Discussiones Mathematicae
Graph Theory 32 (2012) 737–747
doi:10.7151/dmgt.1639

ON PROPERTIES OF MAXIMAL 1-PLANAR GRAPHS[1]

Dávid Hudák, Tomáš Madaras

Institute of Mathematics, Faculty of Sciences
University of P. J. Šafárik
Jesenná 5, 041 54 Košice, Slovak Republic

e-mail: david.hudak@student.upjs.sk
tomas.madaras@upjs.sk

and

Yusuke Suzuki

Department of Mathematics, Faculty of Science
Niigata University
8050, Ikarashi 2-no-cho, Nishi-ku, Niigata, 950-2181, Japan

e-mail: y-suzuki@math.sc.niigata-u.ac.jp

Abstract

A graph is called 1-planar if there exists a drawing in the plane so that each edge contains at most one crossing. We study maximal 1-planar graphs from the point of view of properties of their diagrams, local structure and hamiltonicity.

Keywords: 1-planar graph, maximal graph.

2010 Mathematics Subject Classification: 05C10.

1. Introduction

Throughout this paper, we consider connected graphs without loops or multiple edges; we use the standard graph terminology by [1]. The graphs are represented by drawings in the plane, with vertices being distinct points and edges being arcs that join the points corresponding to their endvertices; the arcs are supposed to

[1]This work was partially supported by the Agency of the Slovak Ministry of Education for the Structural Funds of the EU, under project ITMS:26220120007, by VVGS grant PF 50/2011/M and by Science and Technology Assistance Agency under the contract No. APVV-0023-10.

be simple, not containing vertex points in their interiors, and there is no point of the plane which is an interior point of more than two arcs.

A graph G is called *planar* if there exists its drawing $D(G)$ in the plane so that no two edges of $D(G)$ have an internal point (a *crossing*) in common; the drawing $D(G)$ with this property is called a *plane graph*.

There are several different approaches generalizing the concept of planarity. One of them allows, in a drawing of a graph, a given constant number of crossings per edge. Particularly, if there exists a drawing $D(G)$ of a graph G in the plane in such a way that each edge of $D(G)$ contains at most one crossing, then G is called 1-*planar*. These graphs were first introduced by Ringel [10] in connection with the simultaneous vertex/face colouring of plane graphs (note that the graph of adjacency/incidence of vertices and faces of a plane graph is 1-planar).

In this paper, we concentrate on properties of maximal 1-planar graphs. Recall that a graph G from a family $\mathcal{G}$ of graphs is *maximal* if $G+uv \notin \mathcal{G}$ for any two nonadjacent vertices $u, v \in V(G)$. Pach and Tóth ([9], see also [3]) proved that each 1-planar graph on n vertices has at most $4n-8$ edges and this bound is attained for every $n \geq 12$. An n-vertex 1-planar graph is called *optimal* if it has $4n-8$ edges.

Our results demonstrate that optimal 1-planar graphs are, in certain aspects, similar to maximal planar graphs, however, there are several remarkable differences. In Section 2, we prove that optimal 1-planar graphs are hamiltonian; this is in sharp contrast with the family of maximal planar graphs where exist an infinite sequence $\{G_i\}_{i=1}^{\infty}$ of plane triangulations such that, for each i, every cycle of G_i has length at most $c|V(G_i)|^{\frac{\log 2}{\log 3}}$ (see [8]). On the other hand, optimal 1-planar graphs have similar local structure as maximal planar ones – we prove that each large enough optimal 1-planar graph contains a k-vertex path whose weight (i.e. the sum of degrees of its vertices) is at most $8k-1$, and each large enough maximal 3-connected 1-planar graph contains a k-vertex path with degrees of its vertices being bounded above by $10k$; these results are analogous to the results in [7] (each hamiltonian plane graph containing a k-vertex path contains also a k-vertex path of weight at most $6k-1$) and [2] (each 3-connected plane graph that contains a k-vertex path, contains also a k-vertex path with all vertices of degrees at most $5k$). In Section 3, we study bounds on the number of edges of an n-vertex maximal 1-planar graph; we give constructions showing that there exist maximal 1-planar graphs on n vertices which have about cn edges, where $\frac{8}{3} \leq c \leq 4$. Finally, we present several open problems related to this topic.

