7.1 will also be true. This was essentially the reason why the author became interested in the PPC.

Using Theorem 3.3, Bullock and Frick [22] derived the following bound for $\chi_{\mathcal{W}_m}(\mathcal{W}_k)$.

Theorem 7.2 [22]**.** $\chi_{\mathcal{W}_m}(\mathcal{W}_k) \le \left\lceil \frac{3(k-m)}{2m+2} + 1 \right\rceil$ *for all $m,\ k \ge 1$.*

By a different method, using Theorem 2.8, Broere, Dorfling and Jonck [18] obtained the following bound.

Theorem 7.3 [18]**.** $\chi_{\mathcal{W}_m}(\mathcal{W}_k) \le \left\lceil \frac{3k}{2m+3} \right\rceil$ *for all $m \ge 15,\ k \ge 1$.*

8. The Directed Path Partition Conjecture

Let $\lambda(D)$ denote the number of vertices in a longest directed path of a digraph D. Then the digraph analogue of the Path Partition Conjecture, called the Directed Path Partition Conjecture (or DPPC), may be stated as follows.

Conjecture 8.1 (DPPC)**.** *If D is any digraph and (λ_1, λ_2) any pair of positive integers such that $\lambda_1 + \lambda_2 = \lambda(D)$, then there exists a partition (V_1, V_2) of the vertex set of D such that $\lambda(G[V_i]) \le \lambda_i,\ i = 1, 2$.*

In 1995 Bondy [13] formulated a conjecture that is seemingly stronger than the DPPC, requiring $\lambda(D[V_i]) = \lambda_i$ instead of $\lambda(D[V_i]) \leq \lambda_i$. The first explicit formulation of the DPPC in the literature appeared in 2005 in a paper by van Aardt, Dlamini, Dunbar, Frick and Oellermann [2].
The oriented analogue of the PPC is called the OPPC. Both the PPC and the OPPC may be regarded as special cases of the DPPC.

Bang-Jensen, Nielsen and Yeo [9] proved that the DPPC holds for certain generalizations of tournaments, and recently Arroyo and Galeana-Sanchéz [8] found further generalizations of tournaments for which the DPPC holds. However, as mentioned in [9], the DPPC seems an extremely difficult conjecture to attack for general digraphs, since very little can be said about the structure of longest paths in general digraphs.

While the PPC has been proved for $\lambda_1 \leq 8$, the DPPC has not even been settled for $\lambda_1 = 1$. This special case has turned out to be a difficult and intriguing conjecture in its own right. It is treated, for example, in [2, 31, 32, 37, 42].

Furthermore, while the PPC for t-deficient graphs has been shown to hold for all $t \leq 3$ (and asymptotically for all t), it is not yet known whether the OPPC for t-deficient oriented graphs holds for any $t \geq 1$. Results in support of the special case $t = 1$ are proved in [1, 3, 4, 5, 28] and it is shown in [6] that if the OPPC does hold for 1-deficient oriented graphs, it would be a best possible result (in contrast to Theorem 6.1).

References

[1] S.A. van Aardt, A.P. Burger, J.E. Dunbar, M. Frick, J.M. Harris and J.E. Singleton, *An iterative approach to the traceabiltiy conjecture for oriented graphs*, submitted.

[2] S.A. van Aardt, G. Dlamini, J.E. Dunbar, M. Frick, and O.R. Oellermann, *The directed path partition conjecture*, Discuss. Math. Graph Theory **25** (2005) 331–343. doi:10.7151/dmgt.1286

[3] S.A. van Aardt, J.E. Dunbar, M. Frick, P. Katrenič, M.H. Nielsen, and O.R. Oellermann, *Traceability of k-traceable oriented graphs*, Discrete Math. **310** (2010) 1325–1333.
doi:10.1016/j.disc.2009.12.022

[4] S.A. van Aardt, J.E. Dunbar, M. Frick and M.H. Nielsen, *Cycles in k-traceable oriented graphs*, Discrete Math. **311** (2011) 2085–2094.
doi:10.1016/j.disc.2011.05.032

[5] S.A. van Aardt, J.E. Dunbar, M. Frick, M.H. Nielsen, and O.R. Oellermann, *A traceability conjecture for oriented graphs*, Electron. J. Combin. **15** (2008) #R150.

[6] S.A. van Aardt, M. Frick and C. Whitehead, *Significant differences between path partitions in directed and undirected graphs*, Util. Math. **83** (2010) 179–185.

