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WIENER AND VERTEX PI INDICES OF THE STRONG PRODUCT OF GRAPHS

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Abstract

The *Wiener index* of a connected graph G, denoted by $W(G)$, is defined as $\frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)$. Similarly, the *hyper-Wiener index* of a connected graph G, denoted by $WW(G)$, is defined as $\frac{1}{2}W(G) + \frac{1}{4}\sum_{u,v\in V(G)} d_G^2(u,v)$. The *vertex Padmakar-Ivan* (*vertex PI*) *index* of a graph G is the sum over all edges uv of G of the number of vertices which are not equidistant from u and v. In this paper, the exact formulae for Wiener, hyper-Wiener and vertex PI indices of the strong product $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$, where $K_{m_0,m_1,\ldots,m_{r-1}}$ is the complete multipartite graph with partite sets of sizes $m_0, m_1, \ldots, m_{r-1}$, are obtained. Also lower bounds for Wiener and hyper-Wiener indices of strong product of graphs are established.

Keywords: strong product, Wiener index, hyper-Wiener index, vertex PI index.

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1. Introduction

All the graphs considered in this paper are connected and simple. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G. The *strong product* of graphs G and H, denoted by $G\boxtimes H$, is the graph with vertex set $V(G)\times V(H) = \{(u,v) : u \in V(G), v \in V(H)\}$ and $(u,x)(v,y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$, see Figure 1.

A *topological index* of a graph is a parameter related to the graph, it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacological, toxicological, biological and other properties of chemical compounds [7]. Several types of such indices exist, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index. The Wiener index [26] is one of the oldest molecular-graph-based structure-descriptors [25]. Its chemical applications and mathematical properties are well studied in [6, 20].

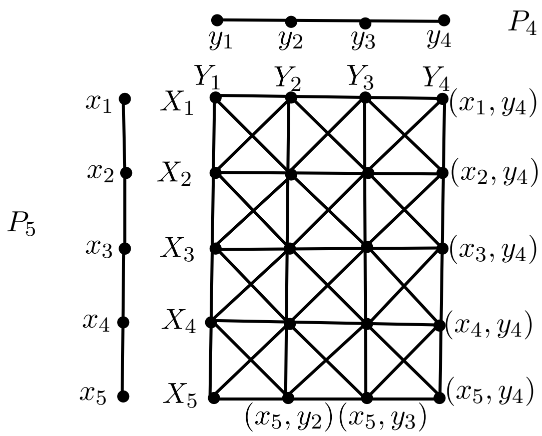

Figure 1. $P_5 \boxtimes P_4$.

Let G be a connected graph. Then *Wiener index* of G, denoted by $W(G)$, is defined as $W(G) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)$ with the summation going over all pairs of vertices of G. The *hyper-Wiener index* of a connected graph G, denoted by $WW(G)$, is defined as $WW(G) = \frac{1}{2}W(G) + \frac{1}{4}\sum_{u,v\in V(G)} d_G^2(u,v)$, where $d_G^2(u,v) = (d_G(u,v))^2$.

The hyper-Wiener index of an acyclic graph was first introduced by Randić [24]. Then Klein *et al.* [16] studied hyper-Wiener index for all connected graphs. Applications of the hyper-Wiener index as well as its calculation are well studied in [15, 17, 18, 19, 21].

Let $e = uv$ be an edge of the graph G. The number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v is denoted by $n_u(e)$. Analogously, $n_v(e)$ is the number of vertices of G whose distance to the vertex v is smaller than the distance to the vertex u; here the vertices equidistant from both the ends of the edge $e = uv$ are not counted. Another topological index, namely, the *vertex Padmakar-Ivan* (*vertex PI*) of G, denoted by $PI(G)$, is defined as follows, $PI(G) = \sum_{e=uv\in E(G)}(n_u(e) + n_v(e))$. For $e = uv$ in G, the number of equidistant vertices of e is denoted by $N_G(e)$. Then the above definition is equivalent to $PI(G) = \sum_{e\in E(G)}(|V(G)| - N_G(e))$.

The vertex PI index is the topological index related to equidistant vertices;

Khadikar *et al.* [11] investigated the chemical applications of the vertex PI index. The mathematical properties of the vertex PI index and its applications in chemistry and nanoscience are well studied in [1, 2, 3, 5, 12, 13, 23]. In [22] we have studied the Wiener, hyper-Wiener and vertex PI indices of the tensor product of graphs. In this paper, we obtain the Wiener, hyper-Wiener and vertex PI indices of the graph $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$, where $K_{m_0,m_1,\ldots,m_{r-1}}$ is the complete multipartite graph with partite sets of sizes $m_0, m_1, \ldots, m_{r-1}$. Also we have obtained lower bounds for Wiener and hyper-Wiener indices of the strong product of graphs.

If $m_0 = m_1 = \cdots = m_{r-1} = s$ in $K_{m_0,m_1,\ldots,m_{r-1}}$, then we denote $K_{m_0,m_1,\ldots,m_{r-1}}$ by $K_{r(s)}$. For $S \subseteq V(G)$, $\langle S\rangle$ denotes the subgraph of G induced by S. A path and cycle on n vertices are denoted by P_n and C_n, respectively. We call C_3 a triangle. For disjoint subsets $S, T \subset V(G)$, by $d_G(S,T)$, we mean the sum of the distances in G from each vertex of S to every vertex of T, that is, $d_G(S,T) = \sum_{s\in S, t\in T} d_G(s,t)$. For disjoint subsets $S, T \subset V(G)$, $E(S,T)$ denotes the set of edges of G having one end in S and the other end in T. Notations and definitions which are not given here can be found in [4] or [9].

In this paper, besides some other results, we prove the following results.

Theorem 1. *Let G be a connected graph with n vertices. Then* $W(G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = n_0^2\, W(G) + n(n_0^2 - q - n_0) + (n_0^2 - 2q - n_0)|E(G)|$, *where* $n_0 = \sum_{i=0}^{r-1} m_i$ *and q is the number of edges of $K_{m_0,m_1,\ldots,m_{r-1}}$.*

Theorem 2. *Let G be a connected graph with n vertices. Then* $WW(G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = n_0^2\, WW(G) + \frac{n}{2}(3n_0^2 - 4q - 3n_0) + 2(n_0^2 - 2q - n_0)|E(G)|$, *where* $n_0 = \sum_{i=0}^{r-1} m_i$ *and q is the number of edges of $K_{m_0,m_1,\ldots,m_{r-1}}$.*

Let $(V_1, V_2, \ldots, V_\chi)$ be a proper $\chi(G)$-colouring of G, where $\chi(G)$ is the chromatic number of G, such that no V_i can be augmented by adding a vertex of V_j, $j \geq i+1$, that is, no vertex of V_j is nonadjacent to all the vertices of V_i, $i < j$, in G. Without loss of generality we assume that $|V_1| \geq |V_2| \geq \cdots \geq |V_r|$. We call such a $\chi(G)$-colouring a *decreasing $\chi(G)$-colouring* of G.

Based on the above results we have obtained the following lower bounds for the Wiener and hyper-Wiener indices of the graph $G \boxtimes G'$, where G and G' are connected graphs.

Theorem 3. *Let G be a connected graph with n vertices and m edges; let G' be a graph with $\chi(G') = r \geq 2$. If the decreasing color classes of G' have $m_0, m_1, \ldots, m_{r-1}$ vertices, then* $W(G \boxtimes G') \geq W(G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = n_0^2\, W(G) + n(n_0^2 - q - n_0) + (n_0^2 - 2q - n_0)m$ *and* $WW(G \boxtimes G') \geq WW(G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = n_0^2\, WW(G) + \frac{n}{2}(3n_0^2 - 4q - 3n_0) + 2(n_0^2 - 2q - n_0)m$, *where* $\sum_{i=0}^{r-1} m_i = n_0$, *$n_0$ is the number of vertices of G' and q is the number of edges of $K_{m_0,m_1,\ldots,m_{r-1}}$.*

2. Wiener Index of $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$

Let G be a simple connected graph with $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and let $K_{m_0,m_1,\ldots,m_{r-1}}$, $r \geq 2$, be the complete multiparite graph with partite sets $V_0, V_1, \ldots, V_{r-1}$ and let $|V_i| = m_i$, $0 \leq i \leq r-1$. In the graph $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$, let $B_{ij} = v_i \times V_j, v_i \in V(G)$ and $0 \leq j \leq r-1$. For our convenience, we write

$$V(G) \times V(K_{m_0,m_1,\ldots,m_{r-1}}) = \bigcup_{i=0}^{n-1}\{v_i \times \bigcup_{j=0}^{r-1} V_j\}$$
$$= \bigcup_{i=0}^{n-1}\{(v_i \times V_0) \cup (v_i \times V_1) \cup \cdots \cup (v_i \times V_{r-1})\}$$
$$= \bigcup_{i=0}^{n-1}\{B_{i0} \cup B_{i1} \cup \cdots \cup B_{i(r-1)}\} = \bigcup_{\substack{i=0\\j=0}}^{\substack{n-1\\r-1}} B_{ij},$$

where $B_{ij} = v_i \times V_j$.