For the purpose of proving the results of this paper, we use specialized notation of [3] and [5]. Given a 1-planar graph G and its 1-planar drawing $D = D(G)$, the *associated plane graph* $D^{\times}$ is the plane graph obtained from D by turning all crossings to new 4-valent vertices; these new vertices are called *false* vertices, other vertices of $D^{\times}$ are *true*.

2. The Results

First, we prove an auxiliary result on the structure of the associated plane graphs of 1-planar drawings of an optimal 1-planar graph:

Lemma 1. *For any optimal* 1*-planar graph* G*, there exists a* 1*-planar drawing such that its associated plane graph is* 4*-connected.*

Proof. Suppose to the contrary that there exists an optimal 1-planar graph G such that each its 1-planar drawing produces the associated plane graph which has vertex connectivity at most 3. By Lemma 15 of [12], G is 4-connected. Also, by Theorem 11 of [12], there exists an 1-planar drawing D of G which can be obtained from a 3-connected plane quadrangulation by inserting a pair of crossing edges into each of its 4-faces. It follows that $D^\times$ is a plane triangulation different from a 3-cycle; thus, we obtain that $D^\times$ has connectivity 3. Now, if $S = \{x, y, z\}$ is a 3-cut of $D^\times$, then (see Lemma 14 of [12]) x, y, z induce a separating 3-cycle C in $D^\times$. If S consists only of true vertices of $D^\times$, then S is also a cut of G , a contradiction. Hence, S contains a false vertex, say x. By the 1-planarity of G, no two false vertices of $D^\times$ are adjacent, thus x is the unique false vertex on C. Since C is a separating 3-cycle of $D^\times$, it follows that, in the interior of C in $D^\times$, there exists an edge xx'; hence, we obtain that the edges yx and xz of $D^\times$ form a crossed edge yz in D. This is, however, a contradiction with the fact that y and z are already joined in D with an edge of C. ■

It is not known whether analogous result holds for maximal 1-planar graphs; however, for 7-connected 1-planar graphs, we have

Lemma 2. *For any* 7*-connected* 1*-planar graph, there exists a* 1*-planar drawing such that its associated plane graph is* 4*-connected.*

Proof. Suppose to the contrary that there exists a 7-connected 1-planar graph G such that, for each of its 1-planar drawings D, the associated plane graph $D^\times$ has vertex connectivity at most 3. Without loss of generality, D can be chosen to have the minimum possible number of crossings. Then, using Lemma 2.1 of [3] on G, we obtain that $D^\times$ has vertex connectivity exactly 3. Let S be a 3-cut in $D^\times$; then $D^\times \setminus S$ consists of exactly two components D_1, D_2.

Assume that S consists of t true vertices $v_1, \ldots, v_t$ and f false vertices $w_1, \ldots, w_f$ with $t + f \leq 3$. Let $w_i \in S$ be a false vertex; as no two false vertices of $D^\times$ are adjacent, all four neighbours of w_i are true. Let $S'(w_i)$ be the subset of neighbours of w_i defined as follows:

- if two neighbours a, b belong to D_1 and another two neighbours belong to D_2, then $S'(w_i) = \{a, b\}$,

- if three neighbours a, b, c belong to D_1 and the remaining neighbour d belongs to D_2, then $S'(w_i) = \{d\}$.

