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This paper is dedicated to Mietek Borowiecki on the occasion of his 70th birthday. We thank him for his warmth and kindness to us, for his inspiration, and for his dedication to graph theory.

ON GRAPHS WITH DISJOINT DOMINATING AND 2-DOMINATING SETS

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Abstract

A DD_2-pair of a graph G is a pair (D, D_2) of disjoint sets of vertices of G such that D is a dominating set and D_2 is a 2-dominating set of G. Although there are infinitely many graphs that do not contain a DD_2-pair, we show that every graph with minimum degree at least two has a DD_2-pair. We provide a constructive characterization of trees that have a DD_2-pair and show that $K_{3,3}$ is the only connected graph with minimum degree at least three for which $D \cup D_2$ necessarily contains all vertices of the graph.

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1. Introduction

In this paper we continue the study of graph theoretic problems in the complement of a dominating set. Domination and its variations in graphs have been studied by many authors. A set D of vertices in a graph $G = (V, E)$ is a *dominating set* if every vertex in $V \setminus D$ has a neighbor in D, while D is a *total dominating set* if every vertex in V has a neighbor in D. For k a positive integer, D is a *k-dominating set* if every vertex in $V \setminus D$ has at least k neighbors in D. The book by Haynes, Hedetmiemi, and Slater [3] surveyed much of the work that has been done on the subject of domination and its variations.

Ore [9] observed that any graph without isolated vertices always contains a pair of disjoint dominating sets. However, Zelinka [12] showed that one cannot guarantee three disjoint dominating (or total dominating) sets in a graph by simply requiring the minimum degree of the graph to be large enough. Thus, studying graphs whose vertex sets admit a partition into two dominating sets is of interest. With that context in mind we now make the following definition. A *dominating pair* of a graph G is a pair (D_1, D_2) of disjoint dominating sets D_1 and D_2 in G. In 2008, Hedetniemi, Hedetniemi, Laskar, Markus, and Slater [2] initiated the study of the *disjoint domination number* of G which they defined as $\gamma\gamma(G) = \min\{|D_1| + |D_2| \colon (D_1, D_2)$ is a dominating pair of $G\}$. Recently, there has been much work on this parameter, including for example in [2, 4, 5, 6].

Henning and Southey [7] showed that the vertex set of every connected graph with minimum degree at least two, with the exception of a 5-cycle, can be partitioned into a dominating set and a total dominating set. A *DT-pair* of a graph G is a pair (D, T) of disjoint sets of vertices of G such that D is a dominating set and T is a total dominating set of G. The parameter $\gamma\gamma_t(G) = \min\{|D| + |T| \colon (D, T)$ is a DT-pair of $G\}$. This parameter has been studied for example in [6, 8, 10]. In particular, it is shown in [6] that if G is a graph with minimum degree at least three, then $\gamma\gamma_t(G) < n$, unless G is the Petersen graph.

In this paper, we consider a *DD_2-pair* of a graph G which is a pair (D, D_2) of disjoint sets of vertices of G such that D is a dominating set and D_2 is a 2-dominating set of G. In contrast to Ore's positive observation that every graph with no isolated vertex contains two disjoint dominating sets, there are infinitely many graphs with no isolated vertex that do not contain a DD_2-pair. For example, if G is the graph obtained by adding a pendant edge to each vertex of an arbitrary graph F (such a graph G is the called the *corona* or 2-*corona* of F in the literature), then G has no DD_2-pair.

We call a graph that has a DD_2-pair a *DD_2-graph* and in this case make the following definition. If G is a DD_2-graph, then let $\gamma\gamma_2(G) = \min\{|D| + |D_2| \colon (D, D_2)$ is a DD_2-pair of $G\}$.

2. Trees with a DD_2-pair

As observed earlier, not every graph is a DD_2-graph. Our first aim is to provide a constructive characterization of DD_2-trees. In general, describing and proving the correctness of constructive characterizations can become quite involved. We will employ the method of labelings that was introduced by Dorfling *et al.* [1], and has since been used successfully by a number of authors. For example, see [8, 11].