Let $\mathscr{B} = \{B_{ij}\}_{\substack{i=0,1,\ldots,n-1\\j=0,1,\ldots,r-1}}$. Let $X_i = \bigcup_{j=0}^{r-1} B_{ij}$ and $Y_j = \bigcup_{i=0}^{n-1} B_{ij}$; we call X_i and Y_j as *layer* and *column* of $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$. Further, if $v_iv_k \in E(G)$, then the induced subgraph $\langle B_{ij} \cup B_{kp}\rangle$ of $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$ is isomorphic to $K_{|V_j||V_p|}$ or, m_p independent edges joining the corresponding vertices of B_{ij} and B_{kj} according to $j \neq p$ or $j = p$. It is used in the proof of the next lemma.

The proof of the following lemma follows easily from the properties and structure of $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$, see Figure 2 and Figure 3. The lemma is used in the proof of the main theorems of this paper.

Lemma 4. *Let G be a connected graph and let $B_{ij},\ B_{kp} \in \mathscr{B}$ of the graph $H = G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$, where $r \geq 2$.*

(i) *If $v_iv_k \in E(G)$, then* $d_H(B_{ij}, B_{kp}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ m_j(2m_j-1), & \text{if } j = p. \end{cases}$

(ii) *If $v_iv_k \notin E(G)$, then $d_H(B_{ij}, B_{kp}) = m_jm_pd_G(v_i, v_k)$.*

(iii) $d_H(B_{ij}, B_{ip}) = \begin{cases} m_jm_p, & if\ j \neq p, \\ 2m_j(m_j-1), & if\ j = p. \end{cases}$

***Proof of Theorem* 1.** Let $H = G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$. Clearly,

$$W(H) = \frac{1}{2}\sum_{B_{ij},B_{kp}\in\mathscr{B}} d_H(B_{ij}, B_{kp})$$
$$= \frac{1}{2}\Big(\sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\j\neq p}}^{r-1} d_H(B_{ij}, B_{ip}) + \sum_{\substack{i,k=0\\i\neq k}}^{n-1}\sum_{j=0}^{r-1} d_H(B_{ij}, B_{kj})$$
$$+ \sum_{\substack{i,k=0\\i\neq k}}^{n-1}\sum_{\substack{j,p=0\\j\neq p}}^{r-1} d_H(B_{ij}, B_{kp}) + \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} d_H(B_{ij}, B_{ij})\Big).$$

$$W(H) = \frac{1}{2}(A_1 + A_2 + A_3 + A_4), \tag{1}$$

where A_1, A_2, A_3 and A_4 are the sums of the terms of the above expression, in order.

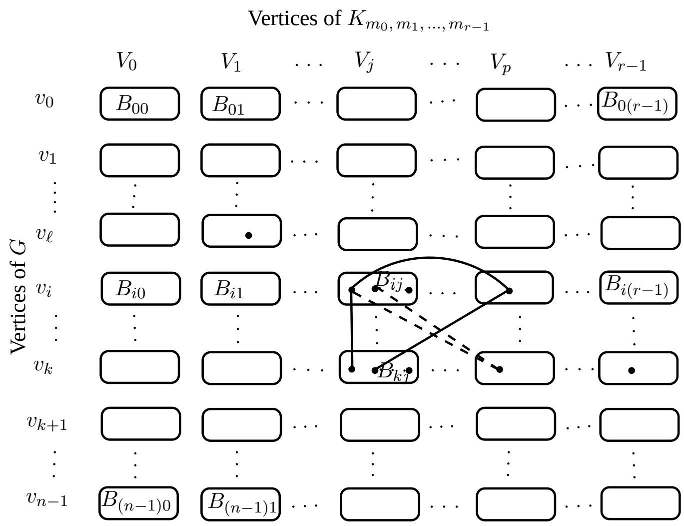

Figure 2 If $v_iv_k \in E(G)$, then shortest paths of length 1 and 2 from B_{ij} to B_{kj} is shown in solid edges. The broken edges give a shortest path of length 2 from a vertex of B_{ij} to another vertex of B_{ij}.

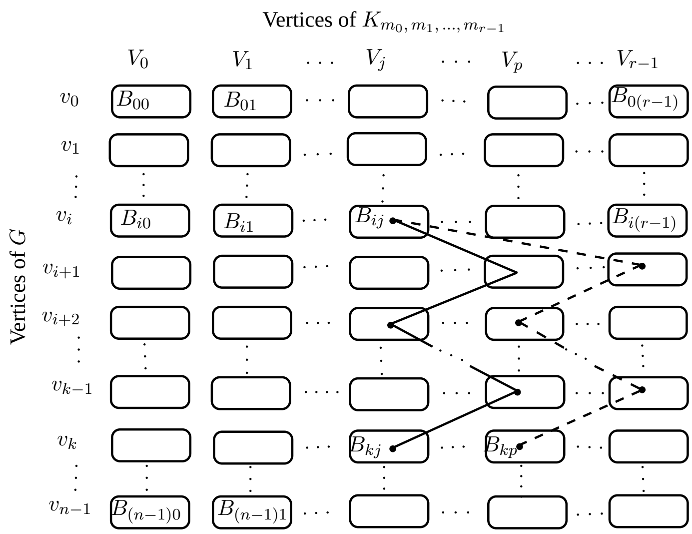

Figure 3. If (v_i, v_k) is a shortest path of length ℓ in G, then a shortest path from any vertex of B_{ij} to any vertex of B_{kj} (resp. any vertex of B_{ij} to any vertex of B_{kp}, $p \neq j$) of length ℓ is shown in solid edges (resp. broken edges).

We shall obtain A_1 to A_4 of (1), separately.

$$A_1 = \sum_{i=0}^{n-1} \sum_{\substack{j,p=0 \\ j\neq p}}^{r-1} d_H(B_{ij}, B_{ip})$$
$$= \sum_{i=0}^{n-1} \Big(\sum_{\substack{p=0 \\ p\neq 0}}^{r-1} d_H(B_{i0}, B_{ip}) + \sum_{\substack{p=0 \\ p\neq 1}}^{r-1} d_H(B_{i1}, B_{ip}) + \cdots + \sum_{\substack{p=0 \\ p\neq r-1}}^{r-1} d_H(B_{i(r-1)}, B_{ip}) \Big)$$
$$= \sum_{i=0}^{n-1} \Big(\sum_{\substack{p=0 \\ p\neq 0}}^{r-1} m_0 m_p + \sum_{\substack{p=0 \\ p\neq 1}}^{r-1} m_1 m_p + \cdots + \sum_{\substack{p=0 \\ p\neq r-1}}^{r-1} m_{r-1} m_p \Big),$$

since $|B_{ip}| = m_p$ and also see Lemma 4, therefore

$$A_1 = n(\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p). \tag{2}$$

$$\begin{aligned}
A_2 &= \sum_{j=0}^{r-1}\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} d_H(B_{ij}, B_{kj})\\
&= \sum_{j=0}^{r-1}(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} d_H(B_{ij}, B_{kj}) + \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} d_H(B_{ij}, B_{kj}))\\
&= \sum_{j=0}^{r-1}(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} m_j(2m_j-1) + \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} m_j^2 d_G(v_i, v_k)),
\end{aligned}$$

by Lemma 4,

$$A_2 = \sum_{j=0}^{r-1}(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} m_j((1+d_G(v_i,v_k))m_j-1) + \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} m_j^2 d_G(v_i, v_k)),$$

where $1 + d_G(v_i, v_k) = 2$ if $v_iv_k \in E(G)$.

$$\begin{aligned}
A_2 &= (\sum_{j=0}^{r-1} m_j^2)(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} d_G(v_i, v_k)) + \sum_{j=0}^{r-1}\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} m_j(m_j-1)\\
&+(\sum_{j=0}^{r-1} m_j^2)(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} d_G(v_i, v_k)) = (\sum_{j=0}^{r-1} m_j^2)(\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} d_G(v_i, v_k))\\
&+(\sum_{j=0}^{r-1} m_j(m_j-1))(2|E(G)|),
\end{aligned}$$

by combining the first and last sums of the above line. By the definition of Wiener index,

$$A_2 = (\sum_{j=0}^{r-1} m_j^2)(2W(G)) + (\sum_{j=0}^{r-1} m_j(m_j-1))(2|E(G)|). \tag{3}$$

$$\begin{aligned}
A_3 &= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0,\\ j\neq p}}^{r-1} d_H(B_{ij}, B_{kp}) = \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}(\sum_{\substack{p=0\\ p\neq 0}}^{r-1} d_H(B_{i0}, B_{kp})\\
&+\sum_{\substack{p=0\\ p\neq 1}}^{r-1} d_H(B_{i1}, B_{kp}) + \cdots + \sum_{\substack{p=0\\ p\neq r-1}}^{r-1} d_H(B_{i(r-1)}, B_{kp}))\\
&= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}(\sum_{\substack{p=0\\ p\neq 0}}^{r-1} m_0 m_p d_G(v_i, v_k) + \sum_{\substack{p=0\\ p\neq 1}}^{r-1} m_1 m_p d_G(v_i, v_k) + \cdots\\
&+\sum_{\substack{p=0\\ p\neq r-1}}^{r-1} m_{r-1} m_p d_G(v_i, v_k)),
\end{aligned}$$

by Lemma 4,

$$A_3 = \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}(\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p) d_G(v_i, v_k).$$

