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*Dedicated to Mietek Borowiecki on the occasion of his seventieth birthday*

## RAINBOW CONNECTION IN SPARSE GRAPHS

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### Abstract

An edge-coloured connected graph  $G = (V, E)$  is called *rainbow-connected* if each pair of distinct vertices of  $G$  is connected by a path whose edges have distinct colours. The *rainbow connection number* of  $G$ , denoted by  $rc(G)$ , is the minimum number of colours such that  $G$  is rainbow-connected. In this

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paper we prove that  $\text{rc}(G) \leq k$  if  $|V(G)| = n$  and  $|E(G)| \geq \binom{n-k+1}{2} + k - 1$  for all integers  $n$  and  $k$  with  $n - 6 \leq k \leq n - 3$ . We also show that this bound is tight.

**Keywords:** rainbow-connected graph, rainbow colouring, rainbow connection number.

**2010 Mathematics Subject Classification:** 05C15.

## 1. INTRODUCTION

We use [1] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-coloured connected graph  $G$  is called *rainbow-connected* if each pair of distinct vertices of  $G$  is connected by a rainbow path, that is, by a path whose edges have pairwise distinct colours. Note that the edge colouring need not to be proper. The *rainbow connection number* of  $G$ , denoted by  $\text{rc}(G)$ , is the minimum number of colours such that  $G$  is rainbow-connected.

This concept of rainbow connection in graphs was introduced by Chartrand et al. in [5]. An easy observation is that if  $G$  has  $n$  vertices then  $\text{rc}(G) \leq n - 1$ , since one may colour the edges of a given spanning tree of  $G$  with different colours and colour the remaining edges with one of the already used colours. Chartrand et al. determined the precise rainbow connection number of several graph classes including complete multipartite graphs [5]. The rainbow connection number has been studied for further graph classes in [2] and for graphs with fixed minimum degree in [2, 10, 15].

There are different applications for such edge colourings of graphs. One interesting example is the secure transfer of classified information between agencies (see, e.g. [6]).

The computational complexity of rainbow connectivity has been studied in [3, 11]. It is proved that the computation of  $\text{rc}(G)$  is NP-hard ([3, 11]). In fact, it is already NP-complete to decide whether  $\text{rc}(G) = 2$ . It is also NP-complete to decide whether a given edge-coloured graph (with an unbounded number of colours) is rainbow-connected [3]. More generally, it has been shown in [11] that for any fixed  $k \geq 2$  it is NP-complete to decide whether  $\text{rc}(G) = k$ .

For the rainbow connection numbers of graphs the following results are known (and obvious).

**Proposition 1.** *Let  $G$  be a connected graph of order  $n$ . Then*

- (1)  $1 \leq \text{rc}(G) \leq n - 1$ ,
- (2)  $\text{rc}(G) \geq \text{diam}(G)$ ,
- (3)  $\text{rc}(G) = 1$  if and only if  $G$  is complete,
- (4)  $\text{rc}(G) = n - 1$  if and only if  $G$  is a tree.

2. RAINBOW CONNECTION AND SIZE OF GRAPHS

In [9] the following problem was introduced.

**Problem 2.** *For all integers  $n$  and  $k$  with  $1 \leq k \leq n - 1$  compute and minimize the function  $f(n, k)$  with the following property: If  $|V(G)| = n$  and  $|E(G)| \geq f(n, k)$ , then  $\text{rc}(G) \leq k$ .*

The following lower bound for  $f(n, k)$  has been shown.

**Proposition 3** [9]. *For  $n$  and  $k$  with  $1 \leq k \leq n - 1$  it holds that  $f(n, k) \geq \binom{n-k+1}{2} + k - 1$ .*

This lower bound is tight what can be seen by the construction of a graph  $G_k$  as follows: Take a  $K_{n-k+1} - e$  and denote the two vertices of degree  $n - k - 1$  with  $u_1$  and  $u_2$ . Now take a path  $P_k$  by vertices labeled  $w_1, w_2, \dots, w_k$  and identify the vertices  $u_2$  and  $w_1$ . The resulting graph  $G_k$  has order  $n$  and size  $|E(G_k)| = \binom{n-k+1}{2} + k - 2$ . For its diameter we obtain  $d(u_1, w_k) = \text{diam}(G_k) = k + 1 = \text{rc}(G_k)$ .

**Problem 4.** *Determine all values of  $n$  and  $k$  such that*

$$(1) \quad f(n, k) = \binom{n - k + 1}{2} + k - 1.$$

It has been shown in [9] that  $f(n, k) = \binom{n-k+1}{2} + k - 1$  for  $k = 1, 2, n - 2$ , and  $n - 1$  and in [12] for  $k = 3$  and  $4$ . This is summarized in the following theorem.

