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HAMILTONIAN-COLORED POWERS OF STRONG DIGRAPHS

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Abstract

For a strong oriented graph D of order n and diameter d and an integer k with $1 \leq k \leq d$, the kth power D^k of D is that digraph having vertex set $V(D)$ with the property that (u, v) is an arc of D^k if the directed distance $\vec{d}_D(u, v)$ from u to v in D is at most k. For every strong digraph D of order $n \geq 2$ and every integer $k \geq \lceil n/2 \rceil$, the digraph D^k is Hamiltonian and the lower bound $\lceil n/2 \rceil$ is sharp. The digraph D^k is distance-colored if each arc (u, v) of D^k is assigned the color i where $i = \vec{d}_D(u, v)$. The digraph D^k is Hamiltonian-colored if D^k contains a properly arc-colored Hamiltonian cycle. The smallest positive integer k for which D^k is Hamiltonian-colored is the Hamiltonian coloring exponent $\text{hce}(D)$ of D. For each integer $n \geq 3$, the Hamiltonian coloring exponent of the directed cycle $\vec{C}_n$ of order n is determined whenever this number exists. It is shown for each integer $k \geq 2$ that there exists a strong oriented graph D_k such that $\text{hce}(D_k) = k$ with the added property that every properly colored Hamiltonian cycle in the kth power of D_k must use all k colors. It is shown for every positive integer p there exists a a connected graph G with two different strong orientations D and D' such that $\text{hce}(D) - \text{hce}(D') \geq p$.

Keywords: powers of a strong oriented graph, distance-colored digraphs, Hamiltonian-colored digraphs, Hamiltonian coloring exponents.

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1. Introduction

For a connected graph G of order n, the *distance* $d_G(u, v)$ between two vertices u and v in G is the *length* of a shortest $u - v$ path in G. A $u - v$ path of length

$d_G(u, v)$ is a $u - v$ *geodesic.* The greatest distance between any two vertices of G is the *diameter* $\mathrm{diam}(G)$ of G. For an integer k with $1 \leq k \leq d = \mathrm{diam}(G)$, the kth *power* G^k of G is that graph with vertex set $V(G)$ and $uv \in E(G^k)$ if $1 \leq d_G(u, v) \leq k$. The graphs G^2 and G^3 are called the *square* and *cube*, respectively, of G, while $G^1 = G$. For an integer $k \geq d$, $G^k = K_n$, the complete graph of order n. We refer to [3] for graph theory notation and terminology not described in this paper.

In 1960 Sekanina [7] proved that the cube of every connected graph G of order at least 3 is Hamiltonian. In fact, he showed that for every such graph G, the graph G^3 is Hamiltonian-connected (every two vertices of G are connected by a Hamiltonian path). In 1971 Fleischner [4] verified a well-known conjecture (at the time) that the square of every 2-connected graph is Hamiltonian.

For a connected graph G, the edge-colored graph G^k is *distance-colored* if each edge uv of G^k is assigned the color i where $i = d_G(u, v)$. The graph G^k is *Hamiltonian-colored* if it contains a properly colored Hamiltonian cycle, that is, a Hamiltonian cycle in which every two adjacent edges are colored differently. There are connected graphs G for which G^k is not Hamiltonian-colored for any positive integer k. Indeed, if G is a graph of order n containing a vertex of degree $n - 1$, then G^k is not Hamiltonian-colored for any positive integer k. On the other hand, if G^k is Hamiltonian-colored for some positive integer k, then the smallest such integer k is called the *Hamiltonian coloring exponent* $\mathrm{hce}(G)$ of G. These concepts were introduced in [1] and studied further in [6]. Applications of Hamiltonian-colored graphs to network communications were studied in [2]. Chartrand, Jones, Kolasinski and Zhang established the following result dealing with the Hamiltonian coloring exponent of a graph (see [1, 6]).

Theorem 1.1. *For each integer $k \geq 2$, there exists a graph G such that* $\mathrm{hce}(G) =$ *k and every properly colored Hamiltonian cycle in G^k must use all k colors.*

In this paper we study the analogous concept of Hamiltonian-colored powers of strong oriented graphs. We begin by presenting some information on powers of strong oriented graphs.

2. Powers of Strong Oriented Graphs

A digraph D is an *oriented graph* if for every two distinct vertices x and y, at most one of the arcs (directed edges) (x, y) and (y, x) belongs to D. The digraph D is *strong* (or *strongly connected*) if for every two vertices u and v, the digraph D contains both a (directed) $u - v$ path and a $v - u$ path. The length of a shortest $u - v$ path in D is the (directed) *distance* $\vec{d}_D(u, v)$ from u to v and a $u - v$ path of length $\vec{d}_D(u, v)$ is a $u - v$ *geodesic.* The maximum value of $\vec{d}_D(x, y)$ among all pairs x, y of vertices of D is the *diameter* $\mathrm{diam}(D)$ of D.

For a strong oriented graph D of order n and diameter d and an integer k with $1 \leq k \leq d$, the *kth power* D^k of D is that digraph (not necessarily oriented graph) having vertex set $V(D)$ with the property that (u, v) is an arc of D^k if $1 \leq \vec{d}_D(u, v) \leq k$. If $k \geq d$, then $D^k = K_n^*$, the complete symmetric digraph of order n. If $n \geq 2$ and $k \geq d$, then D^k is Hamiltonian. Unlike the situation for connected graphs of order at least 3 where there is a fixed constant c (namely $c = 3$) such that G^3 is Hamiltonian for every connected graph G of order at least 3, there is no fixed constant c such that D^c is Hamiltonian for every strong oriented graph D. We will see in Theorem 2.3 that if D is a strong digraph of order $n \geq 2$ and k is an integer such that $k \geq \lceil n/2 \rceil$, then D^k is Hamiltonian. In order to establish this result, we first present a lemma. Obviously, if D is a strong digraph of order $n \geq 2$ and diameter d, then $\operatorname{od} v \geq 1$ and $\operatorname{id} v \geq 1$ for every vertex v of D. Since $D^d = K_n^*$, it follows that $\operatorname{od}_{D^d} v = \operatorname{id}_{D^d} v = n - 1$ for every vertex v of D^d. More generally, we have the following.

Lemma 2.1. *Let D be a strong digraph of order $n \geq 2$ and diameter d. For every integer k with $1 \leq k \leq d$ and every vertex v of D^k, $\operatorname{od}_{D^k} v \geq k$ and $\operatorname{id}_{D^k} v \geq k$.*

Proof. Suppose that the lemma is false. Then there is a smallest positive integer r where $r < d$ such that either $\operatorname{od}_{D^r} v < r$ or $\operatorname{id}_{D^r} v < r$, say the former. Since $\operatorname{od}_D v \geq 1$ and $\operatorname{id}_D v \geq 1$, it follows that $r \geq 2$. Furthermore, because $\operatorname{od}_{D^{r-1}} v \geq r - 1$ and $\operatorname{id}_{D^{r-1}} v \geq r - 1$, it follows that $\operatorname{od}_{D^{r-1}} v = r - 1$. Since $r < d$, it follows that $|N_{D^{r-1}}(v) \cup \{v\}| = r < n$ and so there are vertices of D that do not belong to $N_{D^{r-1}}(v) \cup \{v\}$. Let w be one of these vertices. Since D is strong, there are $v - w$ paths in D. Let P be a $v - w$ geodesic in D and let y be the first vertex of P that does not belong to $N_{D^{r-1}}(v) \cup \{v\}$, where x is the vertex immediately preceding y on P. Thus $d_D(v, x) \leq r - 1$ and $(x, y) \in E(D^{r-1})$. Therefore, $d_D(v, y) = r$ and $y \in N_{D^r}(v)$, a contradiction. ∎

Among the sufficient conditions that exist for a digraph to be Hamiltonian is the following due to Ghouila-Houri [5].

Theorem 2.2 (Ghouila-Houri's Theorem)**.** *If D is a strong digraph of order n such that $\operatorname{od} v + \operatorname{id} v \geq n$ for every vertex v of D, then D is Hamiltonian.*

As a consequence of Lemma 2.1 and Ghouila-Houri's theorem, we have the following.

Theorem 2.3. *For every strong digraph D of order $n \geq 2$ and every integer $k \geq \lceil n/2 \rceil$, the digraph D^k is Hamiltonian. Furthermore, the lower bound $\lceil n/2 \rceil$ is sharp.*

Proof. Let d be the diameter of D. If $k > d$, then D^d is the complete symmetric digraph of order n and so D^k is Hamiltonian. Thus, we may assume that $1 \leq$

$k \leq d$. By Lemma 2.1, $\mathrm{od}_{D^k}\, v \geq \lceil n/2 \rceil$ and $\mathrm{id}_{D^k}\, v \geq \lceil n/2 \rceil$ for every vertex v of D. Therefore, $\mathrm{od}_{D^k}\, v + \mathrm{id}_{D^k}\, v \geq 2\lceil n/2 \rceil \geq n$. By Ghouila-Houri's theorem, D^k is Hamiltonian. Thus, it remains to show that the lower bound $\lceil n/2 \rceil$ is sharp. For a given integer $k \geq 3$, consider the strong oriented graph D_k shown in Figure 1. (If $k = 3$, then we replace the (directed) $u - v$ path $(u, v_1, v_2, \ldots, v_{k-3}, v)$ by the arc (u, v).)