By the definition of Wiener index,

$$A_3 = (\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p)(2W(G)). \tag{4}$$

$$A_4 = \sum_{i=0}^{n-1}(\sum_{j=0}^{r-1} d_H(B_{ij}, B_{ij})) = \sum_{i=0}^{n-1}(\sum_{j=0}^{r-1} 2m_j(m_j-1)),$$

by Lemma 4, and hence

$$A_4 = n(\sum_{j=0}^{r-1} 2m_j(m_j-1)). \tag{5}$$

Using (2), (3), (4) and (5) in (1), we have

$$\begin{aligned} W(H) =& \tfrac{1}{2}(n(\sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p)+(\sum_{j=0}^{r-1} m_j^2)(2W(G))+(\sum_{j=0}^{r-1} m_j(m_j-1))(2|E(G)|) \\ &+(\sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p)(2W(G)) + n(\sum_{j=0}^{r-1} 2m_j(m_j-1))) \\ =& (\sum_{j=0}^{r-1} m_j^2 + \sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p)W(G) + \tfrac{n}{2}(\sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p + \sum_{j=0}^{r-1} 2m_j(m_j-1)) \\ &+(\sum_{j=0}^{r-1} m_j(m_j-1))\,|E(G)|, \end{aligned}$$

by combining terms involving $W(G)$ and the first and last terms,

$$W(H) = \big((n_0^2-2q)+2q)W(G) + \tfrac{n}{2}(2q+2(n_0^2-2q-n_0)\big) + (n_0^2-2q-n_0)|E(G)|,$$

since $\sum_{j=0}^{r-1} m_j^2 = n_0^2 - 2q$, $\sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p = 2q$ and $\sum_{j=0}^{r-1} m_j(m_j-1) = n_0^2-2q-n_0$,

$W(H) = n_0^2\, W(G) + n(n_0^2 - q - n_0) + (n_0^2 - 2q - n_0)|E(G)|$. ■

If $m_i = s,\ 0 \leq i \leq r-1$, in Theorem 1, we have the following

Corollary 5. *Let G be a connected graph with n vertices and m edges. Then $W(G \boxtimes K_{r(s)}) = r^2s^2W(G) + \frac{nrs}{2}(rs+s-2) + rs(s-1)m$.*

As $K_r = K_{r(1)}$, the above corollary gives the following

Corollary 6. *Let G be a connected graph with n vertices and m edges. Then $W(G \boxtimes K_r) = r^2W(G) + \frac{nr}{2}(r-1)$.*

It can be easily verified that $W(K_n) = \frac{n(n-1)}{2}$, $W(P_n) = \frac{n(n^2-1)}{6}$ and

$$W(C_n) = \begin{cases} \frac{n^3}{8}, & n \text{ is even}, \\ \frac{n(n^2-1)}{8}, & n \text{ is odd}. \end{cases}$$

By [10], $W(Q_n) = n2^{2(n-1)}$.

Now using Theorem 1 and Corollaries 5, 6 and the Wiener indices of K_n, P_n, C_n and Q_n we obtain the exact Wiener indices of the following graphs.

1. $W(K_n \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = \frac{n}{2}(2nn_0^2 - 2nq - nn_0 - n_0)$, where $n_0 = \sum_{i=0}^{r-1} m_i$ and q is the number of edges of $K_{m_0,m_1,\ldots,m_{r-1}}$.

2. $W(K_n \boxtimes K_{r(s)}) = \frac{nrs}{2}(nrs + ns - n - 1)$.

3. $W(P_n \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = \frac{n}{6}(n^2n_0^2 + 11n_0^2 - 18q - 12n_0) - (n_0^2 - 2q - n_0)$.

4. $W(P_n \boxtimes K_{r(s)}) = \frac{rs}{6}(n^3rs + 2nrs + 9ns - 12n - 6s + 6)$.

5. $W(P_n \boxtimes K_r) = \frac{nr}{6}(n^2r + 3r - 4)$.

6. $W(C_n \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = \begin{cases} \frac{n}{8}(n^2n_0^2 + 16n_0^2 - 24q - 16n_0), & \text{if } n \text{ is even,} \\ \frac{n}{8}(n^2n_0^2 + 15n_0^2 - 24q - 16n_0), & \text{if } n \text{ is odd.} \end{cases}$

7. $W(C_n \boxtimes K_{r(s)}) = \begin{cases} \frac{nrs}{8}(n^2rs + 4rs + 12s - 16), & \text{if } n \text{ is even,} \\ \frac{nrs}{8}(n^2rs + 3rs + 12s - 16), & \text{if } n \text{ is odd.} \end{cases}$

8. $W(C_n \boxtimes K_r) = \begin{cases} \frac{nr}{8}(n^2r + 4r - 4), & \text{if } n \text{ is even,} \\ \frac{nr}{8}(n^2r + 3r - 4), & \text{if } n \text{ is odd.} \end{cases}$

9. $W(Q_n \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = 2^{n-1}(nn_0^2 2^{n-1} + (n+2)n_0(n_0-1) - 2(n+1)q)$.

10. $W(Q_n \boxtimes K_{r(s)}) = 2^{n-1}((n2^{n-1}+1)r^2s^2 + (n+1)rs^2 - (n+2)rs)$.

11. $W(Q_n \boxtimes K_r) = 2^{n-1}r(n2^{n-1} + r - 1)$.

3. Hyper-Wiener Index of $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$

In this section, we obtain the hyper-Wiener index of the graph $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$. First we give a notation used in the proof of Theorem 2.

For two subsets $S, T \subset V(G)$, we define $d_G^2(S,T) = \sum_{s\in S, t\in T} d_G^2(s,t)$, where $d_G^2(s,t) = (d_G(s,t))^2$.

The proof of the following lemma follows easily from the structure of $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$. The lemma is used in the proof of the main theorem of this section.

Lemma 7. *Let G be a connected graph and let B_{ij}, $B_{kp} \in \mathscr{B}$ of the graph $H = G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$, where $r \geq 2$.*

(i) *If $v_iv_k \in E(G)$, then* $d_H^2(B_{ij}, B_{kp}) = \begin{cases} m_jm_p, & \text{if } j \neq p, \\ m_j(4m_j - 3), & \text{if } j = p. \end{cases}$

(ii) *If $v_iv_k \notin E(G)$, then $d_H^2(B_{ij}, B_{kp}) = m_jm_pd_G^2(v_i, v_k)$.*

(iii) $d_H^2(B_{ij}, B_{ip}) = \begin{cases} m_jm_p, & if\ j \neq p, \\ 4m_j(m_j - 1), & if\ j = p. \end{cases}$

Proof of Theorem 2. Let $H = G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$. By the definition of hyper-Wiener index,

$$\begin{aligned} WW(H) &= \tfrac{1}{2}W(G) + \tfrac{1}{4}\textstyle\sum_{B_{ij},B_{kp}\in\mathscr{B}} d_H^2(B_{ij}, B_{kp}) \\ &= \tfrac{1}{2}W(G) + \tfrac{1}{4}\Big(\textstyle\sum_{i=0}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} d_H^2(B_{ij}, B_{ip}) + \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{j=0}^{r-1} d_H^2(B_{ij}, B_{kj}) \\ &\quad + \textstyle\sum_{\substack{i,k=0\\ i\neq k}}^{n-1}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1} d_H^2(B_{ij}, B_{kp}) + \sum_{i=0}^{n-1}\sum_{j=0}^{r-1} d_H^2(B_{ij}, B_{ij})\Big). \end{aligned}$$

Denote by A_5, A_6, A_7 and A_8 the sums of the terms of the above expression, in order, and hence

$$WW(H) = \frac{1}{2}W(G) + \frac{1}{4}(A_5 + A_6 + A_7 + A_8). \tag{6}$$

We shall obtain A_5 to A_8, in (6), separately.

