Abstract

For a strong oriented graph $D$ of order $n$ and diameter $d$ and an integer $k$ with $1 \leq k \leq d$, the $k$th power $D^k$ of $D$ is that digraph having vertex set $V(D)$ with the property that $(u, v)$ is an arc of $D^k$ if the directed distance $d_D(u, v)$ from $u$ to $v$ in $D$ is at most $k$. For every strong digraph $D$ of order $n \geq 2$ and every integer $k \geq \lceil n/2 \rceil$, the digraph $D^k$ is Hamiltonian and the lower bound $\lceil n/2 \rceil$ is sharp. The digraph $D^k$ is distance-colored if each arc $(u, v)$ of $D^k$ is assigned the color $i$ where $i = d_D(u, v)$. The digraph $D^k$ is Hamiltonian-colored if $D^k$ contains a properly arc-colored Hamiltonian cycle. The smallest positive integer $k$ for which $D^k$ is Hamiltonian-colored is the Hamiltonian coloring exponent $hce(D^k)$ of $D^k$. For each integer $n \geq 3$, the Hamiltonian coloring exponent of the directed cycle $\vec{C}_n$ of order $n$ is determined whenever this number exists. It is shown for each integer $k \geq 2$ that there exists a strong oriented graph $D_k$ such that $hce(D_k) = k$ with the added property that every properly colored Hamiltonian cycle in the $k$th power of $D_k$ must use all $k$ colors. It is shown for every positive integer $p$ there exists a a connected graph $G$ with two different strong orientations $D$ and $D'$ such that $hce(D) - hce(D') \geq p$.

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$d_G(u, v)$ is a $u - v$ geodesic. The greatest distance between any two vertices of $G$ is the diameter $\text{diam}(G)$ of $G$. For an integer $k$ with $1 \leq k \leq d = \text{diam}(G)$, the $k$th power $G^k$ of $G$ is that graph with vertex set $V(G)$ and $uv \in E(G^k)$ if $1 \leq d_G(u, v) \leq k$. The graphs $G^2$ and $G^3$ are called the square and cube, respectively, of $G$, while $G^1 = G$. For an integer $k \geq d$, $G^k = K_n$, the complete graph of order $n$. We refer to [3] for graph theory notation and terminology not described in this paper.

In 1960 Sekanina [7] proved that the cube of every connected graph $G$ of order at least 3 is Hamiltonian. In fact, he showed that for every such graph $G$, the graph $G^3$ is Hamiltonian-connected (every two vertices of $G$ are connected by a Hamiltonian path). In 1971 Fleischner [4] verified a well-known conjecture (at the time) that the square of every 2-connected graph is Hamiltonian.

For a connected graph $G$, the edge-colored graph $G^k$ is distance-colored if each edge $uv$ of $G^k$ is assigned the color $i$ where $i = d_G(u, v)$. The graph $G^k$ is Hamiltonian-colored if it contains a properly colored Hamiltonian cycle, that is, a Hamiltonian cycle in which every two adjacent edges are colored differently. There are connected graphs $G$ for which $G^k$ is not Hamiltonian-colored for any positive integer $k$. Indeed, if $G$ is a graph of order $n$ containing a vertex of degree $n - 1$, then $G^k$ is not Hamiltonian-colored for any positive integer $k$. On the other hand, if $G^k$ is Hamiltonian-colored for some positive integer $k$, then the smallest such integer $k$ is called the Hamiltonian coloring exponent $\text{hce}(G)$ of $G$. These concepts were introduced in [1] and studied further in [6]. Applications of Hamiltonian-colored graphs to network communications were studied in [2]. Chartrand, Jones, Kolasinski and Zhang established the following result dealing with the Hamiltonian coloring exponent of a graph (see [1, 6]).

**Theorem 1.1.** For each integer $k \geq 2$, there exists a graph $G$ such that $\text{hce}(G) = k$ and every properly colored Hamiltonian cycle in $G^k$ must use all $k$ colors.

In this paper we study the analogous concept of Hamiltonian-colored powers of strong oriented graphs. We begin by presenting some information on powers of strong oriented graphs.