Let $S' = \bigcup_{i=1}^{f} S'(w_i) \cup \bigcup_{j=1}^{t} \{v_j\}$. From the properties of S, it follows that S' is a vertex cut in G. By the construction of S', we conclude that $|S'| \leq 6$, a contradiction to the 7-connectivity of G. ∎

Theorem 3. *Each optimal* 1*-planar graph is Hamiltonian.*

Proof. Let G be an optimal 1-planar graph (hence, it has more than six vertices). Then, by Lemma 1, there exists a 1-planar drawing D of G such that $D^\times$ is 4-connected; hence, by Tutte's theorem, $D^\times$ contains a Hamilton cycle C. Let x be a false vertex of $D^\times$ and xy, xz be edges of C incident with x. If xy, xz are not incident with a same 3-face of $D^\times$, then yz is an edge of G. Otherwise, by the maximality of G, y and z are also connected by an edge which does not belong to C. Therefore in the edge set of C, each pair of edges incident with a false vertex of $D^\times$ may be replaced by an edge of G in such a way that the resulting set of edges induces a Hamilton cycle in G. ∎

Note that, from the proof above, it follows that, for each optimal 1-planar graph G, there exists a drawing D and a Hamilton cycle of G which is not self-crossing in D. This proof, together with Lemma 2 (applied to 7-connected maximal 1-planar graphs) gives an analogy of Whitney theorem (see [14]) on hamiltonicity of 4-connected plane triangulations:

Corollary 4. *Each maximal* 7*-connected* 1*-planar graph is Hamiltonian.*

We do not know whether there is an analogy of Tutte's theorem on hamiltonicity of 4-connected planar graphs from [13] for the family of 1-planar graphs (though Lemma 2 guarantees the hamiltonicity of associated plane graph of a particular 1-planar drawing of a 7-connected 1-planar graph). The following construction shows that there are nonhamiltonian 4-connected 1-planar graphs: take the plane drawing of the Barnette-Bosák-Lederberg graph (that is, the smallest nonhamiltonian cubic planar graph) and consider its perfect matching (see Figure 1a). Replace each edge of this matching by a double edge; the obtained multigraph is plane and 4-regular. Now, replace each 4-valent vertex with a copy of a 1-planar drawing of the graph $K_{3,4}$ with the respect to the local orientation around the vertex being replaced (see Figure 1b and 1c). We obtain that the resulting graph is 4-connected and 1-planar. To show that it is nonhamiltonian, we recall the arguments in [11]: any Hamilton cycle would pass, through each copy of $K_{3,4}$, exactly once; however, this would determine a Hamilton cycle in Barnette-Bosák-Lederberg graph, which is impossible.

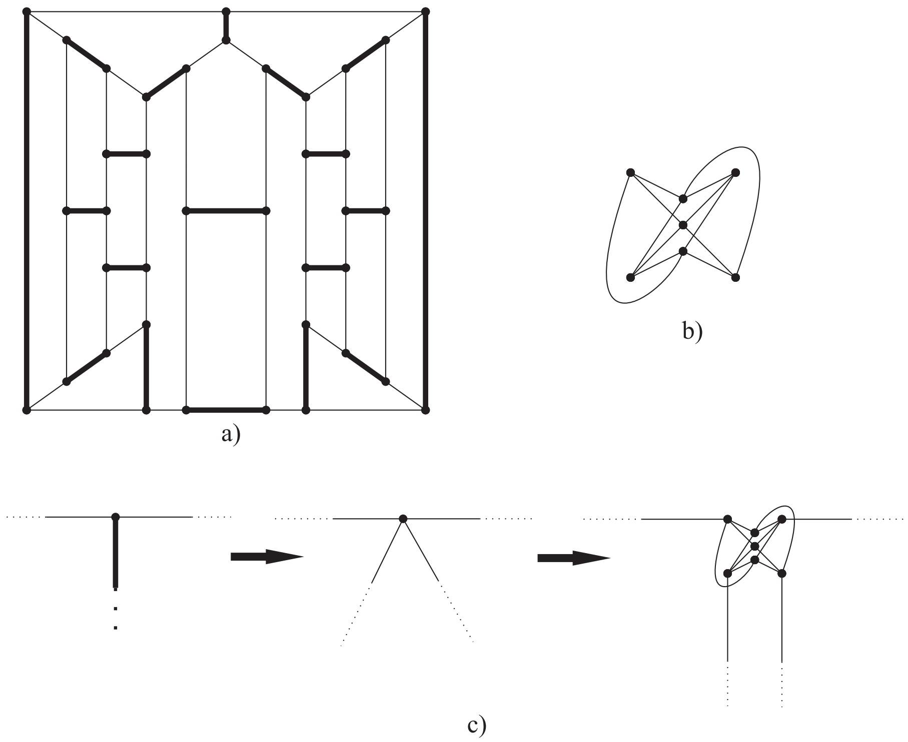

Figure 1. The construction of 4-connected 1-planar nonhamiltonian graph.