A *labeling* of a graph G is a partition $S = (S_A, S_B)$ of $V(G)$. The *label* or *status* of a vertex v, denoted $\mathrm{sta}(v)$, is the letter $x \in \{A, B\}$ such that $v \in S_x$. The key to our constructive characterization is to find a labeling of the vertices that indicates the role each vertex plays in the set associated with each of the parameters.

By a *labeled-*P_3, we shall mean a P_3 with the central vertex labeled A and the two leaves labeled B. Let $\mathcal{T}$ be the minimum family of labeled trees that contains a labeled-P_3 and can be obtained by repeated application of the four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ listed below, which extend a labeled tree (T, S) by attaching a tree to the vertex $v \in V(T)$. These four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ are illustrated in Figure 1.

- **Operation $\mathcal{O}_1$.** Let v be a vertex with $\mathrm{sta}(v) = A$. Add a vertex u and the edge vu, and let $\mathrm{sta}(u) = B$.
- **Operation $\mathcal{O}_2$.** Let v be a vertex with $\mathrm{sta}(v) = B$. Add a path u_1u_2 and the edge vu_1. Let $\mathrm{sta}(u_1) = A$ and $\mathrm{sta}(u_2) = B$.
- **Operation $\mathcal{O}_3$.** Let v be a vertex with $\mathrm{sta}(v) = B$. Add a path $u_1u_2u_3$ and the edge vu_1. Let $\mathrm{sta}(u_1) = \mathrm{sta}(u_3) = B$ and $\mathrm{sta}(u_2) = A$.
- **Operation $\mathcal{O}_4$.** Let v be a vertex with $\mathrm{sta}(v) = A$. Add a path $u_1u_2u_3$ and the edge vu_2. Let $\mathrm{sta}(u_1) = \mathrm{sta}(u_3) = B$ and $\mathrm{sta}(u_2) = A$.

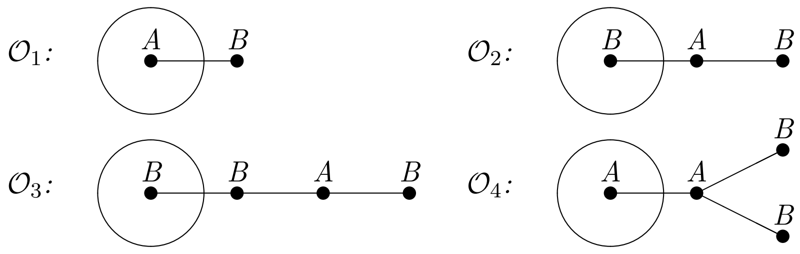

Figure 1. The four operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$.

We recall that a *rooted tree* distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique r–v path, while a *child* of v is any other neighbor of v. We let $C(v)$ denote the set of children of v. A vertex of degree one is called a *leaf* and its neighbor is called a *support* vertex. We shall need the following observation.

Observation 1. *If T is a DD_2-tree and (D, D_2) is a DD_2-pair in T, then every leaf belongs to D_2 while every support vertex belongs to D.*

As remarked in the introductory section, there are infinitely many trees that do not contain a DD_2-pair as may be seen by taking the corona of an arbitrary tree. We are now in a position to establish the following constructive characterization of DD_2-trees that uses labelings.

Theorem 2. *The DD_2-trees are precisely those trees T such that $(T, S) \in \mathcal{T}$ for some labeling S.*

Proof. Suppose first that T is a tree and $(T, S) \in \mathcal{T}$ for some labeling S. By construction, we observe that every vertex of status B is adjacent to a vertex of status A, while every vertex of status A is adjacent to at least two vertices of status B. Thus, (S_A, S_B) is a DD_2-pair in T, and so T is a DD_2-tree. This establishes the sufficiency.