$$A_5 = \sum_{i=0}^{n-1} \sum_{\substack{j,p=0\\ j\neq p}}^{r-1} d_H^2(B_{ij}, B_{ip}) = \sum_{i=0}^{n-1}(\sum_{\substack{p=0\\ p\neq 0}}^{r-1} d_H^2(B_{i0}, B_{ip})$$
$$+ \sum_{\substack{p=0\\ p\neq 1}}^{r-1} d_H^2(B_{i1}, B_{ip}) + \cdots + \sum_{\substack{p=0\\ p\neq r-1}}^{r-1} d_H^2(B_{i(r-1)}, B_{ip}))$$
$$= \sum_{i=0}^{n-1}(\sum_{\substack{p=0\\ p\neq 0}}^{r-1} m_0 m_p + \sum_{\substack{p=0\\ p\neq 1}}^{r-1} m_1 m_p + \ldots + \sum_{\substack{p=0\\ p\neq r-1}}^{r-1} m_{r-1} m_p),$$

by Lemma 7, therefore

$$A_5 = n(\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p). \tag{7}$$

$$A_6 = \sum_{j=0}^{r-1} \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} d_H^2(B_{ij}, B_{kj}) = \sum_{j=0}^{r-1}(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} d_H^2(B_{ij}, B_{kj})$$
$$+ \sum_{\substack{i,k=0\\ i\neq k, v_iv_k\notin E(G)}}^{n-1} d_H^2(B_{ij}, B_{kj})) = \sum_{j=0}^{r-1}(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} m_j(4m_j - 3)$$
$$+ \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} m_j^2 d_G^2(v_i, v_k)),$$

by Lemma 7,

$$A_6 = \sum_{j=0}^{r-1}(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} m_j((3 + d_G^2(v_i, v_k))m_j - 3) + \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} m_j^2 d_G^2(v_i, v_k)),$$

where $3 + d_G^2(v_i, v_k) = 4$ if $v_iv_k \in E(G)$.

$$A_6 = \sum_{j=0}^{r-1}(\sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} (m_j^2 d_G^2(v_i, v_k) + 3m_j(m_j - 1)) + \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\notin E(G)}}^{n-1} m_j^2 d_G^2(v_i, v_k))$$
$$= \sum_{j=0}^{r-1}(\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} m_j^2 d_G^2(v_i, v_k) + \sum_{\substack{i,k=0\\ i\neq k\\ v_iv_k\in E(G)}}^{n-1} 3m_j(m_j - 1)),$$

by combining the first and last sum of the previous line,

$$A_6 = (\sum_{j=0}^{r-1} m_j^2)(\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} d_G^2(v_i, v_k)) + (\sum_{j=0}^{r-1} 3m_j(m_j - 1))(2|E(G)|). \tag{8}$$

$$A_7 = \sum_{\substack{i,k=0\\ i\neq k}}^{n-1} \sum_{\substack{j,p=0,\\ j\neq p}}^{r-1} d_H^2(B_{ij}, B_{kp}) = \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}(\sum_{\substack{p=0\\ p\neq 0}}^{r-1} d_H^2(B_{i0}, B_{kp})$$
$$+ \sum_{\substack{p=0\\ p\neq 1}}^{r-1} d_H^2(B_{i1}, B_{kp}) + \cdots + \sum_{\substack{p=0\\ p\neq r-1}}^{r-1} d_H^2(B_{i(r-1)}, B_{kp}))$$
$$= \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}(\sum_{\substack{p=0\\ p\neq 0}}^{r-1} m_0 m_p d_G^2(v_i, v_k) + \sum_{\substack{p=0\\ p\neq 1}}^{r-1} m_1 m_p d_G^2(v_i, v_k) + \cdots$$
$$+ \sum_{\substack{p=0\\ p\neq r-1}}^{r-1} m_{r-1} m_p d_G^2(v_i, v_k)),$$

by Lemma 7, and hence

$$A_7 = \sum_{\substack{i,k=0\\ i\neq k}}^{n-1}(\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p) d_G^2(v_i, v_k).$$

$$A_7 = (\sum_{\substack{a,p=0\\ a\neq p}}^{r-1} m_a m_p)(\sum_{\substack{i,k=0\\ i\neq k}}^{n-1} d_G^2(v_i, v_k)). \tag{9}$$

$$(10)\qquad A_8 = \sum_{i=0}^{n-1}(\sum_{j=0}^{r-1} d_H^2(B_{ij}, B_{ij})) = \sum_{i=0}^{n-1}(\sum_{j=0}^{r-1} 4m_j(m_j-1)) = n(\sum_{j=0}^{r-1} 4m_j(m_j-1)).$$

Using Theorem 1, and the equations (7), (8), (9) and (10) in (6), we have

$$\begin{aligned}
&WW(H) = \tfrac{1}{2}((\textstyle\sum_{j=0}^{r-1} m_j^2 + \sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p)W(G) + \tfrac{n}{2}(\sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p\\
&+\textstyle\sum_{j=0}^{r-1} 2m_j(m_j-1)) + (\sum_{j=0}^{r-1} m_j(m_j-1))|E(G)|)\\
&+\tfrac{1}{4}((\textstyle\sum_{j=0}^{r-1} m_j^2)(\sum_{\substack{i,k=0\\i\neq k}}^{n-1} d_G^2(v_i,v_k)) + n(\sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p)\\
&+(\textstyle\sum_{j=0}^{r-1} 3m_j(m_j-1))(2|E(G)|) + (\sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p)(\sum_{\substack{i,k=0\\i\neq k}}^{n-1} d_G^2(v_i,v_k))\\
&+n(\textstyle\sum_{j=0}^{r-1} 4m_j(m_j-1))) = (\sum_{j=0}^{r-1} m_j^2 + \sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p)(\tfrac{1}{2}W(G)\\
&+\tfrac{1}{4}\textstyle\sum_{\substack{i,k=0\\i\neq k}}^{n-1} d_G^2(v_i,v_k)) + \tfrac{n}{4}(\sum_{\substack{a,p=0\\a\neq p}}^{r-1} 2m_a m_p + \sum_{j=0}^{r-1} 6m_j(m_j-1))\\
&+2(\textstyle\sum_{j=0}^{r-1} m_j(m_j-1))|E(G)| = (\sum_{j=0}^{r-1} m_j^2 + \sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p)WW(G)\\
&+\tfrac{n}{2}(\textstyle\sum_{\substack{a,p=0\\a\neq p}}^{r-1} m_a m_p + \sum_{j=0}^{r-1} 3m_j(m_j-1)) + 2(\sum_{j=0}^{r-1} m_j(m_j-1))|E(G)|\\
&= ((n_0^2-2q)+2q)WW(G) + \tfrac{n}{2}(2q+3(n_0^2-2q-n_0)) + 2(n_0^2-2q-n_0)|E(G)|\\
&= n_0^2\, WW(G) + \tfrac{n}{2}(3n_0^2-4q-3n_0) + 2(n_0^2-2q-n_0)|E(G)|.
\end{aligned}$$

■

If $m_i = s,\ 0 \le i \le r-1$, in Theorem 2, we have the following

Corollary 8. *If G is a connected graph with n vertices and m edges, then* $WW(G \boxtimes K_{r(s)}) = r^2s^2WW(G) + \frac{nrs}{2}(rs+2s-3) + 2rs(s-1)m$.

As $K_r = K_{r(1)}$, the above corollary gives the following

Corollary 9. *If G is a connected graph with n vertices and m edges, then* $WW(G \boxtimes K_r) = r^2WW(G) + \frac{nr}{2}(r-1)$.

It can be easily verified, see also [8], that $WW(K_n) = \frac{n(n-1)}{2}$, $WW(P_n) = \frac{n^4+2n^3-n^2-2n}{24}$ and

$$WW(C_n) = \begin{cases} \frac{n^2(n+1)(n+2)}{48}, & n \text{ is even},\\ \frac{n(n^2-1)(n+3)}{48}, & n \text{ is odd}. \end{cases}$$

By [10], $W(Q_n) = n2^{2(n-1)}$ and hence $WW(Q_n) = n(n+3)2^{2n-4}$.

Now using Theorem 2 and Corollaries 8, 9 and the hyper-Wiener indices of K_n, P_n, C_n and Q_n, we obtain the exact hyper-Wiener indices of the following graphs.

1. $WW(K_n \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = \frac{n}{2}(3nn_0^2 - 4nq - 2nn_0 - n_0)$, where $n_0 = \sum_{i=0}^{r-1} m_i$ and q is the number of edges of $K_{m_0,m_1,\ldots,m_{r-1}}$.

2. $WW(K_n \boxtimes K_{r(s)}) = \frac{nrs}{2}(nrs + 2ns - 2n - 1)$.

3. $WW(P_n \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = \frac{n}{24}(n^3n_0^2 + 2n^2n_0^2 - nn_0^2 + 82n_0^2 - 84n_0 - 144q)$ $-2(n_0^2 - 2q - n_0)$.

4. $WW(P_n \boxtimes K_{r(s)}) = \frac{nrs}{24}(n^3rs + 2n^2rs - nrs + 10rs + 72s - 84) - 2rs(s-1)$.

5. $WW(P_n \boxtimes K_r) = \frac{nr}{24}(n^3r + 2n^2r - nr + 10r - 12)$.