Next, we turn our attention to local properties of maximal and Hamiltonian 1-planar graphs.

Theorem 5. *Each Hamiltonian* 1*-planar graph on at least k vertices contains a k-vertex path of weight at most* $8k-1$.

Proof. We use an analogue of the proof of Proposition 2.1 from [7]: let G be a Hamiltonian 1-planar graph on $n > k$ vertices and $C = v_1v_2\dots v_n$ be its Hamilton cycle. Let $R_i = v_iv_{i+1}\dots v_{i+k-1} \subseteq C$ (indices are modulo n) be a k-vertex path in C starting in v_i, and let $w(R_i)$ be the sum of degrees of vertices of R_i. We have $\sum_{i=1}^{n} w(R_i) = k\sum_{v\in V(G)} \deg_G(v) = 2k|E(G)| \le 2k(4n-8)$ Hence, the average weight of these paths is at most $\dfrac{2k(4n-8)}{n} = 8k - \dfrac{16k}{n} < 8k$, so there exists $j \in \{1,\dots,n\}$ such that $w(R_j) \le 8k-1$. ■

Corollary 6. *Each optimal* 1*-planar graph on at least k vertices contains a k-vertex path of weight at most* $8k-1$.

There exist, for each integer k, Hamiltonian 1-planar graphs whose k-vertex paths have weights at least $8k-2$. They can be constructed in the following way: take a plane graph of a 3-cube, and replace each of its faces with the $l \times l$ grid, where $l > k$ (see the left graph of Figure 2 for the result of the grid replacement). Into each 4-face of the obtained plane graph, insert a pair of crossing edges. The resulting 1-planar graph Q_l^+ is Hamiltonian (it is not hard to check that Q_l^+ contains, as a spanning subgraph, a 4-connected planar graph), it contains eight vertices of degree 6 which are at distance $> k$; all other vertices are of degree 8. Thus, each of its k-vertex paths is of weight at least $6 + 8(k-1) = 8k - 2$.

We do not know any example of a Hamiltonian 1-planar graph for which the upper bound of Theorem 5 is sharp.

For maximal 1-planar graphs, we have the following analogue of Theorem 1 of [2]:

Theorem 7. *Each maximal* 3*-connected* 1*-planar graph on at least* $2k$ *vertices contains a* k*-vertex path* P *with all vertices of degree at most* $10k$.

Proof. Let G be a maximal 3-connected 1-planar graph having at least $2k$ vertices; without loss of generality, we consider its 1-planar drawing D with the minimum possible number of crossings. Then, by [3], Lemma 2.1, its associated plane graph $D^{\times}$ is 3-connected and contains at least $2k$ vertices. Thus, by [2], Theorem 1, $D^{\times}$ contains a $2k$-vertex path P' with all vertices of degree at most $5 \cdot (2k) = 10k$. Let x be a false vertex of $D^{\times}$ and xy, xz be edges of P' incident with x. If xy, xz are not incident with a 3-face of $D^{\times}$, then yz is an edge of D. Otherwise, the path yzx lies in the boundary of a face of $D^{\times}$ and, by the maximality of G, y and z are also connected by an edge which does not belong to P'. Therefore, in the sequence of edges of P', each pair of edges incident with a false vertex of $G^{\times}$ may be replaced by an edge of G in such a way that the resulting set of edges induces a path P in G on l vertices; as none two false vertices of $D^{\times}$ are adjacent, we have $k \leq l \leq 2k$. Since this procedure does not increase degrees of vertices of P, we obtain that P contains a subpath on k vertices, each of them having degree at most $10k$. ∎