**Theorem 5.** *For all integers  $n$  and  $k$  with  $k = 1, 2, 3, 4, n - 2, n - 1$  it holds that  $f(n, k) = \binom{n-k+1}{2} + k - 1$ .*

The main result of this paper is the solution of Problem 4 for all graphs of order  $n$  for  $k$  satisfying  $n - 6 \leq k \leq n - 3$ .

The proof of this result consists of several parts. First, we prove for 2-connected graphs  $G$  of order  $n$  and size at least  $\binom{n-k+1}{2} + k - 1$  that  $\text{rc}(G) \leq k$  if  $n - 5 \leq k \leq n - 3$ . In the second step we prove this for  $k = n - 6$  and 2-connected graphs  $G$  where the case  $n = 11$  and  $k = 6$  covers most effort. Finally, we prove in the third step the statement for  $n - 6 \leq k \leq n - 3$  and connected graphs that are not 2-connected.

Recently an improved upper bound for the rainbow connection number of 2-connected graphs has been shown.

**Lemma 6** [7]. *Let  $G$  be a 2-connected graph with  $n$  vertices. Then  $\text{rc}(G) \leq \lceil \frac{n}{2} \rceil$ .*

**Theorem 7.** *If  $G$  is a 2-connected graph with  $|V(G)| = n$  and  $|E(G)| \geq \binom{n-k+1}{2} + k - 1$ , then  $\text{rc}(G) \leq k$  if  $n - 5 \leq k \leq n - 3$  or if  $k = n - 6$  and  $n \geq 12$ .*

**Proof.** We may assume that  $k \geq 5$  since (1) holds for  $1 \leq k \leq 4$ . This implies

$$n \geq 8 \text{ if } k = n - 3 \text{ and thus } \text{rc}(G) \leq \lceil \frac{n}{2} \rceil \leq n - 4 < n - 3,$$

$$n \geq 9 \text{ if } k = n - 4 \text{ and thus } \text{rc}(G) \leq \lceil \frac{n}{2} \rceil \leq n - 4,$$

$$n \geq 10 \text{ if } k = n - 5 \text{ and thus } \text{rc}(G) \leq \lceil \frac{n}{2} \rceil \leq n - 5,$$

$$n \geq 11 \text{ if } k = n - 6 \text{ and thus } \text{rc}(G) \leq \lceil \frac{n}{2} \rceil \leq n - 6 \text{ for all } n \geq 12. \quad \blacksquare$$

### 3. $f(11, 5) = 25$

In the proof of the next theorem we will use the following result.

**Lemma 8** [4]. *Let  $G$  be a connected graph with  $\delta(G) \geq 2$  and  $D$  be a dominating set of  $G$  such that  $G[D]$  is connected. Then  $\text{rc}(G) \leq \text{rc}(G[D]) + 3$ .*

Let an *rc colouring* of a graph  $G$  be an edge colouring such that  $G$  is rainbow-connected.

**Lemma 9.** *Let  $G$  be a connected graph with a partition of its vertex set  $V(G)$  into two subsets  $V_1, V_2$  such that  $G[V_1]$  and  $G[V_2]$  are both connected and  $V_1$  is a dominating set in  $G$ . If  $\text{rc}(G[V_1]) \leq k$  and  $\text{rc}(G[V_2]) \leq l$  for integers  $k, l \geq 1$  then  $\text{rc}(G) \leq \max\{k, l\} + 1$ .*

**Proof.** Take an *rc colouring* of  $G[V_1]$  with  $k$  colours and an *rc colouring* of  $G[V_2]$  with  $l$  colours and colour all edges between  $V_1$  and  $V_2$  with an additional colour. Then  $G$  is rainbow-connected and therefore  $\text{rc}(G) \leq \max\{k, l\} + 1$ .  $\blacksquare$

**Theorem 10.** *Let  $G = (V, E)$  be a 2-connected graph of order  $|V| = 11$ , size  $|E| = 25$  and maximum degree  $\Delta(G) \geq 7$ . Then  $\text{rc}(G) \leq 5$ .*

**Proof.** If  $\Delta(G) = 10$  then  $G$  contains a dominating vertex and hence  $\text{rc}(G) \leq 3$  by Lemma 8. If  $\Delta(G) = 9$  then  $G$  contains a dominating  $K_2$  and therefore  $\text{rc}(G) \leq 1 + 3 = 4$ . If  $\Delta(G) = 8$  then  $G$  always contains a dominating  $P_3$  and hence  $\text{rc}(G) \leq 2 + 3 = 5$ .

Suppose now that  $\Delta(G) = 7$ . Let  $w$  be a vertex with  $d(w) = 7$  and let  $F = G[N[w]]$  and  $H = G[V \setminus N[w]]$  which implies  $|V(H)| = 3$ .