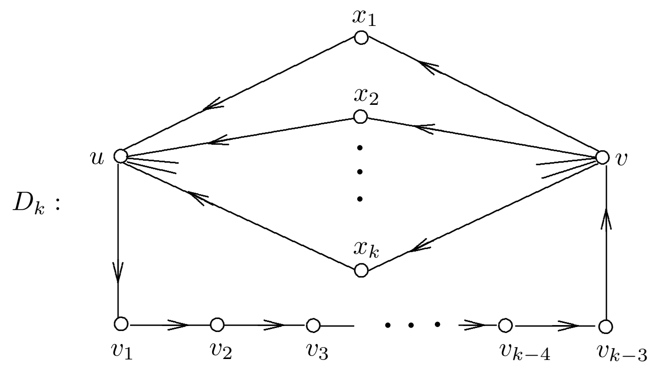

Figure 1. The strong oriented graph D_k in the proof of Theorem 2.3.

Since the order of D_k is $n = 2k - 1$, it follows by the first statement in this theorem that the kth power of D_k is Hamiltonian. The diameter of D_k is k. In fact, the only vertices y and z in D_k for which $\vec{d}_D(y, z) = k$ are distinct vertices of $\{x_1, x_2, \ldots, x_k\}$. In fact, if we let $G = \overline{K}_k + K_{k-1}$ (the join of $\overline{K}_k$ and K_{k-1}), then $D_k^{k-1} = G^*$ (the complete symmetric digraph with underlying graph G). Because G is not Hamiltonian, it follows that D_k^k is Hamiltonian but D_k^{k-1} is not. Therefore, the lower bound $\lceil n/2 \rceil$ is sharp. ■

By Theorem 2.3, unlike the situation for connected graphs of order at least 3, there is no fixed constant c such that D^c is Hamiltonian for every strong oriented graph D.

3. Distance-colored Digraphs

For a strong oriented graph D and a positive integer k, the kth power D^k is called *distance-colored* if each arc (u, v) of D^k is assigned the color i if $\vec{d}_D(u, v) = i$. The digraph D^k is called *Hamiltonian-colored* if D^k contains a properly colored Hamiltonian cycle $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$, that is, the colors of (v_i, v_{i+1}) and (v_{i+1}, v_{i+2}) are distinct for $1 \leq i \leq n$, where $v_{n+2} = v_2$.

If D is a strong oriented graph such that the distance-colored digraph D^2 is Hamiltonian-colored, then D must have even order n. The only strong digraph of order 2 is K_2^*, which is not an oriented graph. There is also no strong oriented graph D of order 4 for which D^2 is Hamiltonian-colored, for suppose, to the contrary, that such a digraph D exists and $C = (u, v, w, x, u)$ is a properly colored Hamiltonian cycle in D^2, where (u, v) and (w, x) are colored 1 and (v, w) and (x, u) are colored 2 (see Figure 2). Since (v, w) belongs to D^2 but not D, $(v, w) \notin E(D)$. Because D is strong and an oriented graph, $(v, x) \in E(D)$. Similarly, $(x, v) \in E(D)$. However then, D is not an oriented graph, a contradiction. The situation for the orders 2 and 4 are the exceptions, however, as we now see.

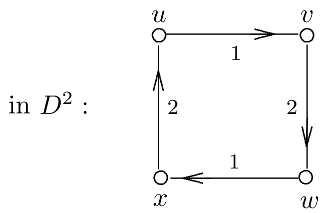

Figure 2. Showing that the square of no strong oriented graph of order 4 is Hamiltonian-colored.

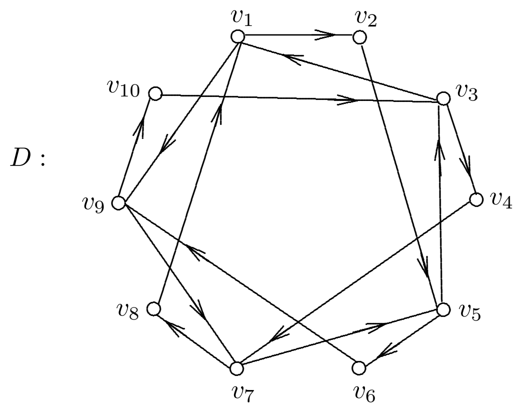

Figure 3. The strong oriented graph D (for $k = 5$) in the proof of Theorem 3.1.

Theorem 3.1. *For every even integer $n \geq 6$, there exists a strong oriented graph D of order n such that D^2 is Hamiltonian-colored.*

Proof. Let D be the strong oriented graph of order $n = 2k \geq 6$ and size $3k$ for which $V(D) = \{v_1, v_2, \ldots, v_{2k}\}$ and $E(D) = \{(v_{2i-1}, v_{2i}) : 1 \leq i \leq k\} \cup \{(v_{2k+3-2i}, v_{2k+1-2i}) : 1 \leq i \leq k\} \cup \{(v_{2i}, v_{2i+3}) : 1 \leq i \leq k\}$, where $v_{2k+1} = v_1$ and $v_{2k+3} = v_3$. (The digraph D is shown in Figure 3 for the case where $k = 5$.) In D^2, the Hamiltonian cycle $(v_1, v_2, \ldots, v_{2k}, v_1)$ is properly colored. ∎

If D is a strong oriented graph such that D^k is Hamiltonian-colored for some positive integer k, then the smallest such integer k is defined as the *Hamiltonian coloring exponent* $\mathrm{hce}(D)$ of D. Thus if $\mathrm{hce}(D) = k$, then D^{k-1} is not Hamiltonian-colored. In particular, Theorem 3.1 shows that if D is a strong oriented graph such that D^2 is Hamiltonian-colored, then $\mathrm{hce}(D) = 2$.

4. Hamiltonian Coloring Exponents of Directed Cycles

We now determine $\mathrm{hce}(\vec{C}_n)$ for the directed cycle $\vec{C}_n$ of order $n \geq 3$. Since $\mathrm{diam}(\vec{C}_n) = n-1$, it follows that if $\mathrm{hce}(\vec{C}_n)$ exists, then $2 \leq \mathrm{hce}(\vec{C}_n) \leq n-1$. Let $D = \vec{C}_n$ where $n \geq 3$. If $\mathrm{hce}(\vec{C}_n)$ exists, let $\mathrm{hce}(D) = k$. Then D^k contains a properly colored Hamiltonian cycle $C' = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $1 \leq \vec{d}_D(v_i, v_{i+1}) \leq k$ for $i = 1, 2, \ldots, n$. Let $\vec{d}_D(v_i, v_{i+1}) = a_i$ for $1 \leq i \leq n$. Thus, corresponding to the properly colored directed cycle C' is the cyclic sequence $s : a_1, a_2, \ldots, a_n$ of colors where $a_i \in \{1, 2, \ldots, k\}$ for $1 \leq i \leq n$. Since C' starts and ends at v_1, it follows that C' proceeds around $\vec{C}_n$ a certain number of times, say p, and so $\sum_{i=1}^{n} \vec{d}_D(v_i, v_{i+1}) = \sum_{i=1}^{n} a_i = pn$.

For a cyclic sequence $s : a_1, a_2, \ldots, a_n$ of length n and any integer t with $1 \leq t \leq n$, the sequence s can also be expressed as $s : a_t, a_{t+1}, \ldots, a_n, a_1, \ldots, a_{t-1}$. A *proper subsequence* $s^\star$ of s is defined as a sequence $s^\star : a_t, a_{t+1}, \ldots, a_{t+n^\star-1}$ of length $n^\star$, where $1 \leq n^\star < n$ and the subscripts are expressed as integers modulo n. There is no proper subsequence $s^\star : a_t, a_{t+1}, \ldots, a_{t+q-1}$ of s for which $\sum_{i=t}^{t+q-1} a_i$ is a multiple of n, for otherwise, the cycle $C^\star = (v_t, v_{t+1}, \ldots, v_{t+q-1}, v_{t+q} = v_t)$ is a cycle of length $q < n$ that is a proper subdigraph of the Hamiltonian cycle C', which is impossible. Consequently, $s : a_1, a_2, \ldots, a_n$ where $a_i \in \{1, 2, \ldots, k\}$ for $1 \leq i \leq n$ is a cyclic sequence of colors of a Hamiltonian-colored digraph D^k with $\mathrm{hce}(D) = k$ if and only if

(1) no two consecutive terms in s are equal,

(2) $\sum_{i=1}^{n} a_i$ is a multiple of n and

(3) the sum of the terms in no proper subsequence of s is a multiple of n.