6. $WW(C_n \boxtimes K_{m_0,m_1,\ldots,m_{r-1}})$
$$= \begin{cases} \frac{n}{48}(n(n+1)(n+2)n_0^2 + 168n_0^2 - 288q - 168n_0), & \text{if } n \text{ is even,} \\ \frac{n}{48}((n^2-1)(n+3)n_0^2 + 168n_0^2 - 288q - 168n_0), & \text{if } n \text{ is odd.} \end{cases}$$

7. $WW(C_n \boxtimes K_{r(s)}) = \begin{cases} \frac{nrs}{48}(nrs(n+1)(n+2) + 24rs + 144s - 168), & \text{if } n \text{ is even,} \\ \frac{nrs}{48}((n^2-1)(n+3)rs + 24rs + 144s - 168), & \text{if } n \text{ is odd.} \end{cases}$

8. $WW(C_n \boxtimes K_r) = \begin{cases} \frac{nr}{48}(n(n+1)(n+2)r + 24r - 24), & \text{if } n \text{ is even,} \\ \frac{nr}{48}((n^2-1)(n+3)r + 24r - 24), & \text{if } n \text{ is odd.} \end{cases}$

9. $WW(Q_n \boxtimes K_{m_0,m_1,\ldots,m_{r-1}})$
$= n_0^2n(n+3)2^{2n-4} + 2^{n-1}(3n_0^2 - 4q - 3n_0 + 2nn_0^2 - 4nq - 2nn_0)$.

10. $WW(Q_n \boxtimes K_{r(s)}) = n(n+3)2^{2n-4}r^2s^2 + 2^{n-1}(rs(rs+2s-3) + 2nrs(s-1))$.

11. $WW(Q_n \boxtimes K_r) = n(n+3)2^{2n-4}r^2 + 2^{n-1}r(r-1)$.

4. Lower Bounds for Wiener and Hyper-Wiener Indices of the Strong Product of Graphs

In this section, we establish lower bounds for Wiener and hyper-Wiener indices of $G \boxtimes G'$.
As the proof of the following lemma is trivial, we just quote the statement.

Lemma 10. *$W(K_{m_0,m_1,\ldots,m_{r-1}}) = n_0^2 - n_0 - q$ and $WW(K_{m_0,m_1,\ldots,m_{r-1}}) = 2n_0^2 - 2n_0 - 3q$, where $\sum_{i=0}^{r-1} m_i = n_0$ and q is the number of edges of $K_{m_0,m_1,\ldots,m_{r-1}}$.*

The following lemma follows as $G \subseteq K_{m_0,m_1,\ldots,m_{r-1}}$.

Lemma 11. *Let G be a connected graph on n vertices with chromatic number $\chi(G) = r \geq 2$. If $\mathscr{C}$ is the decreasing $\chi(G)$-coloring of G with sizes of the colour classes $m_0, m_1, \ldots, m_{r-1}$, then $W(G) \geq W(K_{m_0,m_1,\ldots,m_{r-1}}) = n_0^2 - n_0 - q$ and $WW(G) \geq WW(K_{m_0,m_1,\ldots,m_{r-1}}) = 2n_0^2 - 2n_0 - 3q$, where $\sum_{i=0}^{r-1} m_i = n_0$, where n_0 is the number of vertices of G and q is the number of edges of $K_{m_0,m_1,\ldots,m_{r-1}}$.*

***Proof of Theorem* 3.** By Lemma 11, $W(G') \geq W(K_{m_0,m_1,\ldots,m_{r-1}})$. As G' is a subgraph of $K_{m_0,m_1,\ldots,m_{r-1}}$, we have $W(G \boxtimes G') \geq W(G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}})$, since $d_{G\boxtimes G'}((x_1,y_1),(x_2,y_2)) \geq d_{G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}}((x_1,y_1),(x_2,y_2))$ for any pair of vertices (x_1,y_1) and (x_2,y_2) of $G\boxtimes G'$. Thus, $W(G\boxtimes G') \geq W(G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = n_0^2\ W(G) + n(n_0^2 - q - n_0) + (n_0^2 - 2q - n_0)m$, by Theorem 1.

Similarly, $d^2_{G\boxtimes G'}((x_1,y_1),(x_2,y_2)) \geq d^2_{G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}}((x_1,y_1),(x_2,y_2))$ for any pair of vertices (x_1,y_1) and (x_2,y_2) of $G\boxtimes G'$. Consequently, $WW(G\boxtimes G') \geq WW(G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = n_0^2\ WW(G) + \frac{n}{2}(3n_0^2 - 4q - 3n_0) + 2(n_0^2 - 2q - n_0)m$, by Theorem 2. ■

5. VERTEX PI INDEX OF $G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$

In this section, we compute the vertex PI index of $H = G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$.

First we introduce some notations for our convenience. Let G be a connected graph and $E' \subseteq E(G)$; let $N_G(E') = \sum_{e\in E'} N_G(e)$, where $N_G(e)$ is the number of vertices equidistant from the edge e in G. For $e = v_iv_k \in E(G)$ let $E_{ik}^{jp} = E(B_{ij}, B_{kp})$, where ${B_{ij}}'^s$ are as defined above. We denote $N_H(E_{ik}^{jp}) = \sum_{e'\in E_{ik}^{jp}} N_H(e')$.

For $e = v_iv_k \in E(G)$, we define four sets, namely, $S_1(e)$, $S_2(e)$, $S_3(v_i)$ and $S_3(v_k)$ as follows: let $S_1(e) = \{x \in V(G) \mid d(x,v_i) = 1 = d(x,v_k)\}$, that is, the set of vertices which are lying on a triangle containing the edge e; let $|S_1(e)| = s_1(e)$. Let $S_2(e) = \{x \in V(G) \mid d(x,v_i) = d(x,v_k) = k > 1\}$, that is, the set of vertices which are at distance $k > 1$ from both the ends v_i and v_k of e; let $|S_2(e)| = s_2(e)$. Clearly, $N_G(e) = s_1(e) + s_2(e)$. Let $S_3(v_i) = \{x \in N(v_i) \mid x \text{ is not an isolated vertex in } \langle N(v_i)\rangle_G\}$; let $|S_3(v_i)| = s_3(v_i)$. Similarly, $S_3(v_k) = \{x \in N(v_k) \mid x \text{ is not an isolated vertex in } \langle N(v_k)\rangle_G\}$; let $|S_3(v_k)| = s_3(v_k)$. Let $S(v_i) = S_3(v_i) - S_1(e) - \{v_k\}$ and $S(v_k) = S_3(v_k) - S_1(e) - \{v_i\}$, where $e = v_iv_k$.

Let $T(e) \subset V(G)$ be set of vertices which are equidistant from the edge $e \in E(G)$. For $e = v_iv_k \in E(G)$ and $a \in T(e)$ we define $N_H^a(e')$ as the number of equidistant vertices of $e' \in E(X_i,X_k)$ lying in X_a $(\subseteq V(H))$. Consequently, $N_H^a(E(X_i,X_k)) = \sum_{e'\in E(X_i,X_k)} N_H^a(e')$.

Now we define $N_H^{T(e)}(E(X_i,X_k)) = \sum_{a\in T(e)} N_H^a(E(X_i,X_k))$. For $e \in E'$, $N_G^{E'}(e)$ denotes the number of equidistant vertices of the edge e in G.

In $G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}$, define, $E_1 = \{(u,v)(x,y) \in E(G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) | ux \in E(G) \textit{ and } v = y\}$, $E_2 = \{(u,v)(x,y) \in E(G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) | ux \in E(G) \textit{and } vy \in E(K_{m_0,m_1,\ldots,m_{r-1}})\}$ and $E_3 = \{(u,v)(x,y) \in E(G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) | u = x \textit{ and } vy \in E(K_{m_0,m_1,\ldots,m_{r-1}})\}$.

Clearly, $E_1 \cup E_2 \cup E_3 = E(G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}})$.

Also, $|E_1| = |E(G)||V(K_{m_0,m_1,\ldots,m_{r-1}})|$, $|E_2| = 2|E(G)||E(K_{m_0,m_1,\ldots,m_{r-1}})|$ and $|E_3| = |V(G)||E(K_{m_0,m_1,\ldots,m_{r-1}})|$.

Theorem 12. *Let G be a connected graph on n vertices, then*
$PI(G\boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = n_0(n_0+2q)PI(G)-(n_0^3+n_0^2-n_0-2n_0q-4q-\sum_{h=0}^{r-1} m_h^3)$ $(\sum_{e\in E(G)}(s_3(v_i)+s_3(v_k)-2s_1(e)-2))-2(2q-n_0^3+2n_0q+\sum_{h=0}^{r-1} m_h^3)(\sum_{e\in E(G)} s_1(e))$ $-(2n_0^2-2n_0+16n_0q-6n_0^3+8q+6\sum_{h=0}^{r-1} m_h^3)|E(G)|+n(n_0^3-2n_0q-\sum_{h=0}^{r-1} m_h^3)$.