First assume that  $H \cong 3P_1$  or  $H \cong P_2 \cup P_1$ . If there is a vertex  $w_1 \in N(w)$  such that  $N_H(w_1) = V(H)$  then the vertices  $w$  and  $w_1$  induce a dominating  $P_2$  implying that  $\text{rc}(G) \leq 1 + 3 = 4 < 5$ . If there are two vertices  $w_1, w_2 \in N(w)$  such that  $N_H(w_1) \cup N_H(w_2) = V(H)$  then  $(w_1, w, w_2)$  is a dominating  $P_3$  implying that  $\text{rc}(G) \leq 2 + 3 = 5$ . Otherwise,  $|E(F, H)| \leq 7$ . Then  $|E(F)| \geq 25 - (7 + 1) = 17$  and so  $\text{rc}(F) \leq 3$  by Theorem 5. Take an *rc colouring* of  $F$  with colours 1, 2, 3. If  $H \cong 3P_1$  then colour all edges incident with a vertex of  $H$  with colours 4 and 5 such that each colour occurs at least once at every vertex of  $H$ . If  $H \cong P_2 \cup P_1$ , say  $V(P_2) = \{u_1, u_2\}$ ,  $V(P_1) = \{u_3\}$ , then colour the edge  $u_1u_2$  with colour 1, all

other edges incident to  $u_1$  or  $u_2$  with colour 4, and all edges incident to  $u_3$  with colour 5 to obtain a rainbow colouring of  $G$  with 5 colours.

Next assume that  $H \cong K_3$ . Then there is a dominating  $P_3$  implying  $\text{rc}(G) \leq 2 + 3 = 5$  by Lemma 8.

Finally, assume that  $H \cong P_3$  with  $P_3 = (u_1, u_2, u_3)$ . If  $w_1u_2 \in E(G)$  for a vertex  $w_1 \in N(w)$  then  $\{w, w_1, u_2\}$  is a dominating set implying that  $\text{rc}(G) \leq 2 + 3 = 5$ . Hence we may assume that  $N(u_2) \cap N(w) = \emptyset$ . Then  $N(u_i) \cap N(w) \neq \emptyset$  for  $i = 1$  and  $i = 3$  since  $G$  is 2-connected. If  $w_1u_1, w_1u_3 \in E(G)$  for a vertex  $w_1 \in N(w)$  then  $\{w, w_1, u_1\}$  is a dominating set implying that  $\text{rc}(G) \leq 2 + 3 = 5$ .

Hence we may assume that  $N(u_1) \cap N(u_3) \cap N(w) = \emptyset$ . Then  $|E(F)| \geq 25 - (7 + 2) = 16$ . Let  $d_F(u_i) = d_i$  for  $i = 1, 3$  and  $1 \leq d_1 \leq d_3$ . Then  $d_1 \leq 3$ . Let  $X = \{u_1, u_2\}$  and  $Y = V(G) \setminus X$ . Then  $G[X]$  and  $G[Y]$  are connected,  $Y$  dominates  $X$ , and  $|E(G[Y])| \geq 25 - 5 = 20$ . We then obtain  $\text{rc}(G[Y]) \leq 4$  by Theorem 5, and thus Lemma 9 implies that  $\text{rc}(G) \leq 5$ . ■

We will use the following lemma, which is just a special case of a very strong theorem characterizing  $k$ -connected graphs proven independently by Győri [8] and Lovász [14].

**Lemma 11.** *If  $G = (V, E)$  is a 2-connected graph of order  $n$ , then for every pair of vertices  $v_1, v_2$  and for every pair of positive integers  $n_1, n_2$  with  $n_1 + n_2 = n$  there exists a partition of  $V$  into two subsets  $V_1, V_2$  such that  $v_1 \in V_1, v_2 \in V_2$ ,  $|V_1| = n_1$ ,  $|V_2| = n_2$  and the induced graphs  $G[V_1], G[V_2]$  are connected.*

**Theorem 12.** *Let  $G = (V, E)$  be a 2-connected graph of order  $|V| = 11$ , size  $|E| = 25$ , maximum degree  $\Delta(G) \leq 6$ , and minimum degree  $\delta(G) = 2$ . Then  $\text{rc}(G) \leq 5$ .*

*Proof.* Let  $w \in V(G)$  be a vertex with  $d(w) = 2$  and  $w_1, w_2$  be its two neighbours. Then  $d(w_i) \leq 6$  for  $i = 1, 2$  since  $\Delta(G) \leq 6$ . By Lemma 11 there is a partition of  $V$  into two subsets  $V_1, V_2$  such that  $w \in V_1$ ,  $|V_1| = 2$ ,  $|V_2| = 9$  and the induced graphs  $G[V_1], G[V_2]$  are connected. We conclude that  $|E(G[V_2])| \geq 25 - 7 = 18$ . Hence  $\text{rc}(G[V_2]) \leq 4$  by Theorem 5. Clearly,  $\text{rc}(G[V_1]) = 1 < 4$  and  $V_2$  is a dominating set in  $G$  which induces a connected subgraph. Therefore,  $\text{rc}(G) \leq 5$  by Lemma 9. ■

To organize the proof of the next theorem we provide a couple of lemmas. The following facts are just special cases of Theorem 5.