3. Number of Edges of Maximal 1-planar Graphs

In the following, let $M(\mathcal{G}, n)$ and $m(\mathcal{G}, n)$ denote the maximum and the minimum number of edges of a maximal n-vertex graph from the family $\mathcal{G}$. For the family $\overline{\mathcal{P}}$ of 1-planar graphs, we have, by [9], $M(\overline{\mathcal{P}}, n) = 4n - 8$ for $n \geq 12$. The results of [12] (see also [4]) complete the information on maximal 1-planar graphs for $n \leq 11$; it is shown that $M(\overline{\mathcal{P}}, n) = 4n - 9$ for $n \in \{7, 9\}$, $M(\overline{\mathcal{P}}, n) = 4n - 8$ for $n \in \{8, 10, 11\}$ and $M(\overline{\mathcal{P}}, n) = \binom{n}{2}$ for $n \leq 6$.

Unlike the family $\mathcal{P}$ of planar graphs (where all n-vertex maximal graphs have the same number $3n - 6$ of edges), there exist integers n for which $m(\overline{\mathcal{P}}, n) \neq$

$M(\overline{\mathcal{P}}, n)$. The first examples of maximal 1-planar n-vertex graphs with less than $4n-8$ edges were given in [12] (also in [4]) for $n = 3k, k \geq 3$: take the Cartesian product of a 3-cycle and a k-vertex path and, into each of its 4-cycles, insert a pair of chords. The resulting graph is maximal 1-planar and has $4n-9$ edges. In [4], it was also shown that the graph $K_7 - E(K_{1,3})$ is maximal 1-planar.

Our main result is the following:

Theorem 8. *For each rational number $\frac{p}{q} \in \left[\frac{8}{3}, 4\right]$, there exist infinitely many integers n such that, for each of them, there exists a 2-connected maximal 1-planar graph on n vertices having $\frac{p}{q}(n-2)$ edges.*

Proof. We construct the desired maximal 1-planar graph $H_{p,q}$ in the following way: for any $r \geq 1$, put $k = (p-2q)^r$ and consider the 1-planar drawing of the graph Q_k^+ (defined in the previous section). This graph has $6k^2+2$ vertices and $24k^2$ edges (of which $12k^2$ are not crossed). Next, put $\alpha = \frac{6(4q-p)}{p-2q}$ and select any αk^2 distinct non-crossed edges of Q_k^+. For each such edge xy, add a new vertex z and new edges xz, yz (see Figure 2).

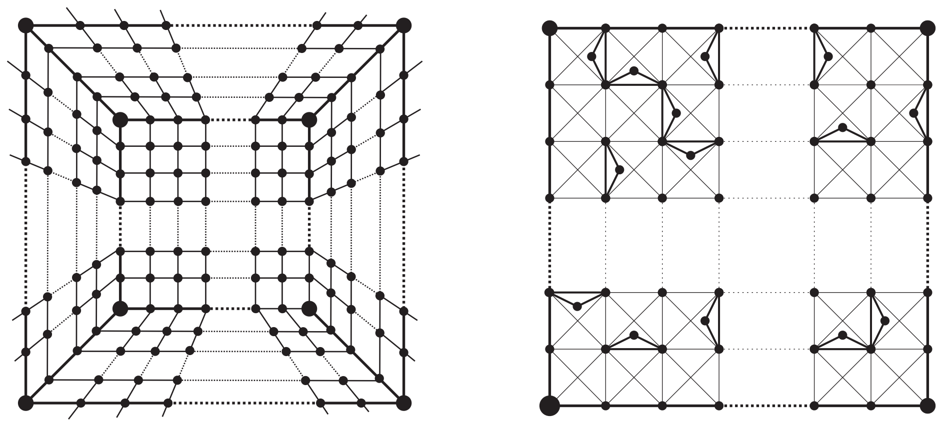

Figure 2. The construction of $H_{p,q}$: a plane subgraph of Q_k^+ and the crossed edges in one $k \times k$ grid.