Proof. An edge $e = v_iv_k \in E(G)$ contributes edges for E_1, E_2 and $E_3 \subseteq E(\langle X_i \cup X_k\rangle)$ in H. We compute $N_H^{E_1}(e')$, for an $e' \in E_1 \cap E(\langle X_i \cup X_k\rangle_H)$, $N_H^{E_2}(e')$, for an $e' \in E_2 \cap E(\langle X_i \cup X_k\rangle_H)$ and $N_H^{E_1}(e')$, for an $e' \in E_3 \cap E(\langle X_i \cup X_k\rangle_H)$, separately, in the items (1), (2) and (3) below.

Case 1. If $e' \in E_1 \cap E(\langle X_i \cup X_k\rangle_H)$, then $e' \in E(B_{ij}, B_{kj})$ for some j.

Case 1(i). If $v_\ell \in S_1(e)$, then every vertex in X_ℓ is an equidistant vertex (of distance one or two) from $e' \in E(\langle X_i \cup X_k\rangle_H)$. Hence, $N_H^{S_1(e)}(e') = (\sum_{h=0}^{r-1} m_h)|S_1(e)| = (\sum_{h=0}^{r-1} m_h)s_1(e)$.

As there are m_j edges of E_1 between B_{ij} and B_{kj},

$$N_H^{S_1(e)}(E_{ik}^{jj}) = \sum\nolimits_{e'\in E_{ik}^{jj}} N_H^{S_1(e)}(e') = m_j(\sum\nolimits_{h=0}^{r-1} m_h)s_1(e). \tag{11}$$

Case 1(ii). If $v_\ell \in S_2(e)$, then every vertex of X_ℓ is an equidistant vertex, of distance $d_G(v_i, v_\ell) = d_G(v_k, v_\ell)$, from $e' \in E(\langle X_i \cup X_k\rangle_H)$. Hence,

$$N_H^{S_2(e)}(e') = (\sum\nolimits_{h=0}^{r-1} m_h)|S_2(e)| = (\sum\nolimits_{h=0}^{r-1} m_h)s_2(e).$$

As there are m_j edges of E_1 in $E(B_{ij}, B_{kj})$, we have

$$N_H^{S_2(e)}(E_{ik}^{jj}) = \sum\nolimits_{e'\in E_{ik}^{jj}} N_H^{S_2(e)}(e') = m_j(\sum\nolimits_{h=0}^{r-1} m_h)s_2(e). \tag{12}$$

Case 1(iii). Let $v_\ell \in S(v_i) \cup S(v_k)$. Without loss of generality, we assume that $v_\ell \in S(v_i)$, the case $v_\ell \in S(v_k)$ is similar. Let $e' = x_{ij}^t x_{kj}^t \in E(B_{ij}, B_{kj})$ for some j, where x_{ij}^t and x_{kj}^t are the t^{th} vertices in B_{ij} and B_{kj}, respectively; the ends of e' are equidistant (of distance two) from all the vertices of $B_{\ell j} - \{x_{\ell j}^t\}$ but not all other vertex of X_ℓ. Consequently, we have $N_H^{\{S(v_i)\cup S(v_k)\}(e)}(e') = (m_j-1)|S(v_i)\cup S(v_k)| = (m_j-1)(|S_3(v_i)|+|S_3(v_k)|-2|S_1(e)|-2) = (m_j-1)(s_3(v_i)+s_3(v_k)-2s_1(e)-2)$.

As there are m_j edges in $E(B_{ij}, B_{kj})$, we have

$$\begin{aligned} N_H^{\{S(v_i)\cup S(v_k)\}(e)}(E_{ik}^{jj}) &= \sum\nolimits_{e'\in E_{ik}^{jj}} N_H^{\{S(v_i)\cup S(v_k)\}(e)}(e') \\ &= m_j(m_j-1)(s_3(v_i)+s_3(v_k)-2s_1(e)-2). \end{aligned} \tag{13}$$

Case 1(iv). We compute $N_H^{\{v_i,v_k\}(e)}(e')$, where $e' = x_{ij}^t x_{kj}^t \in E(B_{ij}, B_{kj})$ for some j. All the vertices of $X_i \cup X_k - \{x_{ij}^t, x_{kj}^t\}$ are equidistant, of distance one or two, from the ends of e'. Hence $N_H^{\{v_i,v_k\}(e)}(e') = |X_i \cup X_k| - 2 = 2(\sum_{h=0}^{r-1} m_h) - 2$.

As there are m_j edges of E_1 in $E(B_{ij}, B_{kj})$, we have

$$N_H^{\{v_i,v_k\}(e)}(E_{ik}^{jj}) = \sum_{e' \in E_{ik}^{jj}} N_H^{\{v_i,v_k\}(e)}(e') = m_j(2(\sum_{h=0}^{r-1} m_h) - 2). \tag{14}$$

Adding (11), (12), (13) and (14), we get

$N_H^{E_1}(E_{ik}^{jj}) = N_H^{S_1(e)}(E_{ik}^{jj}) + N_H^{S_2(e)}(E_{ik}^{jj}) + N_H^{\{S(v_i)\cup S(v_k)\}(e)}(E_{ik}^{jj}) + N_H^{\{v_i,v_k\}(e)}(E_{ik}^{jj})$
$= m_j(\sum_{h=0}^{r-1} m_h)s_1(e) + m_j(\sum_{h=0}^{r-1} m_h)s_2(e) + m_j(m_j - 1)(s_3(v_i) + s_3(v_k)$
$-2s_1(e) - 2) + m_j(2(\sum_{h=0}^{r-1} m_h) - 2) = m_j((\sum_{h=0}^{r-1} m_h)N_G(e) + (m_j - 1)(s_3(v_i)$
$+s_3(v_k) - 2s_1(e) - 2) + 2(\sum_{h=0}^{r-1} m_h) - 2)$, since $N_G(e) = s_1(e) + s_2(e)$.

In particular, if $e' \in E_{ik}^{jj}$, then

$$\begin{aligned} N_H^{E_1}(e') &= (\sum_{h=0}^{r-1} m_h)N_G(e) + (m_j - 1)(s_3(v_i) + s_3(v_k) \\ &- 2s_1(e) - 2) + 2(\sum_{h=0}^{r-1} m_h) - 2, \end{aligned} \tag{15}$$

as there are m_j edges in E_{ik}^{jj}.

Case 2. If $e' \in E_2 \cap E(\langle X_i, X_K\rangle_H)$, then $e' = x_{ij}^t x_{kp}^m \in E(B_{ij}, B_{kp})$ $j \neq p$, for some t and m.

Case 2(i). If $v_\ell \in S_1(e)$, then every vertex in $\cup_{a\neq j,p} B_{\ell a} (\subseteq X_\ell)$ is an equidistant (of distance one) vertex from e'; further, the two vertices $x_{\ell j}^t$ and $x_{\ell p}^t$ are also equidistant (of distance one) from e' in H. Consequently, $N_H^{S_1(e)}(e') = ((\sum_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2)|S_1(e)| = ((\sum_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2)s_1(e)$.

As there are $m_j m_p$ edges of E_2 in $E(B_{ij}, B_{kp})$, $j \neq p$,

$$N_H^{S_1(e)}(E_{ik}^{jp}) = \sum_{e' \in E_{ik}^{jp}} N_H^{S_1(e)}(e') = m_j m_p((\sum_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2)s_1(e). \tag{16}$$

Case 2(ii). If $v_\ell \in S_2(e)$, then every vertex in X_ℓ is an equidistant, of distance $d_G(v_i, v_\ell) = d_G(v_k, v_\ell)$, vertex from e'. Hence,

$$N_H^{S_2(e)}(e') = (\sum_{h=0}^{r-1} m_h)|S_2(e)| = (\sum_{h=0}^{r-1} m_h)s_2(e).$$

As there are $m_j m_p$ edges of E_2 in $E(B_{ij}, B_{kp})$, $j \neq p$,

$$N_H^{S_2(e)}(E_{ik}^{jp}) = \sum_{e' \in E_{ik}^{jp}} N_H^{S_2(e)}(e') = m_j m_p(\sum_{h=0}^{r-1} m_h)s_2(e). \tag{17}$$

Case 2(iii). If $v_\ell \in S(v_i)$, then every vertex of $B_{\ell j} - \{x^t_{\ell j}\}$ is an equidistant of distance two from $e' = x^t_{ij}x^m_{kp}$ and no other vertex of X_ℓ is equidistant from e'. Consequently, we have $N_H^{S(v_i)}(e') = (m_j - 1)|S(v_i)| = (m_j - 1)(|S_3(v_i)| - |S_1(e)| - 1) = (m_j - 1)(s_3(v_i) - s_1(e) - 1)$.