**Proposition 13.** *Let  $G_1$  be a connected graph of order 5 and  $G_2$  be a connected graph of order 6. Then*

- (1)  $\text{rc}(G_1) \leq 3$  if  $|E(G_1)| \geq 5$ ,
- (2)  $\text{rc}(G_1) \leq 2$  if  $|E(G_1)| \geq 7$ ,

(3)  $\text{rc}(G_2) \leq 3$  if  $|E(G_2)| \geq 8$ ,

(4)  $\text{rc}(G_2) \leq 2$  if  $|E(G_2)| \geq 11$ .

**Lemma 14.** *Let  $G = (V, E)$  be a 2-connected graph of order 11, size 25 and  $\delta(G) \geq 3$ . Then there exists a partition of the vertex set  $V$  into two sets,  $X$  and  $Y$ , such that  $|X| = 5$  and  $|Y| = 6$ , the graphs  $G_1 = G[X]$  and  $G_2 = G[Y]$  are connected,  $G_2$  is of size at least 6.*

*Proof.* By Lemma 11 there exists a partition of  $V$  into sets  $X$  and  $Y$  such that  $|X| = 5$ ,  $|Y| = 6$  and  $G_1 = G[X]$ ,  $G_2 = G[Y]$  are connected, and hence  $|E(G_1)| \geq 4$ ,  $|E(G_2)| \geq 5$ . If  $|E(G_2)| \geq 6$ , then we are done, so assume that  $|E(G_2)| = 5$ , hence  $G_2$  is a tree. Let  $y$  be a leaf of  $G_2$ . Since  $\delta(G) \geq 3$ ,  $y$  must have at least two neighbours in  $X$ , and thus  $Y \setminus \{y\}$  and  $X \cup \{y\}$  form a desired partition of  $V$ . ■

Let  $G = (V, E)$  be a 2-connected graph of order 11, size 25 and  $\delta(G) \geq 3$ . Our aim is to give a rainbow colouring of the edges of  $G$  with five colours. Let  $X, Y$  be a partition as in Lemma 14. Then by Theorem 5 four colours, say 1, 2, 3, 4, suffice for an rc colouring of  $G_1$  and  $G_2$ , respectively. The fifth colour, say 5, will be used for the edges of the set  $E(X, Y)$ , the set of edges between vertices of  $X$  and vertices of  $Y$ . If  $X$  dominates  $Y$  or  $Y$  dominates  $X$  then we are done by Lemma 9. If this is not the case then denote by  $X'$  the set of those vertices of  $X$  that are not connected with  $Y$ , and, similarly, denote by  $Y'$  the set of those vertices of  $Y$  that are not connected with  $X$ . Since  $G$  is 2-connected, we have  $1 \leq |X'| \leq 3$  and  $1 \leq |Y'| \leq 4$ . Moreover, since  $\delta(G) \geq 3$ , the vertices in  $X'$  and  $Y'$  are of degree at least three in  $G_1 = G[X]$  and  $G_2 = G[Y]$ , respectively.

Let  $f$  be an rc colouring of the graph  $G_1$  with  $X' \neq \emptyset$ . A colour of an edge of a rainbow path connecting a vertex of  $X'$  with any vertex of  $X \setminus X'$  is called *transit*. We are interested in the minimum number of transit colours sufficient to go from every vertex of  $X'$  to any (i.e., at least one) vertex of  $X \setminus X'$  by a rainbow path using these colours. The minimum is taken over all possible rc colourings of  $G_1$  with (at most) **four** colours. This number will be denoted by  $t(X')$ . We define analogously  $t(Y')$ , where the minimum is taken over all possible rc colourings of  $G_2$  with **four** colours. Evidently, if  $|X'| = 1$ , then  $t(X') = 1$ .

**Lemma 15** (Transit Lemma). *Let  $G$  be a 2-connected graph of order 11, size 25 and  $\delta(G) \geq 3$ , and let  $X, Y$  be a partition of  $V$  as in Lemma 14 with  $G_1 = G[X]$  and  $G_2 = G[Y]$ . Then there is an rc colouring of  $G$  with five colours if*

(1)  $t(X') + \text{rc}(G_2) \leq 4$  or

(2)  $\text{rc}(G_1) + t(Y') \leq 4$ .