To prove the necessity, we proceed by induction on the order $n \geq 3$ of a DD_2-tree T. If $n = 3$, then $T = P_3$ and $(T, S) \in \mathcal{T}$, where S is the labeling of a labeled-P_3. This establishes the base case. For the inductive hypothesis, let $n \geq 4$ and assume that for every DD_2-tree T' of order less than n there exists a labeling S' such that $(T', S') \in \mathcal{T}$.

Let T be a DD_2-tree of order n. Let (D, D_2) be a DD_2-pair in T. We now root the tree T at a leaf, r, of a longest path (of length $\mathrm{diam}(T)$) in T. Necessarily, r is a leaf. Let u be a vertex at maximum distance from r. Necessarily, u is a leaf. Let v be the parent of u, let w be the parent of v. If $w \neq r$, let x be the parent of w. Since u is at maximum distance from the root r, every child of v is a leaf. By Observation 1, we observe that $C(v) \subseteq D_2$ and $v \in D$. In particular, $u \in D_2$.

Suppose that $d_T(v) \geq 3$. Then, v has at least $d_T(v) - 1 \geq 2$ leaf-neighbors in T. Since $n \geq 4$, v has at least three neighbors in T. If $d_T(v) \geq 4$ or if $d_T(v) = 3$ and $w \in D_2$, then we consider the tree $T' = T - u$. The partition $(D, D_2 \setminus \{u\})$ is a DD_2-pair in T', and so T' is a DD_2-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. By Observation 1, $v \in S'_A$. Thus, we can restore the tree T by applying Operation $\mathcal{O}_1$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A, S'_B \cup \{u\})$. Hence we may assume that $d_T(v) = 3$ and $w \in D$. Let $C(v) = \{u, u'\}$.

We now consider the tree $T' = T - \{u, u', v\}$. The partition $(D \setminus \{v\}, D_2 \setminus \{u, u'\})$ is a DD_2-pair in T', and so T' is a DD_2-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. If $w \in S'_A$, then we can restore the tree T by applying Operation $\mathcal{O}_4$ to T'. If $w \in S'_B$, then we can restore the tree T by first applying Operation $\mathcal{O}_2$ to T' and then Operation $\mathcal{O}_1$. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v\}, S'_B \cup \{u, u'\})$.

Hence if $d_T(v) \geq 3$, then $(T, S) \in \mathcal{T}$ for some labeling S. Hence we may assume that $d_T(v) = 2$, for otherwise the desired result follow. By Observation 1, the vertex $w \in D_2$.

Suppose that $d_T(w) \geq 3$. Let $v' \in C(w) \setminus \{v\}$. If v' is a leaf, then by Observation 1, $v' \in D_2$. But then v' is not dominated by D, a contradiction. Hence, $d_T(v') \geq 2$. By our choice of the vertex u, every child of v' is a leaf. As shown above, we may assume that $d_T(v') = 2$. Let u' be the child of v'. Then, u' is a leaf. By Observation 1, $u' \in D_2$ and $v' \in D$. We now consider the tree $T' = T - \{u', v'\}$. The partition $(D \setminus \{v'\}, D_2 \setminus \{u'\})$ is a DD_2-pair in T', and so T' is a DD_2-tree. We remark that since T' contains the three vertices u, v and w, we have $n(T') \geq 3$. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. Necessarily, $w \in S'_B$, and we can therefore restore the tree T by applying Operation $\mathcal{O}_2$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v'\}, S'_B \cup \{u'\})$. Hence if $d_T(w) \geq 3$, then $(T, S) \in \mathcal{T}$ for some labeling S. Hence we may assume that $d_T(w) = 2$, for otherwise the desired result follow.

Suppose that $x \in D$. We now consider the tree $T' = T - \{u, v\}$. The partition $(D \setminus \{v\}, D_2 \setminus \{u\})$ is a DD_2-pair in T', and so T' is a DD_2-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. Since w is a leaf of T', we have that $w \in S'_B$, and we can therefore restore the tree T by applying Operation $\mathcal{O}_2$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v\}, S'_B \cup \{u\})$. Hence we may assume that $x \notin D$. If $x \notin D_2$, we simply add x to D_2. Hence we may assume that $x \in D_2$.