## 2. Powers of Strong Oriented Graphs

A digraph $D$ is an oriented graph if for every two distinct vertices $x$ and $y$, at most one of the arcs (directed edges) $(x, y)$ and $(y, x)$ belongs to $D$. The digraph $D$ is strong (or strongly connected) if for every two vertices $u$ and $v$, the digraph $D$ contains both a (directed) $u - v$ path and a $v - u$ path. The length of a shortest $u - v$ path in $D$ is the (directed) distance $d_D(u, v)$ from $u$ to $v$ and a $u - v$ path of length $d_D(u, v)$ is a $u - v$ geodesic. The maximum value of $d_D(x, y)$ among all pairs $x, y$ of vertices of $D$ is the diameter $\text{diam}(D)$ of $D$. 
For a strong oriented graph $D$ of order $n$ and diameter $d$ and an integer $k$ with $1 \leq k \leq d$, the $k$th power $D^k$ of $D$ is that digraph (not necessarily oriented graph) having vertex set $V(D)$ with the property that $(u,v)$ is an arc of $D^k$ if $1 \leq d_D(u,v) \leq k$. If $k \geq d$, then $D^k = K_n^*$, the complete symmetric digraph of order $n$. If $n \geq 2$ and $k \geq d$, then $D^k$ is Hamiltonian. Unlike the situation for connected graphs of order at least 3 where there is a fixed constant $c$ (namely $c = 3$) such that $G^2$ is Hamiltonian for every connected graph $G$ of order at least 3, there is no fixed constant $c$ such that $D^c$ is Hamiltonian for every strong oriented graph $D$. We will see in Theorem 2.3 that if $D$ is a strong digraph of order $n \geq 2$ and $k$ is an integer such that $k \geq \lceil n/2 \rceil$, then $D^k$ is Hamiltonian. In order to establish this result, we first present a lemma. Obviously, if $D$ is a strong digraph of order $n \geq 2$ and diameter $d$, then od$v \geq 1$ and id$v \geq 1$ for every vertex $v$ of $D$. Since $D^d = K_n^*$, it follows that od$_{D^d}v = id_{D^d}v = n - 1$ for every vertex $v$ of $D^d$. More generally, we have the following.

**Lemma 2.1.** Let $D$ be a strong digraph of order $n \geq 2$ and diameter $d$. For every integer $k$ with $1 \leq k \leq d$ and every vertex $v$ of $D^k$, od$_{D^k}v \geq k$ and id$_{D^k}v \geq k$.

**Proof.** Suppose that the lemma is false. Then there is a smallest positive integer $r$ where $r < d$ such that either od$_{D^r}v < r$ or id$_{D^r}v < r$, say the former. Since od$_Dv \geq 1$ and id$_Dv \geq 1$, it follows that $r \geq 2$. Furthermore, because od$_{D^{r-1}}v \geq r - 1$ and id$_{D^{r-1}}v \geq r - 1$, it follows that od$_{D^{r-1}}v = r - 1$. Since $r < d$, it follows that $|N_{D^{r-1}}(v) \cup \{v\}| = r < n$ and so there are vertices of $D$ that do not belong to $N_{D^{r-1}}(v) \cup \{v\}$. Let $w$ be one of these vertices. Since $D$ is strong, there are $v - w$ paths in $D$. Let $P$ be a $v - w$ geodesic in $D$ and let $y$ be the first vertex of $P$ that does not belong to $N_{D^{r-1}}(v) \cup \{v\}$, where $x$ is the vertex immediately preceding $y$ on $P$. Thus $d_D(v,x) \leq r - 1$ and $(x,y) \in E(D^{r-1})$. Therefore, $d_D(v,y) = r$ and $y \in N_{D^r}(v)$, a contradiction.

Among the sufficient conditions that exist for a digraph to be Hamiltonian is the following due to Ghouila-Houri [5].

**Theorem 2.2** (Ghouila-Houri’s Theorem). If $D$ is a strong digraph of order $n$ such that od$v + idv \geq n$ for every vertex $v$ of $D$, then $D$ is Hamiltonian.

As a consequence of Lemma 2.1 and Ghouila-Houri’s theorem, we have the following.

**Theorem 2.3.** For every strong digraph $D$ of order $n \geq 2$ and every integer $k \geq \lceil n/2 \rceil$, the digraph $D^k$ is Hamiltonian. Furthermore, the lower bound $\lceil n/2 \rceil$ is sharp.

**Proof.** Let $d$ be the diameter of $D$. If $k > d$, then $D^d$ is the complete symmetric digraph of order $n$ and so $D^k$ is Hamiltonian. Thus, we may assume that $1 \leq
By Lemma 2.1, \( od_{D^k} v \geq \lceil n/2 \rceil \) and \( id_{D^k} v \geq \lceil n/2 \rceil \) for every vertex \( v \) of \( D \). Therefore, \( od_{D^k} v + id_{D^k} v \geq 2\lceil n/2 \rceil \geq n \). By Ghouila-Houri’s theorem, \( D^k \) is Hamiltonian. Thus, it remains to show that the lower bound \( \lceil n/2 \rceil \) is sharp. For a given integer \( k \geq 3 \), consider the strong oriented graph \( D_k \) shown in Figure 1. (If \( k = 3 \), then we replace the (directed) \( u - v \) path \( (u, v_1, v_2, \ldots, v_{k-3}, v) \) by the arc \( (u, v) \).