Any cyclic sequence $s : a_1, a_2, \ldots, a_n$ of terms $a_i \in \{1, 2\}$ for $1 \leq i \leq n$ satisfying condition (1) has the property that $n < \sum_{i=1}^{n} a_i < 2n$. Thus condition (2) is not satisfied. Therefore, we have the following observation.

Observation 4.1. *Let $n \geq 3$ be an integer. If $\mathrm{hce}(\vec{C}_n)$ exists, then $\mathrm{hce}(\vec{C}_n) \geq 3$.*

Since $\mathrm{diam}(\vec{C}_3) = 2$, it follows by Observation 4.1 that $\mathrm{hce}(\vec{C}_3)$ does not exist. On the other hand, if $\vec{C}_4 = (v_1, v_2, v_3, v_4, v_1)$, then $C' = (v_1, v_2, v_4, v_3, v_1)$ is a properly colored Hamiltonian cycle in the cube of $\vec{C}_4$ and so $\mathrm{hce}(\vec{C}_4) = 3$. Corresponding to C' is the cyclic sequence $s : 1, 2, 3, 2$ of colors. In fact, not only is $\mathrm{hce}(\vec{C}_4) = 3$ but $\mathrm{hce}(\vec{C}_n) = 3$ for all even integers $n \geq 4$, as we show next.

Theorem 4.2. *For every even integer $n \geq 4$, $\mathrm{hce}(\vec{C}_n) = 3$.*

Proof. We have already observed that $\mathrm{hce}(\vec{C}_4) = 3$ and $\mathrm{hce}(\vec{C}_n) \geq 3$ for all integers $n \geq 3$ (if $\mathrm{hce}(\vec{C}_n)$ exists). Thus, it remains only to show that there is a cyclic sequence $s : a_1, a_2, \ldots, a_n$ of $n \geq 6$ terms with n even and $a_i \in \{1, 2, 3\}$ for $1 \leq i \leq n$ satisfying conditions (1)–(3). We consider two cases.

Case 1. $n \equiv 2 \pmod 4$. So $n = 4r + 2$ for $r \geq 1$. Consider the cyclic sequence $s : 1, 3, 1, 3, \ldots, 1, 3$ of $4r + 2$ terms. Then the sum of the terms of s is $8r + 4 = 2n$. Since the sum of the terms of any subsequence of s is either odd or a multiple of 4, this sum is not n.

Case 2. $n \equiv 0 \pmod 4$. So $n = 4r$ for $r \geq 2$. Consider the cyclic sequence $s : 1, 3, 1, 3, \ldots, 1, 3, 1, 2, 3, 1, 3, 1, \ldots, 3, 1, 3, 2$ of $4r$ terms where there are $2r - 1$ terms between the occurrences of 2 in s. Then the sum of the terms of s is $8r = 2n$. Now observe that the sum of the terms of any subsequence

(i) containing both terms 2 exceeds n,

(ii) containing neither term 2 is less than n and

(iii) containing exactly one term 2 is either odd or is congruent to 2 modulo 4 and consequently is not n. ■

We now consider $\mathrm{hce}(\vec{C}_n)$ where $n \geq 3$ is odd. We saw that $\mathrm{hce}(\vec{C}_3)$ does not exist. In fact, $\mathrm{hce}(\vec{C}_5)$ does not exist either.

Proposition 4.3. *The number $\mathrm{hce}(\vec{C}_5)$ does not exist.*

Proof. Let $D = \vec{C}_5$. Assume, to the contrary, that $\mathrm{hce}(D)$ exists. By Observation 4.1, $3 \leq \mathrm{hce}(D) \leq \mathrm{diam}(D) = 4$, that is, either $\mathrm{hce}(D) = 3$ or $\mathrm{hce}(D) = 4$.

If $\mathrm{hce}(D) = 3$, then there exists a cyclic sequence $s : a_1, a_2, a_3, a_4, a_5$ with $a_i \in \{1, 2, 3\}$, $1 \leq i \leq 5$, satisfying (1)–(3). Necessarily, some term, say a_2, is 3. If either $a_1 = 2$ or $a_3 = 2$, then either $a_1 + a_2 = 5$ or $a_2 + a_3 = 5$, which is impossible. Thus $a_1 = a_3 = 1$. However then, $a_1 + a_2 + a_3 = 5$, also impossible.

If $\mathrm{hce}(D) = 4$, then there exists a cyclic sequence $s : a_1, a_2, a_3, a_4, a_5$ with $a_i \in \{1, 2, 3, 4\}$, $1 \leq i \leq 5$, satisfying (1)–(3). Necessarily, some term, say a_3, is 4. Neither a_2 nor a_4 is 1, for otherwise, either $a_2 + a_3 = 5$ or $a_3 + a_4 = 5$, which is impossible. Also, we cannot have $a_2 = a_4 = 3$ for then $a_2 + a_3 + a_4 = 10$, also impossible. Thus, one of a_2 and a_4 is 2 and the other is 2 or 3. First, suppose that $a_2 = 3$ and $a_4 = 2$. Now $a_5 \neq 1$, for otherwise, $a_2 + a_3 + a_4 + a_5 = 10$, which is impossible. Also, $a_5 \neq 3$, for otherwise, $a_4 + a_5 = 5$. Finally, $a_5 \neq 4$, for otherwise, $a_3 + a_4 + a_5 = 10$. Thus, this case cannot occur. Next suppose that $a_2 = a_4 = 2$. Neither $a_1 = 4$ nor $a_5 = 4$ for otherwise, either $a_1 + a_2 + a_3 = 10$

or $a_3 + a_4 + a_5 = 10$. Also, neither $a_1 = 3$ nor $a_5 = 3$, for otherwise, $a_1 + a_2 = 5$ or $a_4 + a_5 = 5$. Consequently, $a_1 = a_5 = 1$, which contradicts (1). Again, this is impossible. ■

On the other hand, $\mathrm{hce}(\vec{C}_n)$ exists for each odd integer $n \geq 7$. First, we present a lemma.

Lemma 4.4. *For every odd integer* $n \geq 7$, $\mathrm{hce}(\vec{C}_n) \neq 3$.

Proof. Assume, to the contrary, that there is an odd integer $n \geq 7$ such that $\mathrm{hce}(\vec{C}_n) = 3$. Let $D = \vec{C}_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$. Hence there exists a properly colored Hamiltonian cycle $C' = (u_1, u_2, \ldots, u_n, u_{n+1} = u_1)$ in D^3, where $u_1 = v_1$ and where C' proceeds about $\vec{C}_n$ twice. If $s : a_1, a_2, \ldots, a_n$ is the corresponding cyclic sequence of colors for C', then no two consecutive terms in s are equal, $\sum_{i=1}^{n} a_i = 2n$ and no proper subsequence of s has the property that the sum of its terms is n. Since C' is an odd cycle, all three colors $1, 2$ and 3 must appear in s. Furthermore, since the sum $\sum_{i=1}^{n} a_i$ is even and the average term in this sum is 2, the colors 1 and 3 must appear an equal number of times, implying that the color 2 must appear an odd number of times in s.

First, we show that neither $1, 2, 1$ nor $3, 2, 3$ can occur as a subsequence of s. If $1, 2, 1$ occurs as a subsequence of s, then C' contains the path $(v_i, v_{i+1}, v_{i+3}, v_{i+4})$ for some i with $1 \leq i \leq n$ where the subscripts are expressed as integers modulo n. This, however, implies that $(v_{i-1}, v_{i+2}, v_{i+5})$ is a path on C' and that $3, 3$ is a subsequence of s, which is impossible. If $3, 2, 3$ occurs as a subsequence of s, then C' contains the path $(v_i, v_{i+3}, v_{i+5}, v_{i+8})$ for some i $(1 \leq i \leq n)$. Since C' proceeds about $\vec{C}_n$ twice, $(v_{i+1}, v_{i+2}, v_{i+4}, v_{i+6}, v_{i+7})$ is also a path on C' and so $1, 2, 2, 1$ is a subsequence of s, which is impossible.

Therefore, each occurrence of the color 2 in s must occur as $1, 2, 3$ or $3, 2, 1$. If $1, 2, 3$ occurs in s, then C' contains the path $(v_i, v_{i+1}, v_{i+3}, v_{i+6})$ for some i $(1 \leq i \leq n)$, implying that C' also contains $(v_{i-1}, v_{i+2}, v_{i+4}, v_{i+5})$ and so $3, 2, 1$ is a subsequence (later) in s. Similarly, if $3, 2, 1$ occurs in s, then $1, 2, 3$ occurs (later) in s. That is, the subsequences $1, 2, 3$ and $3, 2, 1$ occur in pairs in s, implying that 2 appears an even number of times in s, which is a contradiction. ■

We next show that $\mathrm{hce}(\vec{C}_7) = \mathrm{hce}(\vec{C}_9) = 5$, beginning with $\mathrm{hce}(\vec{C}_7) = 5$.

Proposition 4.5. $\mathrm{hce}(\vec{C}_7) = 5$.