*Proof.* We shall prove only (1). The proof of (2) is analogous. Let  $f_1$  be an rc colouring of  $G_1$  that minimizes the parameter  $t(X')$  and let  $f_2$  be an rc colouring

of  $G_2$  that minimizes the parameter  $\text{rc}(G_2)$ . Without loss of generality, we may assume that the colours used as transit colours in  $G_1$  and the colours used by  $f_2$  form disjoint sets. Let now  $x$  be a vertex of  $X'$ . By definition of  $t = t(X')$  we are able to reach the set  $X'' = X \setminus X'$  (from  $x \in X'$ ) by a rainbow path of length at most  $t$ , i.e., using at most  $t$  colours. Next, we go to a vertex  $y \in Y$  by an edge belonging to  $E(X, Y)$ , i.e., coloured by the fifth colour. From  $y$  we are able to reach all remaining vertices of  $Y$  using at most  $\text{rc}(G_2)$  colours. If  $x \in X \setminus X'$  then we go directly to  $Y$ . ■

In order to apply the transit lemma above, we shall first estimate the parameter  $t$  in distinct cases.

**Lemma 16.** *Let  $G_1$  be a connected graph of order 5 with vertex set  $X$  and let  $X'$  be a nonempty subset of  $X$ . Suppose that all vertices of  $X'$  are of degree at least three in  $G_1$ .*

- (1) *If  $|X'| = 2$ , then  $t(X') = 1$  and  $|E(G_1)| \geq 5$ .*
- (2) *If  $|X'| = 3$ , then  $t(X') \leq 2$  and  $|E(G_1)| \geq 6$ .*

**Proof.** The claimed size of  $|E(G_1)|$  is simply a consequence of the fact that every vertex in  $X'$  is of degree at least 3, while every vertex in  $X'' = X \setminus X'$  must have degree at least 1 by connectedness of  $G_1$ .

Assume first that  $|X'| = 2$  and  $X' = \{x_1, x_2\}$ , hence  $|E(G_1)| \geq 5$ . Then, since  $d(x_1), d(x_2) \geq 3$ ,  $x_1$  and  $x_2$  must have a common neighbour, say  $x$ , such that  $x_1, x, x_2$  lie on a common cycle, i.e., a triangle if  $x_1$  and  $x_2$  are adjacent, or a square otherwise. Thus we may temporarily remove the edge  $xx_2$  and find an rc colouring of the remaining graph with four colours. Then it is sufficient to use the same colour for  $xx_2$  as is used for  $xx_1$  to obtain an rc colouring of  $G_1$  with  $t(X') = 1$ .

Suppose then that  $|X'| = 3$  and  $X' = \{x_1, x_2, x_3\}$ , hence  $|E(G_1)| \geq 6$ . Then, since  $d(x_1), d(x_2), d(x_3) \geq 3$ , it is easy to see that there must exist  $x \in X''$  forming a triangle with two vertices of  $X'$ . Indeed, this is obvious if  $X'$  induces a complete graph in  $G_1$ . Otherwise, if, e.g. (without loss of generality)  $x_1x_2 \in E(G_1)$  and  $x_2x_3 \notin E(G_1)$ , then  $x_1$  must have a neighbour  $x \in X''$ , while  $x_2$  is adjacent with both vertices from  $X''$  (in particular with  $x$ ). Then, analogously as above, we first fix an rc colouring of  $G_1$  with the edge  $xx_2$  removed, and then colour  $xx_2$  with the colour of  $xx_1$ . Since  $x_3$  must have at least one neighbour in  $X''$ , the proof is completed. ■

**Lemma 17.** *Let  $G_2$  be a connected graph of order 6 and size  $\geq 6$  with vertex set  $Y$  and let  $Y'$  be a nonempty subset of  $Y$ . Suppose that all vertices of  $Y'$  are of degree at least three in  $G_2$ .*

- (1) *If  $|Y'| = 2$ , then  $t(Y') = 1$ .*

- (2) If  $|Y'| = 3$ , then  $t(Y') \leq 2$ .  
 (3) If  $|Y'| = 4$ , then  $t(Y') \leq 3$ .

**Proof.** First, note that if  $G_2$  contains a triangle, we may construct an rc colouring of the edges of  $G_2$  with four colours as follows. We contract the edges of this triangle (ignoring multiple edges and loops), then we choose an rc colouring of the resulting connected graph of order 4 with colours 2, 3, 4, subsequently we reverse the contraction process returning to the original graph  $G_2$  (where the meanwhile ignored multiple edges copy the colour of their retained correspondents), and finally we colour the edges of the triangle with colour 1. If, on the other hand,  $G_2$  contains a square, we analogously construct an rc colouring by contracting the set of edges of this square, then using colours 3, 4 for the resulting graph, and finally colouring the edges of the square alternately 1, 2, 1, 2 (and putting arbitrary of the colours on the possibly remaining edges).