The resulting graph $H_{p,q}$ is 2-connected and 1-planar, has $n = 6k^2 + 2 + \alpha k^2$ vertices and $24k^2 + 2\alpha k^2 = \frac{24+2\alpha}{6+\alpha} n - \frac{4\alpha+48}{6+\alpha} = \frac{p}{q} n - \frac{2p}{q}$ edges.

It remains to show that $H_{p,q}$ is maximal 1-planar graph. Since $24k^2 = 4(6k^2+2)-8$, the graph Q_k^+ is optimal. Moreover, by Corollary 4 of [12], this graph has the unique 1-planar drawing in the plane; by a routine check, we can verify that no two non-adjacent vertices of $H_{p,q}$ can be joined by an edge without violating the condition of 1-planarity. ■

Note that the graphs $H_{p,q}$ constructed in the previous proof have connectivity 2;

this brings the question whether an analogy of this theorem for maximal 1-planar graphs of higher connectivity can be formulated, or how low can be the leading coefficient in the edge number (expressed in terms of the vertex number) of these graphs. For 3-connected case, we can construct an infinite family of maximal 1-planar n-vertex graphs with $\frac{29}{8}n - \frac{15}{2}$ edges. The construction begins with the plane graph H on Figure 3 which was considered in [6] as an example of a so called *PN-graph*, being defined as a 3-connected planar graph with the property that each of its drawings in the plane is either a plane graph, or at least one edge is crossed more than once.

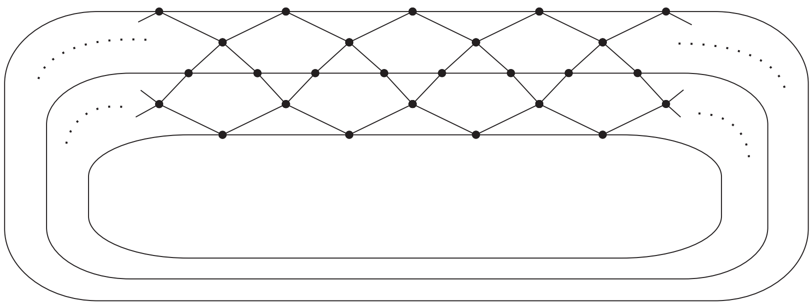

Figure 3. A PN-graph.

By Whitney's theorem (see [1], Theorem 4.3.2), the plane drawing of any PN-graph is unique.

Let $k = 8p + 2$ be a large integer, and let the outerface α of H consist of vertices $x_1, \dots, x_k$ in clockwise order, and, similarly, let the inner face β of H of size k consist of vertices $y_1, \dots, y_k$ in clockwise order. Now, express H as the union of $\frac{k}{2}$ copies of the plane configuration bounded by thick black edges in Figure 4 and add 13 gray dashed edges into the interior of the 12-cycle bounding this configuration, and an extra edge between each two consecutive configurations (joining the vertex x_{i+3} with a neighbour of x_{i+1} in Figure 4, upper part). In addition, add new edges $x_{4j+5}x_{k-4j-4}, y_{4j+5}y_{k-4j-4}$ for $j = 0, \dots, p-2$ and edges $x_{4i+1}x_{k-4i-4}, x_{4i+5}x_{k-4i}, y_{4i+1}y_{k-4i-4}, y_{4i+5}y_{k-4i}$, for $i = 0, \dots, p-1$ (see the middle part of Figure 4); thus, we can express the boundary cycles of α (and similarly, β) as the union of two edges $x_1x_k, x_{4p+1}x_{4p+2}$ and 5-vertex paths of the form $x_{4j+1}\dots x_{4j+5}$ or $x_{4p+4j+2}\dots x_{4p+4j+6}$ for $j = 0, \dots, p-1$. Finally, for each such 5-vertex path $x_{4u+1}\dots x_{4u+5}$, add new edges $x_{4u+1}x_{4u+5}, x_{4u+2}x_{4u+5}, x_{4u+2}x_{4u+4}$, $x_{4u+3}x_{4u+5}$ (see the bottom of Figure 4; also, do the same for 5-vertex paths in the boundary cycle of β).