As there are $m_j m_p$ edges of E_2 in $E(B_{ij}, B_{kp})$, $j \neq p$,

$$N_H^{S(v_i)}(E^{jp}_{ik}) = \sum\nolimits_{e' \in E^{jp}_{ik}} N_H^{\{S(v_i)\}(e)}(e') = m_j m_p (m_j - 1)(s_3(v_i) - s_1(e) - 1). \quad (18)$$

Case 2(iv). If $v_\ell \in S(v_k)$, then every vertex of $B_{\ell p} - \{x^m_{\ell p}\}$ is equidistant (of distance two) from $e' = x^t_{ij}x^m_{kp}$ and no other vertex of X_ℓ is equidistant from e'. Consequently, we have $N_H^{S(v_k)}(e') = (m_p - 1)|S(v_k)| = (m_p - 1)(|S_3(v_k)| - |S_1(e)| - 1) = (m_p - 1)(s_3(v_k) - s_1(e) - 1)$.

As there are $m_j m_p$ edges in $E(B_{ij}, B_{kp})$, $j \neq p$,

$$N_H^{S(v_k)}(E^{jp}_{ik}) = \sum\nolimits_{e' \in E^{jp}_{ik}} N_H^{\{S(v_k)\}(e)}(e') = m_j m_p (m_p - 1)(s_3(v_k) - s_1(e) - 1). \quad (19)$$

Case 2(v). In $\langle X_i \cup X_k \rangle$, all the vertices of B_{is}, $s \neq j, p$ and B_{ks}, $s \neq j, p$ are equidistant (of distance one) from e'; further, the two vertices x^t_{kj} and x^m_{ip} are also equidistant (of distance one) from e'. Thus $N_H^{\{v_i, v_k\}(e)}(e') = ((2\sum_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2)$. As there are $m_j m_p$ edges of E_2 in $E(B_{ij}, B_{kp})$, $j \neq p$,

$$N_H^{\{v_i, v_k\}(e)}(E^{jp}_{ik}) = \sum\nolimits_{e' \in E^{jp}_{ik}} N_H^{\{v_i, v_k\}(e)}(e') = m_j m_p (2(\sum\nolimits_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2). \quad (20)$$

Adding (16), (17), (18), (19) and (20), we get

$$\begin{aligned} &N_H^{E_2}(E^{jp}_{ik}) = N_H^{S_1(e)}(E^{jp}_{ik}) + N_H^{S_2(e)}(E^{jp}_{ik}) + N_H^{(S(v_i))(e)}(E^{jp}_{ik}) + N_H^{\{S(v_k)\}(e)}(E^{jp}_{ik}) \\ &+N_H^{\{v_i,v_k\}(e)}(E^{jp}_{ik}) = m_j m_p((\sum\nolimits_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2)s_1(e) + m_j m_p(\sum\nolimits_{h=0}^{r-1} m_h)s_2(e) \\ &+m_j m_p(m_j - 1)(s_3(v_i) - s_1(e) - 1) + m_j m_p(m_p - 1)(s_3(v_k) - s_1(e) - 1) \\ &+(m_j m_p)(2(\sum\nolimits_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2) = m_j m_p((\sum\nolimits_{h=0}^{r-1} m_h)N_G(e) + (2 - m_j - m_p)s_1(e) \\ &+(m_j - 1)(s_3(v_i) - s_1(e) - 1) + (m_p - 1)(s_3(v_k) - s_1(e) - 1) \\ &+2(\sum\nolimits_{h=0}^{r-1} m_h - m_j - m_p) + 2), \quad \text{since } s_1(e) + s_2(e) = N_G(e). \end{aligned}$$

In particular, if $e' \in E^{jp}_{ik}$, then

$$N_H^{E_2}(e') = (\sum\nolimits_{h=0}^{r-1} m_h)N_G(e) + (2 - m_j - m_p)s_1(e) + (m_j - 1)(s_3(v_i) - s_1(e) - 1) + (m_p - 1)(s_3(v_k) - s_1(e) - 1) + 2(\sum\nolimits_{h=0}^{r-1} m_h - m_j - m_p) + 2. \quad (21)$$

Case 3. For every vertex $v_i \in V(G)$, and every edge $e' = x_{ij}^t x_{ip}^m \in E_3 \cap E(\langle X_i, X_K\rangle_H)$, $j \neq p$, in $\langle X_i\rangle$, we obtain the equidistant vertices of e' in H.

First we find the equidistant vertices of $e' = x_{ij}^t x_{ip}^m$, in H, where $v_iv_k \in E(G)$.

(i) In X_i, the ends of e' are commonly adjacent to the vertices of $X_i - \{B_{ij}, B_{ip}\}$, that is, the ends of e' are commonly adjacent to $\sum_{\substack{h=0\\h\neq j,p}}^{r-1} m_h$ vertices in X_i.

(ii) In X_k, the ends of e' are commonly adjacent to all the vertices of $X_k - \{B_{kj} \cup B_{kp}\}$; further, the two vertices x_{kj}^t and x_{kp}^m are also equidistant, of distance one, from e', where $v_iv_k \in E(G)$. As $d_G(v_i)$ edges are incident with v_i in G, the ends of e' are commonly adjacent to $d_G(v_i)((\sum_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2)$ vertices in H.

Next we find the equidistant vertices of $e' \in E_3$ in H, where $d_G(v_i, v_k) \geq 2$. The ends of e' are equidistant to all the vertices of X_k; that is, $\sum_{h=0}^{r-1} m_h$ vertices of X_k are equidistant from e'. As there are $n - d_G(v_i) - 1$ vertices with $d_G(v_i, v_j) \geq 2$, $(n - d_G(v_i) - 1)(\sum_{h=0}^{r-1} m_h)$ vertices are equidistant from e'.
Hence

$$\begin{aligned} N_H^{E_3}(E_{ii}^{jp}) &= \sum_{e' \in E_{ii}^{jp}} N_H(e') = m_j m_p((\sum_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + d_G(v_i) \\ &\quad ((\sum_{\substack{h=0\\h\neq j,p}}^{r-1} m_h) + 2) + (n - d_G(v_i) - 1)(\sum_{h=0}^{r-1} m_h)) \\ &= m_j m_p((n\sum_{h=0}^{r-1} m_h - m_j - m_p) + d_G(v_i)(2 - m_j - m_p)). \end{aligned} \tag{22}$$

Now we obtain the vertex PI index of the graph H.

$$\begin{aligned} PI(H) &= \sum_{e' \in E(H)} (|V(H)| - N_H(e')) = \sum_{e' \in E_1} (|V(H)| - N_H(e')) \\ &+ \sum_{e' \in E_2} (|V(H)| - N_H(e')) + \sum_{e' \in E_3} (|V(H)| - N_H(e')) \\ &= A_1 + A_2 + A_3, \end{aligned} \tag{23}$$

where $A_1,\ A_2$ and A_3 are the sums of the above term, in order.

We shall calculate $A_1,\ A_2$ and A_3 separately.

$$\begin{aligned} &A_1 = \sum_{e' \in E_1} (|V(H)| - N_H(e')) = \sum_{e \in E(G)} ((\sum_{j=0}^{r-1} m_j)(V(H) - N_H^{E_1}(e'))) \\ &= \sum_{e \in E(G)} ((\sum_{j=0}^{r-1} m_j)(|V(G)|(\sum_{h=0}^{r-1} m_h) - (\sum_{h=0}^{r-1} m_h) N_G(e)) \\ &-(\sum_{j=0}^{r-1} m_j)((m_j - 1)(s_3(v_i) + s_3(v_k) - 2s_1(e) - 2) + 2(\sum_{h=0}^{r-1} m_h) - 2)) \\ &= \sum_{e \in E(G)} ((\sum_{j=0}^{r-1} m_j)(\sum_{h=0}^{r-1} m_h)(|V(G)| - N_G(e))) \\ &- \sum_{e \in E(G)} (((\sum_{j=0}^{r-1} m_j^2) - (\sum_{j=0}^{r-1} m_j))(s_3(v_i) + s_3(v_k) - 2s_1(e) - 2)) \end{aligned}$$

$$+\sum_{e\in E(G)}(-2(\sum_{j=0}^{r-1}m_j)(\sum_{h=0}^{r-1}m_h)+2(\sum_{j=0}^{r-1}m_j))$$
$$=n_0^2(\sum_{e\in E(G)}(|V(G)|-N_G(e)))-(n_0^2-n_0-2q)$$
$$(\sum_{v_iv_k=e\in E(G)}(s_3(v_i)+s_3(v_k)-2s_1(e)-2))-2n_0(n_0-1)|E(G)|.$$