[7] R.E.L. Aldred and C. Thomassen, *Graphs with not all possible path-kernels*, Discrete Math. **285** (2004) 297–300.
doi:10.1016/j.disc.2004.02.012

[8] A. Arroyo and H. Galeana-Sánchez, *The Path Partition Conjecture is true for some generalizations of tournaments*, Discrete Math. **313** (2013) 293–300.
doi:10.1016/j.disc.2012.10.014

[9] J. Bang-Jensen, M.H. Nielsen, and A. Yeo, *Longest path partitions in generalizations of tournaments*, Discrete Math. **306** (2006) 1830–1839.
doi:10.1016/j.disc.2006.03.063

[10] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, Second Edition (Springer-Verlag, London, 2009).

[11] L.W. Beineke, J.E. Dunbar and M. Frick, *Detour-saturated graphs*, J. Graph Theory **49** (2005) 116–134.
doi:10.1002/jgt.20069

[12] G. Benade, I. Broere, B. Jonck and M. Frick, *Uniquely* $(m, k)^{\tau}$*-colourable graphs and* k*-*τ*-saturated graphs*, Discrete Math. **162** (1996) 13–22.
doi:10.1016/0012-365X(95)00301-C

[13] J.A. Bondy, *Basic graph theory: Paths and circuits*, in: Handbook of Combinatorics, R.L. Graham, M. Grötschel, and L. Lovász, (Eds.), (The MIT Press, Cambridge, MA 1995) Vol I.

[14] O.V. Borodin, *On decomposition of graphs into degenerate subgraphs*, Diskret. Analiz. **28** (1976) 3–11.

[15] M. Borowiecki, I. Broere, M. Frick, P. Mihók, G. Semanišin, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory, **17** (1997) 5–50.
doi:10.7151/dmgt.1037

[16] S. Brandt, O. Favaron and Z. Ryjáček, *Closure and stable hamiltonian properties in claw-free graphs*, J. Graph Theory **349** (2000) 31–41.

[17] I. Broere, M. Dorfling, J.E. Dunbar and M. Frick, *A Path(ological) Partition Problem*, Discuss. Math. Graph Theory **18** (1998) 113–125.
doi:10.7151/dmgt.1068

[18] I. Broere, S. Dorfling and E. Jonck, *Generalized chromatic numbers and additive hereditary properties of graphs*, Discuss. Math. Graph Theory **22** (2002) 259–270.
doi:10.7151/dmgt.1174

[19] I. Broere, M. Frick and G. Semanišin, *Maximal graphs with respect to hereditary properties*, Discuss. Math. Graph Theory **17** (1997) 51–66.
doi:0.7151/dmgt.1038

[20] I. Broere, P. Hajnal and P. Mihók, *Partition problems and kernels of graphs*, Discuss. Math. Graph Theory **17** (1997) 311–313
doi:10.7151/dmgt.1058

[21] F. Bullock, J.E. Dunbar and M. Frick, *Path partitions and P_n-free sets*, Discrete Math. **289** (2004) 145–155.
doi:10.1016/j.disc.2004.07.012

[22] F. Bullock and M. Frick, *Detour chromatic numbers of graphs*, Discuss. Math. Graph Theory **21** (2001) 283–291.
doi:0.7151/dmgt.1150

[23] G. Chartrand, D.P. Geller and S. Hedetniemi, *A generalization of the chromatic number*, Math. Proc. Cambridge Philos. Soc. **64** (1968) 265–271.
doi:10.1017/S0305004100042808

[24] A. Dudek and P. Wojda, *P_m-saturated graphs with minimum size*, Opuscula Math. **24** (2004) 43–55.

[25] J.E. Dunbar and M. Frick, *Path kernels and partitions*, J. Combin. Math. Combin. Comput. **31** (1999) 137–149.

[26] J.E. Dunbar and M. Frick, *The path partition conjecture is true for claw-free graphs*, Discrete Math. **307** (2007) 1285–1290.
doi:10.1016/j.disc.2005.11.065

[27] M. Frick, *A survey on (m, k)-colorings*, in: Quo Vadis, Graph Theory?, J.Gimbel, J.W. Kennedy and L.V. Quintas (Eds.), Annals Discrete Math. **55** (1993) 45–48.

[28] M. Frick and P. Katrenič, *Progress on the Traceability Conjecture for oriented graphs*, Discrete Math. Theor. Comput. Sci. **10** (2008) 105–114.

[29] M. Frick and I. Schiermeyer, *An asymptotic result on the path partition conjecture* Electron. J. Combin. **12** (2005) #R48.