We now consider the tree $T' = T - \{u, v, w\}$. The partition $(D \setminus \{v\}, D_2 \setminus \{u, w\})$ is a DD_2-pair in T', and so T' is a DD_2-tree. Applying the inductive hypothesis to T', there exists a labeling $S' = (S'_A, S'_B)$ such that $(T', S') \in \mathcal{T}$. If $x \in S'_A$, we can restore the tree T by first applying Operation $\mathcal{O}_1$ to T' and then Operation $\mathcal{O}_2$. If $x \in S'_B$, we can restore the tree T by applying Operation $\mathcal{O}_3$ to T'. Therefore, $(T, S) \in \mathcal{T}$, where S is the labeling $(S'_A \cup \{v\}, S'_B \cup \{u, w\})$. ■

By Theorem 2, if T is a DD_2-tree, then $(T, S) \in \mathcal{T}$ for some labeling S. However we remark that such a labeling is not necessarily unique. Perhaps the simplest example is to take $T = P_9$ and note that T can be obtained from a labeled-P_3 by applying operation $\mathcal{O}_2$ three times or T can be obtained from a labeled-P_3 by applying operation $\mathcal{O}_3$ twice.

3. Minimum Degree at Least Two

Although not every graph is a DD_2-graph, if we restrict the minimum degree to at least two, then $\gamma\gamma_2(G)$ is well defined as the following result shows.

Theorem 3. *If G is a graph with minimum degree at least two, then G is a DD_2-graph.*

Proof. Let $G = (V, E)$ be a graph with minimum degree at least two. Let D be a maximal independent set in G. Then, D is a dominating set, while $V \setminus D$ is a 2-dominating set in G. Thus, $(D, V \setminus D)$ is a DD_2-pair in G. ■

By Theorem 3, $\gamma\gamma_2(G)$ is well defined, implying that $\gamma\gamma_2(G) \leq |V(G)|$. Next we study graphs G satisfying $\gamma\gamma_2(G) = |V(G)|$. A characterization of such graphs seems difficult to obtain since there are several families each containing infinitely many graphs that satisfy this equation. For example, take any graph or multi-graph with minimum degree at least 3 and subdivide every edge at least once.

A second infinite class of examples can be constructed in the following way. Let G be obtained from an arbitrary graph F as follows: For each vertex v of F, add a path $v_1v_2v_3v_4v_5$ and join v to v_1, v_3 and v_5. Finally subdivide every edge of the original graph F at least once. Each such graph G satisfies $\gamma\gamma_2(G) = |V(G)|$. An example of such a graph G is illustrated in Figure 2, where here the original graph F is a path P_3 and every edge of F is subdivided four times.

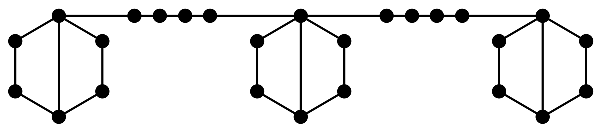

Figure 2. A graph G satisfying $\gamma\gamma_2(G) = |V(G)|$.

Surprisingly, when we increase the degree condition from two to three, then there is only one graph G with $\gamma\gamma_2(G) = |V(G)|$.

Theorem 4. *Let G be a connected graph with $\delta(G) \geq 3$. Then, $\gamma\gamma_2(G) = |V(G)|$ if and only if $G = K_{3,3}$.*

Proof. Let $G = (V, E)$ be a connected graph of order n. If $G = K_{3,3}$, then it is straightforward to verify that $\gamma\gamma_2(G) = n$. Suppose, then, that $\gamma\gamma_2(G) = n$. We show that necessarily $G = K_{3,3}$. Let (A, B) be a partition of V into two sets with the maximum number of edges between the two sets. Then each vertex $v \in V$ has at least $d_G(v)/2$ neighbors in the other set, since otherwise the vertex v can be moved to the other set. Thus each vertex has at least as many neighbors in the other set than in its own set. (We remark that this well-known fact is attributed to Lovász or Erdős.) Since G has minimum degree at least three, this implies that both A and B are 2-dominating sets in G.