Proof. Let $D = \vec{C}_7 = (v_1, v_2, \ldots, v_7, v_1)$. Since the cyclic sequence

$$s : 1, 5, 3, 2, 1, 5, 4$$

corresponds to the properly colored Hamiltonian cycle

$$(v_1, v_2, v_7, v_3, v_5, v_6, v_4, v_1)$$

in D^5, it follows that $\mathrm{hce}(\vec{C}_7) \leq 5$. By Lemma 4.4, $\mathrm{hce}(\vec{C}_7) \geq 4$. Thus $\mathrm{hce}(\vec{C}_7) = 4$ or $\mathrm{hce}(\vec{C}_7) = 5$. We show that $\mathrm{hce}(\vec{C}_7) = 5$.

Assume, to the contrary, that $\mathrm{hce}(\vec{C}_7) = 4$. Then D^4 contains a properly colored Hamiltonian cycle C'. Corresponding to C' is a cyclic sequence of colors $s: a_1, a_2, \ldots, a_7$, where $\sum_{i=1}^{7} a_i = 14$ or $\sum_{i=1}^{7} a_i = 21$. Necessarily, at least one of the terms in s is the color 4, say $a_4 = 4$. Since the sum of the terms in no proper subsequence of s is a multiple of 7, it follows that (1) neither a_3 nor a_5 is 3 and (2) $\{a_3, a_5\} \neq \{1, 2\}$. Hence either $a_3 = a_5 = 1$ or $a_3 = a_5 = 2$. First, assume that $a_3 = a_5 = 1$. Thus either $a_1 + a_2 + a_6 + a_7 = 8$ or $a_1 + a_2 + a_6 + a_7 = 15$. Since no two consecutive terms in s are 4, it follows that $a_1 + a_2 + a_6 + a_7 = 8$. If one of the colors a_1, a_2, a_6 and a_7 is 4, then two of them are 1, contradicting the assumption of the case. Again, the assumption of the case implies that no two the colors a_1, a_2, a_6, a_7 can be 1. Consequently, we may assume that $s: 1, 2, 1, 4, 1, 3, 2$. Since $a_2 + a_3 + a_4 = 7$, a contradiction is produced. Next, assume that $a_3 = a_5 = 2$. First, we observe that neither a_2 nor a_6 is 1 since the sum of the terms in no proper subsequence of s is 7. Also, since the sum of the terms in no proper subsequence of s is 14, it cannot occur that $a_2 = a_6 = 3$. Therefore, either $a_2 = a_6 = 4$ or we may assume that $a_2 = 3$ and $a_6 = 4$. If $a_2 = a_6 = 4$, then $a_1 \notin \{1, 2, 3, 4\}$, for otherwise, the sum of the terms in a proper subsequence of s is a multiple of 7; if $a_2 = 3$ and $a_6 = 4$, then $a_7 \notin \{1, 2, 3, 4\}$, a contradiction. ■

Proposition 4.6. $\mathrm{hce}(\vec{C}_9) = 5$.

Proof. Let $D = \vec{C}_9 = (v_1, v_2, \ldots, v_9, v_1)$. Since the cyclic sequence

$$s: 1, 4, 3, 4, 3, 5, 2, 3, 2$$

corresponds to the properly colored Hamiltonian cycle

$$(v_1, v_2, v_6, v_9, v_4, v_7, v_3, v_5, v_8, v_1)$$

in D^5, it follows that $\mathrm{hce}(\vec{C}_9) \leq 5$. By Lemma 4.4, $\mathrm{hce}(\vec{C}_9) \geq 4$. Thus $\mathrm{hce}(\vec{C}_9) = 4$ or $\mathrm{hce}(\vec{C}_9) = 5$. We show that $\mathrm{hce}(\vec{C}_9) = 5$.

Assume, to the contrary, that $\mathrm{hce}(\vec{C}_9) = 4$. Then D^4 contains a properly colored Hamiltonian cycle C'. Corresponding to C' is a cyclic sequence of colors $s: a_1, a_2, \ldots, a_9$, where $\sum_{i=1}^{9} a_i = 18$ or $\sum_{i=1}^{9} a_i = 27$. (There is no no proper subsequence of s, the sum of whose terms is a multiple of 9.) We consider two cases.

Case 1. $\sum_{i=1}^{9} a_i = 18$. Then the cycle C' proceeds about $\vec{C}_9$ exactly twice. Since at least one of the terms in s is the color 4, we may assume that (v_1, v_5) is a path on C'. However then, (v_2, v_3, v_4) is also path on C', implying that $1, 1$ is a subsequence of s, which is impossible.

Case 2. $\sum_{i=1}^{9} a_i = 27$. Consider the three subsequences of s,

$$s_1: a_1, a_2, a_3,\ s_2: a_4, a_5, a_6,\ s_3: a_7, a_8, a_9,$$

where σ_i is the sum of the terms in s_i for $i = 1, 2, 3$. Necessarily, no σ_i has the value 9. Since $\sigma_1 + \sigma_2 + \sigma_3 = 27$, two of the numbers $\sigma_1, \sigma_2, \sigma_3$ exceed 9 or two are less than 9. First assume that two of the numbers $\sigma_1, \sigma_2, \sigma_3$ exceed 9, say σ_1 and σ_2. Thus each of σ_1 and σ_2 is 10 or 11. If $\sigma_1 = 11$, then $s_1 : 4, 3, 4$. If $\sigma_1 = 10$, then $s_1 : 4, 2, 4$ or $s_1 : 3, 4, 3$. Since $a_3 \neq a_4$, we may assume that $s_1 : 3, 4, 3$ and either $s_2 : 4, 2, 4$ or $s_2 : 4, 3, 4$. Since $a_3+a_4+a_5 \neq 9$, it follows that $s_1 : 3, 4, 3$ and $s_2 : 4, 3, 4$. Thus $\sigma_3 = 6$, which implies that $a_7 + a_8 + a_9 + a_1 = 9$, producing a contradiction. Next, assume that two of the numbers $\sigma_1, \sigma_2, \sigma_3$ are less than 9, say σ_1 and σ_3. Thus $\sigma_2 = 11$, which implies that $s_2 : 4, 3, 4$. Hence $\sigma_1 = \sigma_3 = 8$. Consequently, s_1 is one of (1) $4, 3, 1$, (2) $4, 1, 3$ or (3) $1, 4, 3$; while s_3 is one of (1') $1, 3, 4$, (2') $3, 1, 4$ or (3') $3, 4, 1$. Since $a_1 \neq a_9$, $a_9 + a_1 + a_2 \neq 9$ and $a_8 + a_9 + a_1 \neq 9$, none of these are possible. ■

We now show that $\text{hce}(\vec{C}_n) = 5$ for each odd integer $n \geq 7$.

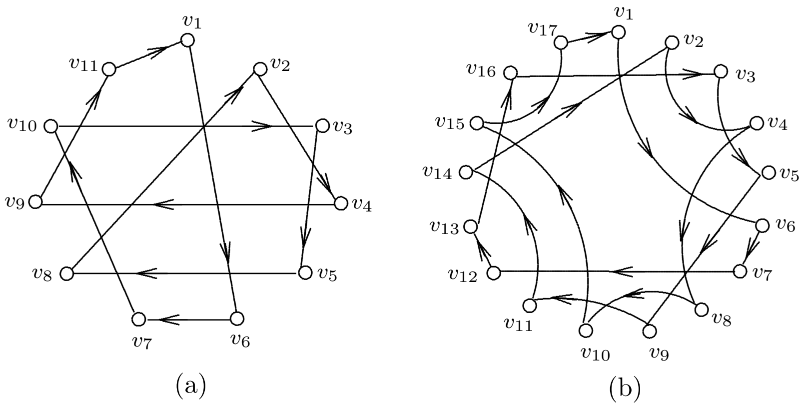

Figure 4. Properly colored Hamiltonian cycles in the 5th powers of $\vec{C}_{11}$ and $\vec{C}_{17}$.

Theorem 4.7. *For every odd integer $n \geq 7$, $\text{hce}(\vec{C}_n) = 5$.*

Proof. Let $D = \vec{C}_n = (v_1, v_2, \ldots, v_n, v_1)$. We have seen by Propositions 4.5 and 4.6 that $\text{hce}(\vec{C}_7) = \text{hce}(\vec{C}_9) = 5$. Hence we may assume that $n \geq 11$. We first show that $\text{hce}(\vec{C}_n) \leq 5$. There are hree cases, according to whether n is congruent to 5, 1 or 3 modulo 6.