![Figure 1. The strong oriented graph \( D_k \) in the proof of Theorem 2.3.](image)

Since the order of \( D_k \) is \( n = 2k - 1 \), it follows by the first statement in this theorem that the \( k \)th power of \( D_k \) is Hamiltonian. The diameter of \( D_k \) is \( k \). In fact, the only vertices \( y \) and \( z \) in \( D_k \) for which \( d_D(y, z) = k \) are distinct vertices of \( \{x_1, x_2, \ldots, x_k\} \). In fact, if we let \( G = \overline{K}_k + K_{k-1} \) (the join of \( \overline{K}_k \) and \( K_{k-1} \)), then \( D_k^{k-1} = G^* \) (the complete symmetric digraph with underlying graph \( G \)). Because \( G \) is not Hamiltonian, it follows that \( D_k^k \) is Hamiltonian but \( D_k^{k-1} \) is not. Therefore, the lower bound \( \lceil n/2 \rceil \) is sharp.

By Theorem 2.3, unlike the situation for connected graphs of order at least 3, there is no fixed constant \( c \) such that \( D^c \) is Hamiltonian for every strong oriented graph \( D \).

### 3. Distance-colored Digraphs

For a strong oriented graph \( D \) and a positive integer \( k \), the \( k \)th power \( D^k \) is called distance-colored if each arc \( (u, v) \) of \( D^k \) is assigned the color \( i \) if \( d_D(u, v) = i \). The digraph \( D^k \) is called Hamiltonian-colored if \( D^k \) contains a properly colored Hamiltonian cycle \( C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1) \), that is, the colors of \( (v_i, v_{i+1}) \) and \( (v_{i+1}, v_{i+2}) \) are distinct for \( 1 \leq i \leq n \), where \( v_{n+2} = v_2 \).

If \( D \) is a strong oriented graph such that the distance-colored digraph \( D^2 \) is Hamiltonian-colored, then \( D \) must have even order \( n \). The only strong digraph
of order 2 is $K_2^*$, which is not an oriented graph. There is also no strong oriented graph $D$ of order 4 for which $D^2$ is Hamiltonian-colored, for suppose, to the contrary, that such a digraph $D$ exists and $C = (u, v, w, x, u)$ is a properly colored Hamiltonian cycle in $D^2$, where $(u, v)$ and $(w, x)$ are colored 1 and $(v, w)$ and $(x, u)$ are colored 2 (see Figure 2). Since $(v, w)$ belongs to $D^2$ but not $D$, $(v, w) \notin E(D)$. Because $D$ is strong and an oriented graph, $(v, x) \in E(D)$. Similarly, $(x, v) \notin E(D)$. However then, $D$ is not an oriented graph, a contradiction. The situation for the orders 2 and 4 are the exceptions, however, as we now see.

![Figure 2. Showing that the square of no strong oriented graph of order 4 is Hamiltonian-colored.](image)

![Figure 3. The strong oriented graph $D$ (for $k = 5$) in the proof of Theorem 3.1.](image)

**Theorem 3.1.** For every even integer $n \geq 6$, there exists a strong oriented graph $D$ of order $n$ such that $D^2$ is Hamiltonian-colored.

**Proof.** Let $D$ be the strong oriented graph of order $n = 2k \geq 6$ and size $3k$ for which $V(D) = \{v_1, v_2, \ldots, v_{2k}\}$ and $E(D) = \{(v_{2i-1}, v_{2i}) : 1 \leq i \leq k\} \cup \{(v_{2k+3-2i}, v_{2i+1-2i}) : 1 \leq i \leq k\} \cup \{(v_{2i}, v_{2i+3}) : 1 \leq i \leq k\}$, where $v_{2k+1} = v_1$ and $v_{2k+3} = v_3$. (The digraph $D$ is shown in Figure 3 for the case where $k = 5$.) In $D^2$, the Hamiltonian cycle $(v_1, v_2, \ldots, v_{2k}, v_1)$ is properly colored. 

If $D$ is a strong oriented graph such that $D^k$ is Hamiltonian-colored for some positive integer $k$, then the smallest such integer $k$ is defined as the Hamiltonian coloring exponent $hce(D)$ of $D$. Thus if $hce(D) = k$, then $D^{k-1}$ is not Hamiltonian-colored. In particular, Theorem 3.1 shows that if $D$ is a strong oriented graph such that $D^2$ is Hamiltonian-colored, then $hce(D) = 2$.