Suppose  $|Y'| = 2$ ,  $Y' = \{y_1, y_2\}$ . Since  $d(y_1), d(y_2) \geq 3$  then, if  $y_1y_2 \notin E(G_2)$ ,  $G_2$  contains a square with (opposite) vertices  $y_1, y_2$ . Then the assertion follows by the contraction construction above. If, on the other hand,  $y_1y_2 \in E(G_2)$  then, if  $G_2$  contains a triangle or a square including  $y_1$  and  $y_2$ , we are again done by the construction above. Otherwise,  $N(y_1)$  and  $N(y_2)$  are disjoint (hence  $|N(y_1) \setminus \{y_2\}| = 2$  and  $|N(y_2) \setminus \{y_1\}| = 2$ ) and there are no edges between  $N(y_1) \setminus \{y_2\}$  and  $N(y_2) \setminus \{y_1\}$ . Since  $|E(G_2)| \geq 6$ , the vertices from  $N(y_1) \setminus \{y_2\}$  or (symmetrically)  $N(y_2) \setminus \{y_1\}$  must form an edge in  $G_2$ . Then colour the edges incident to  $y_1$  but not to  $y_2$  with colours 1 and 2, also the edges incident to  $y_2$  but not to  $y_1$  with colours 1 and 2, the edge connecting the two vertices of  $N(y_1) \setminus \{y_2\}$  (or of  $N(y_2) \setminus \{y_1\}$ ) with colour 3, and the edge  $y_1y_2$  with colour 4.

Suppose now that  $|Y'| = 3$  and  $Y' = \{y_1, y_2, y_3\}$ . If  $Y'$  induces a triangle in  $G_2$ , then we apply the contraction construction above using this triangle. Then from every  $y_i$  we can reach  $Y'' = Y \setminus Y'$  by (optionally) first going through an edge coloured with 1 and then through any (previously fixed) edge joining  $Y'$  and  $Y''$ . Otherwise, since  $d(y_1), d(y_2), d(y_3) \geq 3$ , it is very easy to verify that  $G_2$  must contain a triangle with two vertices from  $Y'$  or a square with two elements from  $Y'$  as opposite vertices, hence we are done by the contraction construction above.

Let finally  $|Y'| = 4$ ,  $Y' = \{y_1, y_2, y_3, y_4\}$  and  $Y'' = Y \setminus Y' = \{y_5, y_6\}$ . Without loss of generality, assume that  $d(y_6) \leq d(y_5)$ . Then it is easy to verify that the graph  $G_2 - y_6$  is connected and has size at least 5 (since  $d(y_1), d(y_2), d(y_3), d(y_4) \geq 3$ ), hence, by Proposition 13, we may choose an rc colouring using colours 1, 2, 3 which implies  $t(Y') \leq 3$ . Then we are done by using colour 4 on the edges incident to  $y_6$ . ■

**Theorem 18.** *Let  $G = (V, E)$  be a 2-connected graph of order  $|V| = 11$ , size  $|E| = 25$ , and minimum degree  $\delta(G) \geq 3$ . Then  $\text{rc}(G) \leq 5$ .*



**Proof.** Let  $X, Y$  be a partition as in Lemma 14 with  $X', Y'$  being the subsets of those vertices from  $X, Y$  which have no neighbours in  $Y, X$ , respectively. Set  $X'' = X \setminus X', Y'' = Y \setminus Y'$ . If  $X' = \emptyset$  or  $Y' = \emptyset$ , then the result is obvious. Assume then that  $|X'| \geq 1, |Y'| \geq 1$  ( $|X'| \leq 3, |Y'| \leq 4$  since  $G$  is 2-connected). We will consider four cases with respect to the size of  $|Y'|$ .

*Case 1.*  $|Y'| = 4$ . Consider first the case where  $|X'| = 1$  or  $|X'| = 2$ . Then  $t(X') = 1$  by Lemma 16. If  $\text{rc}(G_2) \leq 3$ , we are done by Lemma 15. So, we may assume that either  $\text{rc}(G_2) \leq 3$  and  $|X'| \geq 3$  or  $\text{rc}(G_2) = 4$ .

If  $\text{rc}(G_2) \leq 3$  and  $|X'| \geq 3$ , then  $|E(X, Y)| \leq 4$  (since  $|Y'| = 4$ ) and, in consequence,  $|E(G_2)| \geq 11$  (since  $|E(G_1)| \leq 10$ ). Then  $\text{rc}(G_2) \leq 2$  by Proposition 13. So, we can apply Lemma 15(1) since  $t(X') \leq 2$ .

If, on the other hand,  $\text{rc}(G_2) = 4$ , then  $|E(G_2)| \leq 7$  by Proposition 13. Since  $|E(X, Y)| \leq 8$  (because of  $|X'| \geq 1$ ) we have  $|E(G_1)| = 10$ , i.e.,  $G_1 \cong K_5$ . Then  $\text{rc}(G_1) = 1$ . Moreover, by Lemma 17,  $t(Y') \leq 3$ , hence we are done by Lemma 15(1).