If G is not a bipartite graph, then at least one of the sets A and B is not independent in G. We may assume that $v \in A$ is adjacent to at least one other vertex in A. But then $(A \setminus \{v\}, B)$ is a DD_2-pair, and so $\gamma\gamma_2(G) \leq |A|+|B|-1 = n-1$, a contradiction. Hence, G is bipartite with partite sets A and B. In particular, we note that each of A and B is a 3-dominating set.

Let $v \in A$ and let $N_v = \{v_1, v_2, v_3\}$ be a set of three neighbors of v. Then, $N_v \subseteq B$. Let $A' = (A \setminus \{v\}) \cup \{v_1, v_2\}$ and let $B' = (B \setminus N_v) \cup \{v\}$. Then, $|A'| = |A| + 1$ and $|B'| = |B| - 2$, and so $|A'| + |B'| = n - 1$. By construction the set A' is a 2-dominating set of G. If B' is a dominating set of G, then $\gamma\gamma_2(G) \leq |A'| + |B'| = n - 1$, a contradiction. Hence, B' is not a dominating set. Thus there must exist a vertex $v' \in A \setminus \{v\}$ such that $N(v') = N_v$. In particular, $d_G(v') = 3$. This is true for every vertex v of A. Therefore for every vertex $v \in A$, there exists a vertex $v' \in A \setminus \{v\}$ with $d_G(v') = 3$ such that v and v' have three common neighbors in B. Analogously, for every vertex $v \in B$, there exists a vertex $v' \in B \setminus \{v\}$ with $d_G(v') = 3$ such that v and v' have three common neighbors in B. In particular, we note that each of A and B has a vertex of degree 3 in G. Choosing the vertex $v \in A$ to have degree exactly three in G, there therefore exists a vertex $v' \in A$ such that $N(v) = N(v') = N_v$, where $N_v = \{v_1, v_2, v_3\}$.

We now consider the vertex v_1 and let $u \in N(v_1) \setminus \{v, v'\}$. Then there is a vertex $w \in B \setminus \{v_1\}$ such that $N(w) = \{v, v', u\}$. Renaming v_2 and v_3, if necessary, we may assume that $w = v_2$. There is therefore a vertex $z \in B \setminus \{v_2\}$ such that $N(z) = \{v, v', u\}$. Renaming v_1 and v_3, if necessary, we may assume that $z = v_1$. Thus, $N(v_1) = N(v_2) = \{v, v', u\}$ and $N(v) = N(v') = \{v_1, v_2, v_3\}$.

We now consider the set $A^* = (A \setminus \{v, v'\}) \cup \{v_3\}$ and $B^* = (B \setminus \{v_2, v_3\}) \cup \{v, v'\}$. Then, $|A^*| = |A| - 1$ and $|B^*| = |B|$, and so $|A_1| + |B_1| = n - 1$. Since G is bipartite with partite sets A and B, and since $\delta(G) \geq 3$, each vertex in $B \setminus \{v_3\}$ is adjacent to at least one vertex of $A \setminus \{v, v'\} \subset A^*$. Further both v and v' are adjacent to $v_3 \in A^*$. Hence every vertex not in A^* is adjacent to at least one vertex of A^*, implying that A^* is a dominating set. If u is not adjacent to v_3 or if $d(u) \geq 4$, then B^* is a 2-dominating set and $\gamma\gamma_2(G) \leq |A^*| + |B^*| = n - 1$, a contradiction. Hence, u is adjacent only to v_1, v_2, and v_3. Analogously, v_3 is adjacent only to u, v and v'. Hence since G is connected, $G = K_{3,3}$. ∎

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