Case 1. $n \equiv 5 \pmod 6$. First, observe that the cyclic sequence

$$s_{11} : 5, 1, 3, 4, 2, 3, 5, 2, 5, 2, 1$$

corresponds to the properly colored Hamiltonian cycle

$$C'_{11} = (v_1, v_6, v_7, v_{10}, v_3, v_5, v_8, v_2, v_4, v_9, v_{11}, v_1)$$

shown in Figure 4(a) in the 5th power of $\vec{C}_{11}$; while the cyclic sequence

$$s_{17} : 5, 1, 5, 1, 3, 4, 2, 4, 2, 3, 5, 2, 4, 2, 5, 2, 1$$

corresponds to the properly colored Hamiltonian cycle

$$(1) \qquad C'_{17} = (v_1, v_6, v_7, v_{12}, v_{13}, v_{16}, v_3, v_5, v_9, v_{11}, v_{14}, v_2, v_4, v_8, v_{10}, v_{15}, v_{17}, v_1)$$

shown in Figure 4(b) in the 5th power of $\vec{C}_{17}$. Thus $\mathrm{hce}(\vec{C}_{11}) \leq 5$ and $\mathrm{hce}(\vec{C}_{17}) \leq 5$. For the cycle C'_{17} (in (1) and in Figure 4(b)), let $n = 17$ and relabel v_i

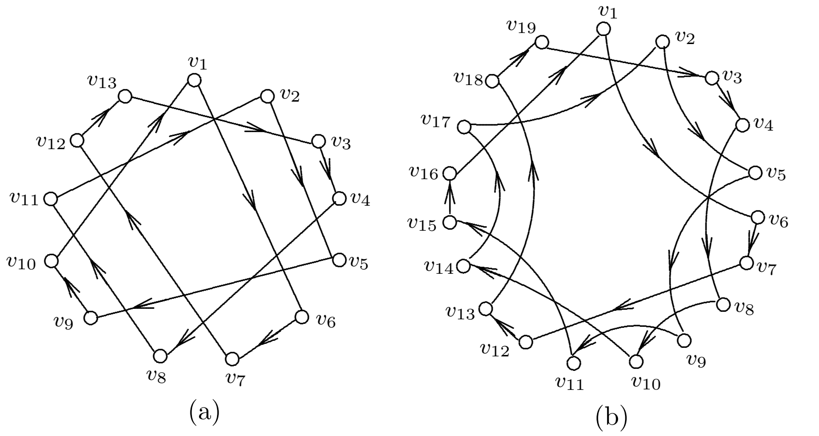

Figure 5. Properly colored Hamiltonian cycles in the 5th powers of $\vec{C}_{13}$ and $\vec{C}_{19}$.

$(1 \leq i \leq 17 = n)$ as v_{i+6} and delete the arcs (v_{n+6}, v_7), (v_{n+5}, v_9), (v_{n+3}, v_8). We next add vertices $v_1, v_2, \ldots, v_6$ along with all arcs of C'_{17} incident with and directed away from $v_1, v_2, \ldots, v_6$. Finally, we add the arcs (v_{n+6}, v_1), (v_{n+5}, v_3), (v_{n+3}, v_2). This produces a properly colored Hamiltonian cycle C' for the 5th power of $\vec{C}_{23}$. Corresponding to this cycle is the cyclic sequence

$$s' : \ 5, 1, 5, 1, 5, 1, 3, 4, 2, 4, 2, 4, 2, 3, 5, 2, 4, 2, 4, 2, 5, 2, 1.$$

By first letting $n = 23$ and then proceeding successively as above, we obtain a properly colored Hamiltonian cycle in the 5th power of $\vec{C}_n$ for each $n \geq 29$ such that $n \equiv 5 \pmod 6$. Such a cycle also corresponds to the cyclic sequence obtained by inserting in s' (a) the sequence $5, 1$ between $5, 1$ and $3, 4$, (b) the sequence $2, 4$ between $2, 4$ and $2, 3$ and (c) the sequence $2, 4$ between $2, 4$ and $2, 5$.

Case 2. $n \equiv 1 \pmod 6$. First, observe that the cyclic sequence

$$s_{13} : 5, 1, 5, 1, 3, 1, 4, 3, 4, 3, 4, 1, 4$$

corresponds to the properly colored Hamiltonian cycle

$$C'_{13} = (v_1, v_6, v_7, v_{12}, v_{13}, v_3, v_4, v_8, v_{11}, v_2, v_5, v_9, v_{10}, v_1)$$

shown in Figure 5(a) in the 5th power of $\vec{C}_{13}$; while the cyclic sequence

$$s_{19} : 5, 1, 5, 1, 5, 1, 3, 1, 4, 2, 4, 3, 4, 3, 4, 2, 4, 1, 4$$

corresponds to the properly colored Hamiltonian cycle

$$C'_{19} = (v_1, v_6, v_7, v_{12}, v_{13}, v_{18}, v_{19}, v_3, v_4, v_8, v_{10}, v_{14}, v_{17}, v_2, v_5, v_9, v_{11}, v_{15}, v_{16}, v_1) \tag{2}$$

shown in Figure 5(b) in the 5th power of $\vec{C}_{19}$. Thus $\mathrm{hce}(\vec{C}_{13}) \leq 5$ and $\mathrm{hce}(\vec{C}_{19}) \leq 5$. For the cycle C'_{19} (in (2) and in Figure 5(b)), let $n = 19$ and relabel v_i

$(1 \le i \le 19 = n)$ as v_{i+6} and delete the arcs (v_{n+6}, v_9), (v_{n+4}, v_8), (v_{n+3}, v_7). We next add vertices $v_1, v_2, \ldots, v_6$ along with all arcs of C'_{19} incident with and directed away from $v_1, v_2, \ldots, v_6$. Finally, we add the arcs (v_{n+6}, v_1), (v_{n+5}, v_3), (v_{n+4}, v_2). This produces a properly colored Hamiltonian cycle C' for the 5th power of $\vec{C}_{25}$. Corresponding to this cycle is the cyclic sequence

$$s' : \ 5,1,5,1,5,1,5,1,3,1,4,2,4,2,4,3,4,3,4,2,4,2,4,1,4.$$

By first letting $n = 25$ and then proceeding successively as above, we obtain a properly colored Hamiltonian cycle in the 5th power of $\vec{C}_n$ for every integer $n \ge 31$ such that $n \equiv 1 \pmod 6$. Such a cycle also corresponds to the cyclic sequence obtained by inserting in s' (a) the sequence $5,1$ between between $5,1$ and $3,1,4$, (b) the sequence $2,4$ after $3,1,4$ and (c) the sequence $2,4$ after $3,4,3,4$.

Case 3. $n \equiv 3 \pmod 6$. First, observe that the cyclic sequence

$$s_{15} : 5,1,5,1,5,1,4,2,5,2,3,4,2,3,2$$

corresponds to the properly colored Hamiltonian cycle

$$C'_{15} = (v_1, v_6, v_7, v_{12}, v_{13}, v_3, v_4, v_8, v_{10}, v_{15}, v_2, v_5, v_9, v_{11}, v_{14}, v_1)$$

shown in Figure 6(a) in the 5th power of $\vec{C}_{15}$; while the cyclic sequence

$$s_{21} : 5,1,5,1,5,1,5,1,4,2,4,2,5,2,3,4,2,4,2,3,2$$

corresponds to the properly colored Hamiltonian cycle

$$\begin{aligned} C'_{21} = (&v_1, v_6, \\ &v_7, v_{12}, v_{13}, v_{18}, v_{19}, v_3, v_4, v_8, v_{10}, v_{14}, v_{16}, v_{21}, v_2, v_5, v_9, v_{11}, v_{15}, v_{17}, v_{20}, v_1) \end{aligned} \tag{3}$$

shown in Figure 6(b) in the 5th power of $\vec{C}_{21}$. Thus $\mathrm{hce}(\vec{C}_{15}) \le 5$ and $\mathrm{hce}(\vec{C}_{21}) \le 5$.

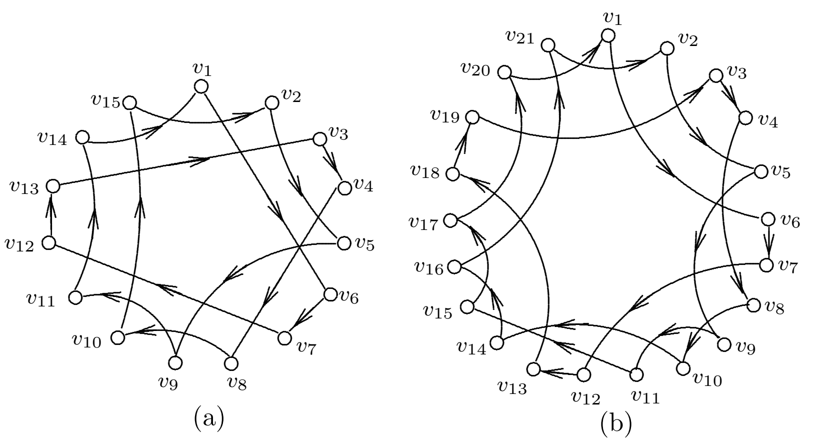

Figure 6. Properly colored Hamiltonian cycles in the 5th powers of $\vec{C}_{15}$ and $\vec{C}_{21}$.