[30] M. Frick and C. Whitehead, *A new approach to the Path Partition Conjecture*, Util. Math. **99** (2006) 195–206.

[31] H. Galeana-Sánchez, R. Gómez and J.J. Montellano-Ballesteros, *Independent transversals of longest paths in locally semicomplete and locally transitive digraphs*, Discuss. Math. Graph Theory **29** (2009) 469–480.
doi:10.7151/dmgt.1458

[32] H. Galeana-Sánchez, H.A. Rincón-Meja, *Independent sets which meet all longest paths*, Discrete Math. **152** (1996) 141–145.
doi:10.1016/0012-365X(94)00261-G

[33] A.N. Glebov and D.J. Zambalaeva, *Path partitioning planar graphs*, Siberian Electron. Math. Reports (http://semr.math.nsc.ru) **4** (2007) 450–459 (in Russian, English abstract).

[34] W. He and B. Wang, *A note on path kernels and partitions*, Discrete Math. **310** (2010) 3040–3042.
doi:10.1016/j.disc.2010.06.029

[35] J.M. Harris, J.L. Hirst and M.J. Mossinghoff, Combinatorics and Graph Theory (Second Edition) (Springer Science+Business Media, LLC, New York, 2008).

[36] P. Hajnal, Graph Partitions, Thesis supervised by L. Lovász, (J.A. University, Szeged, 1984) (in Hungarian).

[37] F. Havet, *Stable set meeting every longest path*, Discrete Math. **289** (2004) 169–173. doi:10.1016/j.disc.2004.07.013

[38] T.R. Jensen and B. Toft, Graph Coloring Problems (Wiley-Interscience Publications, New York, 1995).

[39] P. Katrenič and G. Semanišin,*On a tree-partition problem*, Electron. Notes Discrete Math. **28** (2007) 325–330.
doi:10.1016/j.endm.2007.01.046

[40] P. Katrenič and G. Semanišin, *A note on the Path Kernel Conjecture*, Discrete Math. **309** (2009) 2551–2554.
doi:10.1016/j.disc.2008.05.002

[41] L. Kászonyi and Zs. Tuza, *Saturated graphs with minimal number of edges*, J. Graph Theory **10** (1986) 203–210.
doi:10.1002/jgt.3190100209

[42] J.M. Laborde, C. Payan and N.H. Xuong, *Independent sets and longest directed paths in digraphs*, in: Graphs and Other Combinatorial Topics (Prague, 1982), 173–177 (Teubner-Texte Math. **59** (1983)).

[43] D.R. Lick and A.T. White, *k-degenerate graphs*, Canad. J. Math. **22** (1970) 1082–1096.
doi:10.4153/CJM-1970-125-1

[44] L. Lovász, *On decomposition of graphs*, Studia Sci. Math. Hungar. **1** (1966) 237–238.

[45] L.S. Mel'nikov and I.V. Petrenko, *On path kernels and partitions of undirected graphs*, Diskretn. Anal. Issled. Oper. **9** (2002) 21–35 (in Russian).

[46] L.S. Mel'nikov and I.V. Petrenko, *Path kernels and cycle length in undirected graphs*, in: V.N. Kassyanov (Ed.), Modern Problems of Program Construction, (ISI SB Russiam Academy of Science, Novosibirsk, 2002) 222–231 (in Russian).

[47] L.S. Mel'nikov and I.V. Petrenko, *Path kernels and partitions of graphs with small cycle length*, In: Methods and Tools of Program Construction and Optimization, V.N. Kasyanov (Ed.), (ISI SB Russian Academy of Science, Novosibirsk, 2005) 145–160 (in Russian).

[48] P. Mihók, *Problem* 4, in: M. Borowiecki and Z. Skupień (Eds.), Graphs, Hypergraphs and Matroids, (Zielona Góra, 1985) p. 86.

[49] M.H. Nielsen, *On a cycle partition problem*, Discrete Math. **308** (2008) 6339–6347. doi:10.1016/j.disc.2007.12.002

[50] Z. Ryjáček, *On a closure concept in claw-free graphs*, J. Combin. Theory (B) **70** (1997) 217–224.
doi:10.1006/jctb.1996.1732

[51] J. Vronka, *Vertex sets of graphs with prescribed properties*, Thesis supervised by P. Mihók, (P.J. Šafárik University, Košice 1986), (in Slovak).

[52] F. Yang and E. Vumar, *A note on a cycle partition problem*, Appl. Math. Lett. **24** (2011) 1181–1184.
doi:10.1016/j.aml.2011.02.003

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