*Case 2.*  $|Y'| = 3$ . Then  $t(Y') \leq 2$ , and, if  $\text{rc}(G_1) \leq 2$ , then we are done by Lemma 15(2). So, assume that  $\text{rc}(G_1) \geq 3$ . Then, in particular,  $|E(G_1)| \leq 6$  by Proposition 13.

Consider first the case  $|X'| = 3$ . Then  $|E(G_1)| + |E(X, Y)| \leq 12$ , which implies that  $|E(G_2)| \geq 13$ . But then  $\text{rc}(G_2) \leq 2$  by Proposition 13 and, since  $t(X') \leq 2$ , we can again apply Lemma 15(1).

Now suppose that  $|X'| = 2$ . Then  $t(X') = 1$  by Lemma 17 and we are done if  $\text{rc}(G_2) \leq 3$ . If not, then  $\text{rc}(G_2) = 4$  and  $|E(G_2)| \leq 7$  by Proposition 13. But then  $|E(G_1)| \geq 9$ , and hence  $\text{rc}(G_1) \leq 2$  again by Proposition 13. Thus we are done by Lemma 15(2).

Finally, consider the case  $|X'| = 1$  (which implies  $t(X') = 1$ ). Let  $X' = \{a\}$  and denote by  $x$  one of its neighbours,  $x \in X''$ . As above, we are done if  $\text{rc}(G_2) \leq 3$ . So, suppose that  $\text{rc}(G_2) = 4$ . By Proposition 13 we have  $|E(G_2)| \leq 7$ . Thus  $|E(G_1)| \geq 6$ . If  $|E(G_1)| \geq 7$ , then  $\text{rc}(G_1) \leq 2$  and, since  $t(Y') \leq 2$ , we are done. So,  $|E(G_1)| = 6$ , and thus the set  $E(X'', Y'')$  has all possible edges. In consequence,  $x$  is connected by an edge to all vertices in  $Y''$ , and the result follows.

*Case 3.*  $|Y'| = 2$ . Then  $t(Y') = 1$  by Lemma 17 and we are done if  $\text{rc}(G_1) \leq 3$ . So, assume that  $\text{rc}(G_1) = 4$ . Thus  $G_1$  is a tree,  $|E(G_1)| = 4$ . But then, by Lemma 16,  $|X'| = 1$ . Let  $X' = \{a\}$  and  $Y' = \{b, c\}$ . By Lemma 17 we may choose an rc colouring of  $G_2$  with four colours such that we have two edges  $by_1, cy_2$  with  $y_1, y_2 \in Y''$  coloured the same (in particular we may have  $y_1 = y_2$ ). As above, we are done if  $\text{rc}(G_2) \leq 3$ . So, suppose that  $G_2$  has less than 8 edges. Then there are at least 14 edges between  $X$  and  $Y$ . So, we can have at most two edges ‘missing’ in  $E(X'', Y'')$ , hence at least one of (at least) three neighbours of

$a$ , say  $x \in X''$ , is joined to both  $y_1$  and  $y_2$ , and the result follows.

*Case 4.*  $|Y'| = 1$ . Let  $Y' = \{b\}$  and denote by  $y$  any neighbour of  $b$ ,  $y \in Y''$ . As above, we are done if  $\text{rc}(G_1) \leq 3$ . So, assume that  $\text{rc}(G_1) = 4$ . Thus  $G_1$  is a tree, and hence  $|X'| = 1$  by Lemma 16. Say  $X' = \{a\}$ . Since  $d(a) \geq 3$ , for each vertex  $x$  of  $X''$  there is a path of length at most 2 (i.e., with at most 2 colours in any rainbow colouring of the tree  $G_1$ ) from this vertex  $x$  to  $a$ . On the other hand, by the definition of  $Y'$ , at least one vertex from  $X''$  is joined to  $y$ , and again the result follows. ■

#### 4. $G$ IS NOT 2-CONNECTED

**Lemma 19** (Bridge reduction). *Let  $G$  be a connected graph,  $e = uw$  be a bridge of  $G$ , and let  $G'$  be the graph obtained from  $G$  by contracting the edge  $e$ . If  $\text{rc}(G') = p$ , then  $\text{rc}(G) \leq p + 1$ .*

**Proof.** Choose an rc colouring of  $G'$  with colours  $1, 2, \dots, p$ . Now colour the edge  $e$  with colour  $p + 1$ . Then  $G$  is rainbow-connected and  $\text{rc}(G) \leq p + 1$ . ■

**Theorem 20.** *If  $G$  is a connected graph which is not 2-connected with  $|V(G)| = n$  and  $|E(G)| \geq \binom{n-k+1}{2} + k - 1$ , then  $\text{rc}(G) \leq k$  if  $n - 6 \leq k \leq n - 3$ .*

**Proof.** Setting  $g(n, k) = \binom{n-k+1}{2} + k - 1$  and  $k = n - t$  we obtain  $g(n, n - t) = n + \binom{t}{2} - 1$ .