For the cycle C'_{21} (in (3) and in Figure 6(b)), let $n = 21$ and relabel v_i $(1 \le i \le 21 = n)$ as v_{i+6} and delete the arcs (v_{n+6}, v_8), (v_{n+5}, v_7), (v_{n+4}, v_9). We next add vertices $v_1, v_2, \ldots, v_6$ along with all arcs of C'_{21} incident with and directed away from $v_1, v_2, \ldots, v_6$. Finally, we add the arcs (v_{n+6}, v_2), (v_{n+5}, v_1), (v_{n+4}, v_3). This produces a properly colored Hamiltonian cycle C' for the 5th power of $\vec{C}_{27}$. Corresponding to this cycle is the cyclic sequence

$$s' : \; 5,1,5,1,5,1,5,1,5,1,4,2,4,2,4,2,5,2,3,4,2,4,2,4,2,3,2.$$

By first letting $n = 27$ and then proceeding successively as above, we obtain a properly colored Hamiltonian cycle in the 5th power of $\vec{C}_n$ for every integer n such that $n \geq 33$ and $n \equiv 3 \pmod 6$. Such a cycle also corresponds to the cyclic sequence obtained by inserting in s' (a) the sequence $5, 1$ between $5, 1$ and $4, 2$, (b) the sequence $2, 4$ between $2, 4$ and $2, 5$ and (c) the sequence $2, 4$ between $2, 4$ and $2, 3$.

Next, we show that $\mathrm{hce}(\vec{C}_n) \geq 5$. We have seen by Lemma 4.4 that $\mathrm{hce}(\vec{C}_n) \geq 4$ for every odd integer $n \geq 7$. Thus it remains only to show that $\mathrm{hce}(\vec{C}_n) \neq 4$ for all such integers n. Assume, to the contrary, that the distance-colored digraph D^4 contains a properly colored Hamiltonian cycle C, which we assume begins and ends at v_1. Thus the arcs of C are colored with elements of the set $\{1, 2, 3, 4\}$. Since $\mathrm{hce}(\vec{C}_n) \geq 4$, at least one arc of C is colored 4, say (v_i, v_{i+4}) is colored 4 for some i. If the cycle C proceeds about $\vec{C}_n$ only twice, then C must contain the path $(v_{i+1}, v_{i+2}, v_{i+3})$, which implies that two consecutive arcs of C are colored 1, which is impossible. Consequently, C proceeds about $\vec{C}_n$ exactly three times.

We claim that no arc of C is colored 1. Suppose that this is not the case. Then one or more arcs of C are colored 1. We may assume that (v_1, v_2) is colored 1 and this is the first arc of C. Thus (v_2, v_3) is not an arc of C. Let v_{k+1} $(2 \leq k \leq n)$ be the next vertex of C that is incident with an arc colored 1, where $v_{n+1} = v_1$. Therefore, no arc of C that is incident with any of $v_3, v_4, \ldots, v_k$ is colored 1. We refer to the set $\{v_1, v_2, \ldots, v_k\}$ of vertices as a block of C, where the block is even or odd according to whether k is even or odd. We show that this block is even.

First, we show that (v_n, v_4) and (v_{n-1}, v_3) are arcs of C. Certainly, v_n is adjacent to either v_3 or v_4. If (v_n, v_3) is an arc of C, then (v_1, v_2) and (v_n, v_3) belong to two of the three distinct paths that pass by v_1 as we proceed about $\vec{C}_n$ on C. However then, the third path that passes by v_1 must contain an arc (v_j, v_ℓ), where $j < n$ and $\ell > 3$, which is impossible. Hence (v_n, v_4) is an arc on C, which implies that (v_{n-1}, v_3) is an arc on C.

In summary then, the cycle C contains the arc (v_1, v_2) colored 1 and the arcs (v_n, v_4) and (v_{n-1}, v_3), both colored 4. The vertex v_2 is adjacent to either v_5 or v_6. We consider these two cases.

Case 1. *(v_2, v_5) is an arc on C.* In this case, (v_3, v_6) and (v_4, v_7) are arcs of C. This implies that (v_5, v_9) is an arc of C (see Figure 7). The vertex v_{n-2} is adjacent to either v_{n-1}, v_n or v_1. We consider these three subcases.

Subcase 1.1. *(v_{n-2}, v_{n-1}) is an arc of C.* Since (v_{n-2}, v_{n-1}) is an arc of C colored 1, it follows by the previous discussion that (v_{n-3}, v_n) is not an arc of C

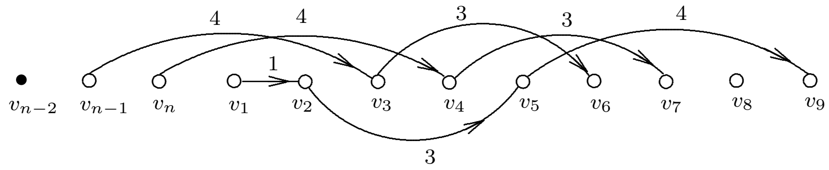

Figure 7. Illustrating Case 1.

and so (v_{n-3}, v_1) is an arc of C. This, however, implies that (v_{n-4}, v_n) is an arc of C colored 4, which is impossible since (v_n, v_4) is also an arc of C colored 4.

Subcase 1.2. *(v_{n-2}, v_n) is an arc of C.* If (v_7, v_8) is an arc of C colored 1, then the block is even. Thus, we may assume that (v_7, v_8) is not an arc of C. This implies that (v_7, v_{11}) is an arc of C. If $n = 11$, then a contradiction is produced since $(v_n, v_4) = (v_{11}, v_4)$ is also an arc of C. Thus, $n \geq 13$ and then (v_6, v_8) must be an arc of C. This implies that (v_8, v_{10}) cannot be an arc of C. Thus (v_9, v_{10}) is an arc of C colored 1 and the block is even.

Subcase 1.3. *(v_{n-2}, v_1) is an arc of C.* If (v_7, v_8) is an arc of C, then the block is even; otherwise, (v_7, v_{11}) is an arc of C. As we saw in Subcase 1.2, a contradiction is produced if $n = 11$. Thus, $n \geq 13$. In this case, (v_6, v_8) and (v_8, v_{12}) are arcs of C. From this, it follows that (v_9, v_{10}) is an arc of C and the block is even.

Case 2. *(v_2, v_6) is an arc of C.* In this case, (v_3, v_5) and (v_4, v_7) are also arcs of C. See Figure 8. Then v_5 is adjacent to either v_8 or v_9. We consider these two subcases.

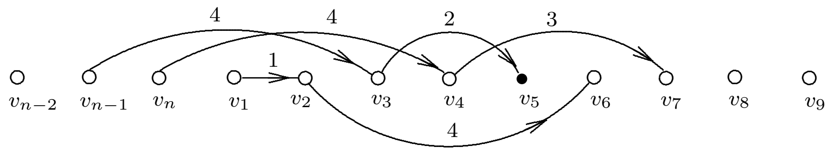

Figure 8. Illustrating Case 2.

Subcase 2.1. *(v_5, v_8) is an arc of C.* Here, both (v_6, v_9) and (v_7, v_{11}) are arcs of C. Again, if $n = 11$, then a contradiction is produced since $(v_n, v_4) = (v_{11}, v_4)$ is an arc of C. Thus, $n \geq 13$. If (v_9, v_{10}) is an arc of C, then the block is even; otherwise, (v_9, v_{13}) is an arc of C as is (v_8, v_{10}), which implies that (v_{11}, v_{12}) is an arc of C and once again the block is even.

Subcase 2.2. *(v_5, v_9) is an arc of C.* If (v_7, v_8) is an arc of C, then the block is even; otherwise, (v_6, v_8) is an arc of C, which implies that (v_7, v_{11}) is an arc of

C. As we have seen that $n \neq 11$. Thus $n \geq 13$ and then (v_8, v_{12}) is an arc of C. From this, it follows that (v_9, v_{10}) is an arc of C and so the block is even.

Therefore, each arc of C colored 1 belongs to an even block. Since the distinct blocks produce a partition of $V(\vec{C}_n)$, it follows that n is even, which is a contradiction. Hence no arc of C is colored 1. Consequently, each arc of a properly colored Hamiltonian cycle C of the distance-colored digraph D^4 is colored 2, 3 or 4.

Let $s : a_1, a_2, \ldots, a_n$ be the corresponding cyclic sequence of colors of C, where, as we noted, $a_i \in \{2, 3, 4\}$ for each i $(1 \leq i \leq n)$. Also $\sum_{i=1}^{n} a_i = 3n$. Since $(\sum_{i=1}^{n} a_i)/n = 3$ and n is odd, the color 3 appears an odd number of times in s and the colors 2 and 4 occur an equal number of times.

First, we show that $2, 3$ is not a subsequence of s, for suppose that it is. We may assume that (v_3, v_5) and (v_5, v_8) are arcs of C. Observe that (v_4, v_7) and (v_2, v_6) are arcs of C. Then v_1 is adjacent to no vertex of D on C, a contradiction.