Hence

$$\begin{aligned} A_1 &= n_0^2PI(G)-(n_0^2-n_0-2q)(\sum_{v_iv_k=e\in E(G)}(s_3(v_i)+s_3(v_k)-2s_1(e)-2)) \\ &- 2n_0(n_0-1)|E(G)|. \quad (24) \end{aligned}$$

$$A_2=\sum_{e'\in E_2}(|V(H)|-N_H(e'))=\sum_{e\in E(G)}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p(|V(H)|-N_H(e'))$$
$$=\sum_{e\in E(G)}\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p(|V(G)|\sum_{h=0}^{r-1}m_h-(\sum_{h=0}^{r-1}m_h)N_G(e)$$
$$-(2-m_j-m_p)s_1(e)-(m_j-1)(s_3(v_i)-s_1(e)-1)-(m_p-1)(s_3(v_k)-s_1(e)-1)$$
$$-2(\sum_{h=0}^{r-1}m_h-m_j-m_p)-2)=(\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p)(\sum_{h=0}^{r-1}m_h)\sum_{e\in E(G)}(|V(G)|$$
$$-N_G(e))-(2\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p-\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_j^2m_p-\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p^2)\sum_{e\in E(G)}s_1(e)$$
$$-(\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_j^2m_p-\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p)\sum_{v_iv_k=e\in E(G)}(s_3(v_i)-s_1(e)-1)$$
$$-(\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p^2-\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p)\sum_{v_iv_k=e\in E(G)}(s_3(v_k)-s_1(e)-1)$$
$$-2\sum_{e\in E(G)}((\sum_{h=0}^{r-1}m_h)\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p-\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_j^2m_p-\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p^2$$
$$+\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p).$$

Thus

$$\begin{aligned} A_2 &= 2qn_0PI(G)-2(2q-n_0^3+2n_0q+\sum_{i=0}^{r-1}m_i^3)\sum_{e\in E(G)}s_1(e) \\ &- (n_0^3-2n_0q-(\sum_{i=0}^{r-1}m_i^3)-2q)\sum_{v_iv_k=e\in E(G)}(s_3(v_i)+s_3(v_k)-2s_1(e)-2) \\ &- 4(3n_0q-n_0^3+(\sum_{i=0}^{r-1}m_i^3)+q)|E(G)|. \quad (25) \end{aligned}$$

$$A_3=\sum_{e'\in E_3}(|V(H)|-N_H(e'))=\sum_{v_i\in V(G)}\frac{1}{2}((\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p)$$
$$(|V(G)|\sum_{i=0}^{r-1}m_i-n\sum_{i=0}^{r-1}m_i+m_j+m_p-(2-m_j-m_p)d_G(v_i)))$$
$$=\sum_{v_i\in V(G)}\frac{1}{2}((\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_j^2m_p)+(\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p^2)-(2(\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p)$$
$$-(\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_j^2m_p)-(\sum_{\substack{j,p=0\\ j\neq p}}^{r-1}m_jm_p^2))d_G(v_i))$$
$$=\sum_{v_i\in V(G)}\frac{1}{2}(2(n_0^3-2n_0q-\sum_{i=0}^{r-1}m_i^3)-(4q-2(n_0^3-2n_0q-\sum_{i=0}^{r-1}m_i^3))d_G(v_i))$$
$$=n(n_0^3-2n_0q-\sum_{i=0}^{r-1}m_i^3)-(2q-(n_0^3-2n_0q-\sum_{i=0}^{r-1}m_i^3))\sum_{v_i\in V(G)}d_G(v_i).$$

$$A_3=n(n_0^3-2n_0q-\sum_{i=0}^{r-1}m_i^3)-2(2q-n_0^3+2n_0q+\sum_{i=0}^{r-1}m_i^3)|E(G)|. \quad (26)$$

Using (24), (25) and (26) in (23), we have

$$PI(H) = A_1 + A_2 + A_3 = n_0^2 PI(G) - (n_0^2 - n_0 - 2q)$$
$$(\sum_{v_iv_k=e\in E(G)}(s_3(v_i) + s_3(v_k) - 2s_1(e) - 2)) - 2n_0(n_0 - 1)|E(G)|$$
$$+2qn_0 PI(G) - 2(2q - n_0^3 + 2n_0q + \sum_{h=0}^{r-1} m_h^3)$$
$$(\sum_{e\in E(G)} s_1(e)) - (n_0^3 - 2n_0q - (\sum_{h=0}^{r-1} m_h^3) - 2q)$$
$$(\sum_{v_iv_k=e\in E(G)}(s_3(v_i)+ s_3(v_k) - 2s_1(e)- 2))-4(3n_0q - n_0^3 + q + \sum_{h=0}^{r-1} m_h)|E(G)|$$
$$+n(n_0^3 - 2n_0q - \sum_{h=0}^{r-1} m_h^3) -2(2q - n_0^3 +2n_0q +\sum_{h=0}^{r-1} m_h^3)|E(G)|$$
$$= n_0(n_0 + 2q)PI(G) - (n_0^3 + n_0^2 - n_0 - 2n_0q - 4q - \sum_{h=0}^{r-1} m_h^3)$$
$$(\sum_{v_iv_k=e\in E(G)}(s_3(v_i) + s_3(v_k) - 2s_1(e) - 2)) - 2(2q - n_0^3 + 2n_0q+\sum_{h=0}^{r-1} m_h^3)$$
$$(\sum_{e\in E(G)} s_1(e)) -(2n_0^2 - 2n_0+ 16n_0q - 6n_0^3 + 8q +6\sum_{h=0}^{r-1} m_h^3)|E(G)|$$
$$+n(n_0^3 - 2n_0q - \sum_{h=0}^{r-1} m_h^3).$$

■

If $m_i = s,\ 0 \le i \le r-1$, in Theorem 12, we have the following

Corollary 13. *If G is a connected graph on n vertices and m edges, then* $PI(G \boxtimes K_{r(s)}) = r^2s^2(rs - s + 1)PI(G) - rs(rs^2 - rs - 2s - s^2 - 1)$ $(\sum_{v_iv_k=e\in E(G)}(s_3(v_i) + s_3(v_k) - 2s_1(e) - 2)) - 2rs^2(r - rs + s - 1)(\sum_{e\in E(G)} s_1(e))$ $- 2rs(r^2s^2 - 4rs^2 + 3rs - 2s + 3s^2 - 1)m + nrs^2(rs - 1)$.

As $K_r = K_{r(1)}$, the following corollary follows from the above corollary.

Corollary 14. *If G is a connected graph with n vertices and m edges, then* $PI(G\boxtimes K_r) = r^3PI(G) + 4r(\sum_{v_iv_k=e\in E(G)}(s_3(v_i) + s_3(v_k) - 2s_1(e) - 2)) - 2r^2(r - 1)m + nr(r - 1)$.

Theorem 12 gives the following corollary, since $s_3(v_i)$, $s_3(v_k)$ and $s_1(e)$ are all zero for a triangle free graph.

Corollary 15. *If G is a triangle free graph on n vertices and m edges, then* $PI(G \boxtimes K_{m_0,m_1,\ldots,m_{r-1}}) = n_0(n_0 + 2q)PI(G) - (2n_0^2 - 2n_0 + 16n_0q - 6n_0^3 + 8q + 6\sum_{h=0}^{r-1} m_h^3)m + n(n_0^3 - 2n_0q - \sum_{h=0}^{r-1} m_h^3)$.

If $m_i = s,\ 0 \le i \le r-1$, in the above corollary, we have the following

Corollary 16. *If G is a triangle free graph on n vertices and m edges, then* $PI(G \boxtimes K_{r(s)}) = rs(r^2s^2 + rs - rs^2)PI(G) - 2rs(r^2s^2 - 4rs^2 + 3rs + 3s^2 - 2s - 1)m + nrs^3(r - 1)$.

If $s = 1$ in the above corollary, we obtain the following

Corollary 17. *If G is a triangle free graph on n vertices and m edges, then* $PI(G \boxtimes K_r) = r^3PI(G) - 2r^2(r - 1)m + nr(r - 1)$.

To obtain the exact vertex PI indices of some graphs given below we just quote the following lemma.

Lemma 18. [8] *For* $n \geq 3$,

(1) $PI(C_n)= \begin{cases} n(n-1), & if\ n\ is\ odd, \\ n^2, & if\ n\ is\ even. \end{cases}$

(2) *For* $n \geq 2$, $PI(P_n) = n(n-1)$.

(3) *For* $n \geq 2$, $PI(K_n) = n(n-1)$.

Now using Corollary 17 and Lemma 18, we obtain the exact vertex PI indices of the graphs $C_n \boxtimes K_r$ and $P_n \boxtimes K_r$.

$$PI(C_n \boxtimes K_r) = \begin{cases} nr(nr^2 - 3r^2 + 3r - 1), & \text{if } n \geq 3 \text{ is odd,} \\ nr(nr^2 - 2r^2 + 3r - 1), & \text{if } n \geq 4 \text{ is even.} \end{cases}$$

$$PI(P_n \boxtimes K_r) = r^2(n-1)(nr - 2r + 2) + nr(r-1).$$

Let Q_n and T_n denote the hypercube of dimension n and a tree with n vertices, respectively. From [14], $PI(Q_n) = 2^{2n-1}n$ and $PI(T_n) = n(n-1)$.

Now using Corollary 17 and $PI(Q_n)$,$PI(T_n)$, we obtain the exact vertex PI indices of the following graphs:

$$PI(Q_n \boxtimes K_r) = n2^{2n-1} - 2^n(nr-1)r(r-1).$$

$$PI(T_n \boxtimes K_r) = (n-1)r^2(nr - 2r + 2) + nr(r-1).$$

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