The resulting graph $\overline{H}$ has $n = 6k = 48p + 12$ vertices, and $174p + 36 = \frac{29}{8}n - \frac{15}{2}$ edges; from its construction, it follows that $\overline{H}$ is 3-connected and 1-planar. As $H \subseteq \overline{H}$ and H is a PN-graph, we obtain that, in any 1-planar drawing of $\overline{H}$, H appears in a unique way, essentially being drawn as on Figure

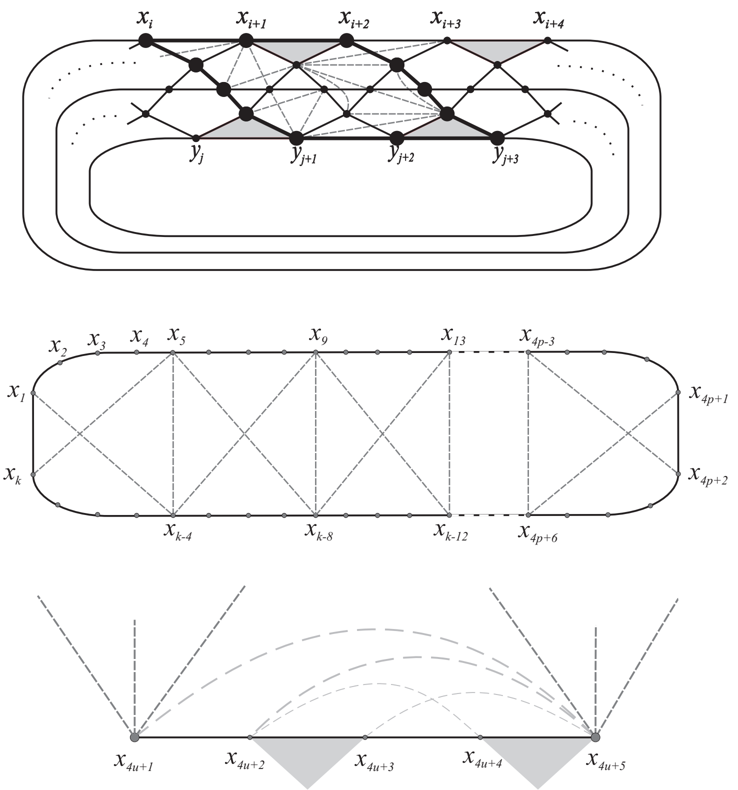

Figure 4. Forming a maximal 1-planar graph from PN-graph.

3 (the only difference may be the degree of outerface of H). This enforces the remaining gray dashed edges of $\overline{H}$ being drawn essentially as on Figure 4. We conclude that $\overline{H}$ has a unique 1-planar drawing, and, by routine checking, it is easy to show that any additional edge joining two nonadjacent vertices in $\overline{H}$ violates the 1-planarity of $\overline{H}$. Hence, $\overline{H}$ is also a maximal 1-planar graph.

We believe that the following is true.

Conjecture 9. *For the family $\overline{\mathcal{P}^\star}$ of 3-connected maximal 1-planar graphs, $m(\overline{\mathcal{P}^\star}, n) = \frac{18}{5}n + c$, where c is a constant.*

The leading coefficient $\frac{18}{5}$ follows from the following construction (Fabrici, personal communication): take a 3-connected plane graph consisting only of triangular and quadrangular faces in such a way that no two faces of the same kind are adjacent; then, insert into each quadrangular face a pair of crossing diagonals. This yields 1-planar graph with the desired number of edges. However, it is not clear whether such a graph is maximal, although its drawing is maximal 1-planar: there exist examples showing that maximal and 1-planar drawings do not always guarantee maximal 1-planar graphs, as seen from Figure 5 (when moving the edge xy to another gray region, we can join vertices z, w with a new edge; the maximality of the left drawing can be verified by hand). Based on this example, it is possible to construct, for any integer k, a maximal 1-planar drawing which can be redrawn in such a way that at least k new edges may be added without violating the 1-planarity (take k copies of the left drawing of Figure 5 with missing edge uv, identify all vertices corresponding to u and, similarly, to v, and add a new edge between vertices resulted from this identification).