Consequently, each term 3 in s is immediately preceded by 4 in s. Since the number of terms 2 and the number of terms 4 are equal, each subsequence of s between consecutive occurrences of 3 must alternate 2 and 4, beginning with 2 and ending with 4. In particular, each occurrence of 3 in s is immediately followed by $2, 4$, that is, $3, 2, 4$ is a subsequence of s. We may assume therefore that C contains the arcs (v_1, v_4), (v_4, v_6) and (v_6, v_{10}). Note that (v_2, v_5) and (v_3, v_7) must be arcs on C. However then, v_5 is adjacent to no vertex of D on C, a contradiction.

Hence, D^4 contains no properly colored Hamiltonian cycle. Therefore, $\text{hce}(\vec{C}_n) \geq 5$ and so $\text{hce}(\vec{C}_n) = 5$ for each odd integer $n \geq 7$. ■

In summary, $\text{hce}(\vec{C}_3)$ and $\text{hce}(\vec{C}_5)$ do not exist and

$$\text{hce}(\vec{C}_n) = \begin{cases} 3 & \text{if } n \geq 4 \text{ is even,} \\ 5 & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

5. Distance-colored Digraphs with Prescribed Hamiltonian Coloring Exponent

We saw that there are strong oriented graphs D for which $\text{hce}(D)$ does not exist. On the other hand, for each integer $k \geq 2$, there exists a strong oriented graph D such that $\text{hce}(D) = k$. In fact, more can be said. We now present a result that is analogous to Theorem 1.1.

Theorem 5.1. *For each integer $k \geq 2$, there exists a strong oriented graph D_k such that $\text{hce}(D_k) = k$. Furthermore, every properly colored Hamiltonian cycle in the kth power of D_k must use all k colors.*

Proof. By Theorem 3.1, we may assume that $k \geq 3$. We consider two cases, according to whether k is even or k is odd.

Case 1. *k is even.* First, we define four oriented graphs H_1, H_2, H_3 and H_4 as follows:

- H_1 is a transitive tournament of order $2k$ with the Hamiltonian path $(u_1, u_2, \ldots, u_{2k})$,
- $H_2 = (v_1, v_2, \ldots, v_k)$ is a directed path of order k,
- H_3 is a transitive tournament of order $2k$ with the Hamiltonian path $(w_1, w_2, \ldots, w_{2k})$,
- $H_4 = (x_1, x_2, \ldots, x_k)$ is a directed path of order k.

The oriented graph D_k is then constructed from H_1, H_2, H_3 and H_4 by adding the arcs (u_{2k}, v_1), (v_k, w_1), (w_{2k}, x_1) and (x_k, u_1) (see Figure 9). Since

$$(u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_{2k}, x_1, x_2, \ldots, x_k, u_1)$$

is a Hamiltonian cycle in D_k, it follows that D_k is a strong oriented graph.

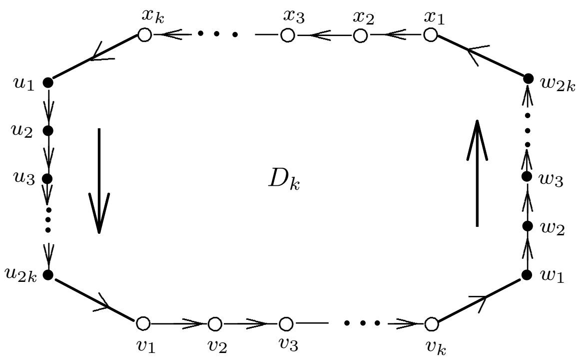

Figure 9. The strong oriented graph D_k where k is even.

We first show that $\text{hce}(D_k) \geq k$. Assume, to the contrary, that the distance-colored digraph D_k^{k-1} contains a properly colored Hamiltonian cycle C^*. Since, for each pair i, j with $1 \leq i, j \leq 2k$ and $i < j$, we have $d_{D_k}(w_i, w_j) = 1$ and $d_{D_k}(w_j, w_i) > k$, at most two vertices of H_3 can appear consecutively on C^*. On the other hand, $v_2, v_3, \ldots, v_k$ are the only vertices of D_k that are adjacent to vertices of H_3 in D_k^{k-1}. This implies that C^* encounters H_3 at most $k-1$ times and so C^* contains at most $2(k-1)$ vertices of H_3, which is a contradiction. Next, we show that $\text{hce}(D_k) \leq k$ by constructing a properly colored Hamiltonian cycle in D_k^k. Consider the k directed paths $P_i = (u_{k+i}, v_i, w_i)$, $1 \leq i \leq k$, of order 3 in D_k^k. Observe that $d_{D_k}(u_{k+i}, v_i) = 1 + i$ for $1 \leq i \leq k-1$, $d_{D_k}(u_{2k}, v_k) = k$, $d_{D_k}(v_1, w_1) = k$ and $d_{D_k}(v_i, w_i) = k + 2 - i$ for $2 \leq i \leq k$. Also, $k \geq 4$ is even and so $k+1$ is odd. These observations imply that

(1) $2 \leq d_{D_k}(u_{k+i}, v_i) \leq k$ and $2 \leq d_{D_k}(v_i, w_i) \leq k$ for $1 \leq i \leq k$,

(2) $d_{D_k}(u_{k+i}, v_i) \neq d_{D_k}(v_i, w_i)$ for $1 \leq i \leq k$.

Similarly, consider the k directed paths $Q_i = (w_{k+i}, x_i, u_i)$, $1 \leq i \leq k$, of order 3 in D_k^k. By symmetry, we have

(3) $2 \leq d_{D_k}(w_{k+i}, x_i) \leq k$ and $2 \leq d_{D_k}(x_i, u_i) \leq k$ for $1 \leq i \leq k$,

(4) $d_{D_k}(w_{k+i}, x_i) \neq d_{D_k}(x_i, u_i)$ for $1 \leq i \leq k$.

Since $d_{D_k}(u_i, u_{k+1+i}) = 1$ for $1 \leq i \leq k-1$, $d_{D_k}(w_i, w_{k+i}) = 1$ for $1 \leq i \leq k$ and $d_{D_k}(u_k, u_{k+1}) = 1$, it follows by (1)–(4) that $(P_1, Q_1, P_2, Q_2, \ldots, P_k, Q_k, u_{k+1})$ is a properly colored Hamiltonian cycle in D_k^k.

It remains to show that every properly colored Hamiltonian cycle in the kth power of D_k must use all colors $1, 2, \ldots, k$. Let C be any properly colored Hamiltonian cycle in D_k^k. As we saw, at most two vertices of H_3 can appear consecutively on C. Thus C must encounter H_3 at least k times. On the other hand, since $v_1, v_2, \ldots, v_k$ are the only vertices that are adjacent to vertices of H_3 in D_k^k, it follows that C encounters H_3 exactly k times. Moreover, C enters H_3 immediately after encountering a vertex v_i for some i with $1 \leq i \leq k$. Hence, C contains an arc (v_i, w) for each i with $1 \leq i \leq k$ and for some $w \in V(H_3)$. Since $d_{D_k}(v_1, w_j) > k$ for $2 \leq j \leq k$, it follows that (v_1, w_1) is an arc of C. Also, we saw that $d_{D_k}(v_i, w_j) = k + 2 - i$ for all i, j with $2 \leq i \leq k$ and $2 \leq j \leq k$. This implies that C contains at least one arc colored by each of the colors $2, 3, \ldots, k$. Furthermore, the order of H_3 is $2k$ and so two vertices of H_3 must appear consecutively on C, which implies that C contains at least one arc colored 1.

Case 2. *k is odd.* We construct a strong oriented graph D_k in the same fashion as the one in Case 1. First, we define four oriented graphs H_1, H_2, H_3 and H_4 as follows:

- H_1 is a transitive tournament of order $2k$ with the Hamiltonian path $(u_1, u_2, \ldots, u_{2k})$,
- $H_2 = (v_1, v_2, \ldots, v_{k-1})$ is a directed path of order $k-1$,
- H_3 is a transitive tournament of order $2k$ with the Hamiltonian path $(w_1, w_2, \ldots, w_{2k})$,
- $H_4 = (x_1, x_2, \ldots, x_{k-1})$ is a directed path of order $k-1$.

The oriented graph D_k is then constructed from H_1, H_2, H_3 and H_4 by adding the arcs (u_{2k}, v_1), (v_{k-1}, w_1), (w_{2k}, u_1), and (x_{k-1}, u_1). (See Figure 9, where we replace v_k by v_{k-1} and replace x_k by x_{k-1}.) Since

$$(u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_{k-1}, w_1, w_2, \ldots, w_{2k}, x_1, x_2, \ldots, x_{k-1}, u_1)$$

is a Hamiltonian cycle in D_k, it follows that D_k is a strong oriented graph.