The proof will be by induction on the number  $n$  of vertices. For  $n = t + 1$  we obtain  $g(t + 1, 1) = \binom{t+1}{2} = t + 1 + \binom{t}{2} - 1$ .

Consider for the induction step from  $n$  to  $n + 1$  a graph  $G$  of order  $n + 1$  with  $n + 1 + \binom{t}{2} - 1$  edges.

If  $G$  contains a bridge, say  $e$ , then we apply the bridge reduction of Lemma 19 and obtain  $|E(G')| = (n + 1 + \binom{t}{2} - 1) - 1 = g(n, n - t)$ . Hence  $\text{rc}(G') \leq n - t$  by induction hypothesis and therefore  $\text{rc}(G) \leq n + 1 - t$  by Lemma 19.

If  $G$  contains no bridge but a cut vertex, say  $w$ , then  $G$  can be decomposed into two subgraphs  $G_1, G_2$  such that  $V(G_1) \cup V(G_2) = V(G)$  and  $V(G_1) \cap V(G_2) = \{w\}$ . Let  $n_i = |V(G_i)|$  and  $m_i = |E(G_i)| = n_i + t_i$  for  $i = 1, 2$ . Since  $G$  has no bridges we conclude that  $n_i \geq 3$  and  $t_i \geq 0$  for  $i = 1, 2$ .

For each  $t_i$  there exists an integer  $s_i \geq 2$  such that  $t_i + 1 \geq \binom{s_i}{2}$ . Choose  $s_i$  maximal with this property. Then  $t_i + 1 \leq \binom{s_i+1}{2} - 1$ .

Assume that  $s_1 + s_2 \leq t$ . With  $|E(G)| = n + 1 + \binom{t}{2} - 1 = (n_1 + t_1) + (n_2 + t_2)$  we obtain

$$\begin{aligned}
 t_1 + t_2 + 1 &= \binom{t}{2} = \sum_{j=1}^{t-1} j \geq \sum_{j=1}^{s_1+s_2-1} j = \sum_{j=1}^{s_1} j + \sum_{j=s_1+1}^{s_1+s_2-1} j \\
 &> \sum_{j=1}^{s_1-1} j + s_1 + \sum_{j=s_1+1}^{s_1+s_2-1} (j - (s_1 - 1)) \\
 &= \sum_{j=1}^{s_1-1} j + (s_1 - 1) + \sum_{j=1}^{s_2} j \\
 &> \sum_{j=1}^{s_1-1} j + (s_1 - 1) + \sum_{j=1}^{s_2-1} j + (s_2 - 1) \\
 &= \binom{s_1+1}{2} - 1 + \binom{s_2+1}{2} - 1 \geq (t_1 + 1) + (t_2 + 1),
 \end{aligned}$$

a contradiction. Hence we have  $s_1 + s_2 \geq t + 1$ .

Now we apply the induction hypothesis. We have  $|E(G_i)| = n_i + t_i = n_i + (t_i + 1) - 1 \geq n_i + \binom{s_i}{2} - 1$ . Hence  $\text{rc}(G_i) \leq n_i - s_i$  for  $i = 1, 2$ . Choose for  $i = 1, 2$  an rc colouring of  $G_i$  with  $n_i - s_i$  distinct colours where the colour sets for  $G_1$  and  $G_2$  are disjoint. Note that if  $s_i = 2$  this is possible by Theorem 5 and if  $3 \leq s_i \leq t$  this is possible by induction if  $G_i$  is not 2-connected and by Theorems 7, 10, 12, 18 if  $G_i$  is 2-connected. This colouring is an rc colouring of  $G$ . Moreover,  $\text{rc}(G) \leq (n_1 - s_1) + (n_2 - s_2) = (n + 1) - (s_1 + s_2) \leq (n + 1) - (t + 1) = n - t$  which concludes the proof. ■

Summarizing the results of Theorems 7, 10, 12, 18, 20 we obtain together with the remark after Proposition 3 our main theorem.

**Theorem 21.** *For all integers  $n$  and  $k$  with  $n - 6 \leq k \leq n - 3$  it holds that  $f(n, k) = \binom{n-k+1}{2} + k - 1$ .*

It would be an interesting task to determine additional values of  $n$  and  $k$  (beside those of Theorems 5 and 21) such that  $f(n, k) = \binom{n-k+1}{2} + k - 1$ . Of course, partial results can be obtained by applying Lemma 6.

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