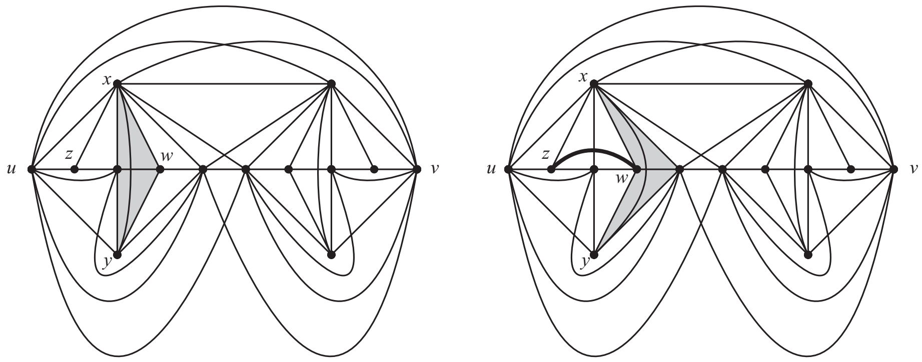

Figure 5. The maximal 1-planar drawing with underlying 1-planar non-maximal graph.

References

[1] R. Diestel, Graph Theory (Springer, 2006).

[2] I. Fabrici and S. Jendrol', *Subgraphs with restricted degrees of their vertices in planar 3-connected graphs*, Graphs Combin. **13** (1997) 245–250.

[3] I. Fabrici and T. Madaras, *The structure of 1-planar graphs*, Discrete Math. **307** (2007) 854–865. doi:10.1016/j.disc.2005.11.056

[4] D. Hudák: Štruktúra 1-planárnych grafov, Master Thesis, P.J. Šafárik University, Košice, 2009

[5] V. P. Korzhik, *Minimal non-1-planar graphs*, Discrete Math. **308** (2008) 1319–1327. doi:10.1016/j.disc.2007.04.009

[6] V. P. Korzhik and B. Mohar, *Minimal obstructions for 1-immersions and hardness of 1-planarity testing*, Springer Lecture Notes in Computer Science **5417** (2009) 302–312. doi:10.1007/978-3-642-00219-9_29

[7] B. Mohar, *Light paths in 4-connected graphs in the plane and other surfaces*, J. Graph Theory **34** (2000) 170–179. doi:10.1002/1097-0118(200006)34:2⟨170::AID-JGT6⟩3.0.CO;2-P

[8] J.W. Moon and L. Moser, *On hamiltonian bipartite graphs*, Israel J. Math **1** (1963) 163–165.
doi:10.1007/BF02759704

[9] J. Pach and G. Tóth, *Graphs drawn with few crossings per edge*, Combinatorica **17** (1997) 427–439.
doi:10.1007/BF01215922

[10] G. Ringel, *Ein Sechsfarbenproblem auf der Kugel*, Abh. Math. Sem. Univ. Hamburg **29** (1965) 107p–117.
doi:10.1007/BF02996313

[11] T. Kaiser, D. Král, M. Rosenfeld, Z. Ryjáček and H.-J. Voss: *Hamilton cycles in prisms over graphs*, http://cam.zcu.cz/~ryjacek/publications/files/60.pdf

[12] Y. Suzuki, *Re-embeddings of maximum* 1*-planar graphs*, SIAM J. Discrete Math. **24** (2010) 1527–1540.
doi:10.1137/090746835

[13] W. T. Tutte, *A theorem on planar graphs*, Trans. Am. Math. Soc. **82** (1956) 99–116.
doi:10.1090/S0002-9947-1956-0081471-8

[14] H. Whitney, *A theorem on graphs*, Ann. Math. **32** (1931) 378-390.
doi:10.2307/1968197

Received 23 May 2011
Revised 17 January 2012
Accepted 18 January 2012