We first show that $\mathrm{hce}(D_k) \geq k$. Assume, to the contrary, that the distance-colored digraph D_k^{k-1} contains a properly colored Hamiltonian cycle C^*. Since

$v_1, v_3, \ldots, v_{k-1}$ are the only vertices of D_k that are adjacent to vertices of H_3 in D_k^{k-1}, it follows that that C^* encounters H_3 at most $k-1$ times and so C^* contains at most $2(k-1)$ vertices of H_3, which is a contradiction. Next, we show that $\text{hce}(D_k) \leq k$ by constructing a properly colored Hamiltonian cycle in D_k^k. Consider the k directed paths $P_i = (u_{k+i}, v_i, w_i)$, $1 \leq i \leq k-1$, and $P_k = (u_{2k}, w_1)$ of order 3 in D_k^k. Observe that $d_{D_k}(u_{k+i}, v_i) = 1+i$ for $1 \leq i \leq k-1$, $d_{D_k}(v_i, w_i) = k+1-i$ for $1 \leq i \leq k-1$ and $d_{D_k}(u_{2k}, w_1) = k$. Furthermore, $k \geq 3$ is odd and $k+1$ is even. Thus

(1) $2 \leq d_{D_k}(u_{k+i}, v_i) \leq k$ and $2 \leq d_{D_k}(v_i, w_i) \leq k$ for $1 \leq i \leq k$,

(2) $d_{D_k}(u_{k+i}, v_i) \neq d_{D_k}(v_i, w_i)$ for $1 \leq i \leq k-1$.

Similarly, consider the k directed paths $Q_i = (w_{k+i}, x_i, u_i)$ $(1 \leq i \leq k-1)$ and $Q_k = (w_{2k}, u_1)$ of order 3 in D_k^k. By symmetry, we have

(3) $2 \leq d_{D_k}(w_{k+i}, x_i) \leq k$ and $2 \leq d_{D_k}(x_i, u_i) \leq k-1$ for $1 \leq i \leq k-1$,

(4) $d_{D_k}(w_{k+i}, x_i) \neq d_{D_k}(x_i, u_i)$ for $1 \leq i \leq k-1$.

Since $d_{D_k}(u_i, u_{k+1+i}) = 1$ for $1 \leq i \leq k-1$, $d_{D_k}(w_i, w_{k+i}) = 1$ for $1 \leq i \leq k$ and $d_{D_k}(u_k, u_{k+1}) = 1$, it follows by (1)–(4) that $(P_1, Q_1, P_2, Q_2, \ldots, P_k, Q_k, u_{k+1})$ is a properly colored Hamiltonian cycle in D_k^k.

It remains to show that every properly colored Hamiltonian cycle in the kth power of D_k must use all colors $1, 2, \ldots, k$. Let C be any properly colored Hamiltonian cycle in D_k^k. An argument similar to the one in Case 1 shows that C must enter H_3 exactly k times. Since $u_{2k}, v_1, v_3, \ldots, v_{k-1}$ are the only vertices of D_k that are adjacent to vertices of H_3 in D_k^k, each of the vertices $u_{2k}, v_1, v_3, \ldots, v_{k-1}$ is immediately followed by a vertex of H_3 on C. This, however, requires that C contains (u_{2k}, w_1) and an arc (v_i, w) for each i with $1 \leq i \leq k-1$ and for some $w \in V(H_3)$. Since $d_{D_k}(u_{2k}, w_1) = k$ and $d_{D_k}(v_i, w_j) = k+1-i$ for $1 \leq i \leq k-1$ and $2 \leq j \leq k$, it follows that C contains at least one arc colored by each of the colors $2, 3, \ldots, k$. Furthermore, the order of H_3 is $2k$ and so two vertices of H_3 must appear consecutively on C. Hence C contains an arc colored 1. ■

6. On the Existence of Graphs Having Distinct Strong Orientations with Different Hamiltonian Coloring Exponents

By Theorem 5.1, there exists for each integer $k \geq 2$ a strong oriented graph D such that $\text{hce}(D) = k$. Equivalently, there exists a connected graph G possessing a strong orientation D such that $\text{hce}(D) = k$. It is possible, however, that there may be another strong orientation of G, resulting in a digraph D' whose Hamiltonian coloring exponent is far differ from that of D. In fact, for two different strong orientations D and D' of a connected graph, the difference between $\mathrm{hce}(D)$ and $\mathrm{hce}(D')$ can be arbitrarily large.

Theorem 6.1. *For every positive integer p there exists a connected graph G with strong orientations D and D' such that* $\mathrm{hce}(D) - \mathrm{hce}(D') \geq p$.

Proof. For a positive integer p, let k be an integer such that $k \geq p + 3$ and $k \equiv 0 \pmod 4$. Now let G be the underlying graph of the strong oriented graph D_k in the proof of Theorem 5.1 when k is even. Following the same vertex labeling for D_k and the same notation for the subdigraphs H_1, H_2, H_2 and H_4 in D_k (as described in the proof of Theorem 5.1), let D'_k be the orientation of G obtained from D by replacing the two arcs (u_1, u_{2k}) and (w_1, w_{2k}) by (u_{2k}, u_1) and (w_{2k}, w_1). Now let $D = D_k$ and $D' = D'_k$. By Theorem 5.1, $\mathrm{hce}(D) = k$. In fact, $\mathrm{hce}(D') = 3$ as we show next.

First, we show that the cube of D' is Hamiltonian-colored. To construct a properly colored Hamiltonian cycle in the cube of D', we first define eight vertex-disjoint properly colored subpaths $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ in the cubes of the subdigraphs H_1, H_2, H_2 and H_4 of D', respectively, as follows:

• In the cube of H_1, define two vertex-disjoint properly colored paths P_{u_1} and P_{u_2} of order $k-2$ as $P_{u_1} = (u_k, u_3, u_{k-1}, u_4, \ldots, u_{\frac{k-2}{2}+3}, u_{\frac{k-2}{2}+2})$, $P_{u_2} = (u_{2k-2}, u_{k+1}, u_{2k-3}, u_{k+2}, \ldots, u_{k+\frac{k-2}{2}+1}, u_{k+\frac{k-2}{2}})$. Let $A_1 = (u_2, P_{u_1}, u_{2k-1})$ and $A_2 = (u_1, P_{u_2}, u_{2k})$ be the subpaths of order k in the cube of H_1. Then $V(A_1) \cup V(A_2) = V(H_1)$, each of the initial and terminal arcs of A_1 and A_2 is colored 1 and A_1 and A_2 are properly colored.

• In the cube of H_2, define two vertex-disjoint paths B_1 and B_2 of order $k/2$ as $B_1 = (v_1, v_2, v_5, v_6, v_9, v_{10}, v_{13}, \ldots, v_{k-6}, v_{k-3}, v_{k-2})$, $B_2 = (v_3, v_4, v_7, v_8, v_{11}, v_{12}, v_{15}, \ldots, v_{k-4}, v_{k-1}, v_k)$. Observe that $V(B_1) \cup V(B_2) = V(H_2)$ and each of the initial and terminal arcs of B_1 and B_2 is colored 1. The arcs of B_1 and B_2 are colored 1 and 3 alternatively.

• In the cube of H_3, define two vertex-disjoint properly colored paths P_{w_1} and P_{w_2} of order $k-2$ as $P_{w_1} = (w_k, w_3, w_{k-1}, w_4, \ldots, w_{\frac{k-2}{2}+3}, w_{\frac{k-2}{2}+2})$, $P_{w_2} = (w_{2k-2}, w_{k+1}, w_{2k-3}, w_{k+2}, \ldots, w_{k+\frac{k-2}{2}+1}, w_{k+\frac{k-2}{2}})$. Let $C_1 = (w_1, P_{w_1}, w_{2k-1})$ and $C_2 = (w_2, P_{w_2}, w_{2k})$ be the subpaths of order k in the cube of H_3. Then $V(C_1) \cup V(C_2) = V(H_3)$, each of the initial and terminal arcs of C_1 and C_2 is colored 1 and C_1 and C_2 are properly colored.

• In the cube of H_4, define two vertex-disjoint paths D_1 and D_2 of order $k/2$ as $D_1 = (x_1, x_2, x_5, x_6, x_9, x_{10}, x_{13}, \ldots, x_{k-6}, x_{k-3}, x_{k-2})$, $D_2 = (x_3, x_4, x_7, x_8, x_{11}, x_{12}, x_{15}, \ldots, x_{k-4}, x_{k-1}, x_k)$. Observe that $V(D_1) \cup V(D_2) = V(H_3)$ and each of the initial and terminal arcs of D_1 and D_2 is colored 1. The arcs of D_1 and D_2 are colored 1 and 3 alternatively.

Then $(A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, u_2)$ is a properly colored Hamiltonian cycle in the cube of D' and so $\text{hce}(D') \leq 3$. On the other hand, D' contains an induced path $\vec{P}_4$ and so it can be shown that the square of D' is not Hamiltonian-colored. Thus $\text{hce}(D') = 3$.

Consequently, $\text{hce}(D) - \text{hce}(D') = k - 3 \geq p$ as desired. ■

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