4. Hamiltonian Coloring Exponents of Directed Cycles

We now determine $hce(\vec{C}_n)$ for the directed cycle $\vec{C}_n$ of order $n \geq 3$. Since $diam(\vec{C}_n) = n - 1$, it follows that if $hce(\vec{C}_n)$ exists, then $2 \leq hce(\vec{C}_n) \leq n - 1$. Let $D = \vec{C}_n$ where $n \geq 3$. If $hce(\vec{C}_n)$ exists, let $hce(D) = k$. Then $D^k$ contains a properly colored Hamiltonian cycle $C' = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ where $1 \leq d_D(v_i, v_{i+1}) \leq k$ for $i = 1, 2, \ldots, n$. Let $d_D(v_i, v_{i+1}) = a_i$ for $1 \leq i \leq n$. Thus, corresponding to the properly colored directed cycle $C'$ is the sequence $s : a_1, a_2, \ldots, a_n$ of colors where $a_i \in \{1, 2, \ldots, k\}$ for $1 \leq i \leq n$. Since $C'$ starts and ends at $v_1$, it follows that $C'$ proceeds around $\vec{C}_n$ a certain number of times, say $p$, and so $\sum_{i=1}^{n} d_D(v_i, v_{i+1}) = \sum_{i=1}^{n} a_i = pn$.

For a cyclic sequence $s : a_1, a_2, \ldots, a_n$ of length $n$ and any integer $t$ with $1 \leq t \leq n$, the sequence $s$ can also be expressed as $s : a_t, a_{t+1}, \ldots, a_n, a_1, \ldots, a_{t-1}$. A proper subsequence $s^*$ of $s$ is defined as a sequence $s^* : a_t, a_{t+1}, \ldots, a_{t+n^*-1}$ of length $n^*$, where $1 \leq n^* < n$ and the subscripts are expressed as integers modulo $n$. There is no proper subsequence $s^* : a_t, a_{t+1}, \ldots, a_{t+q-1}$ of $s$ for which $\sum_{i=t}^{t+q-1} a_i$ is a multiple of $n$, for otherwise, the cycle $C^* = (v_1, v_{t+1}, \ldots, v_{t+q-1}, v_{t-q} = v_1)$ is a cycle of length $q < n$ that is a proper subdiagram of the Hamiltonian cycle $C'$, which is impossible. Consequently, $s : a_1, a_2, \ldots, a_n$ where $a_i \in \{1, 2, \ldots, k\}$ for $1 \leq i \leq n$ is a cyclic sequence of colors of a Hamiltonian-colored digraph $D^k$ with $hce(D) = k$ if and only if

1. no two consecutive terms in $s$ are equal,
2. $\sum_{i=1}^{n} a_i$ is a multiple of $n$ and
3. the sum of the terms in no proper subsequence of $s$ is a multiple of $n$.

Any cyclic sequence $s : a_1, a_2, \ldots, a_n$ of terms $a_i \in \{1, 2\}$ for $1 \leq i \leq n$ satisfying condition (1) has the property that $n < \sum_{i=1}^{n} a_i < 2n$. Thus condition (2) is not satisfied. Therefore, we have the following observation.

Observation 4.1. Let $n \geq 3$ be an integer. If $hce(\vec{C}_n)$ exists, then $hce(\vec{C}_n) \geq 3$.

Since $diam(\vec{C}_3) = 2$, it follows by Observation 4.1 that $hce(\vec{C}_3)$ does not exist. On the other hand, if $\vec{C}_4 = (v_1, v_2, v_3, v_4, v_1)$, then $C' = (v_1, v_2, v_3, v_1)$ is a properly colored Hamiltonian cycle in the cube of $\vec{C}_4$ and so $hce(\vec{C}_4) = 3$. Corresponding
to \( C' \) is the cyclic sequence \( s : 1, 2, 3, 2 \) of colors. In fact, not only is \( \text{hce}(\vec{C}_4) = 3 \) but \( \text{hce}(\vec{C}_n) = 3 \) for all even integers \( n \geq 4 \), as we show next.

**Theorem 4.2.** For every even integer \( n \geq 4 \), \( \text{hce}(\vec{C}_n) = 3 \).

**Proof.** We have already observed that \( \text{hce}(\vec{C}_4) = 3 \) and \( \text{hce}(\vec{C}_n) \geq 3 \) for all integers \( n \geq 3 \) (if \( \text{hce}(\vec{C}_n) \) exists). Thus, it remains only to show that there is a cyclic sequence \( s : a_1, a_2, \ldots, a_n \) of \( n \geq 6 \) terms with \( n \) even and \( a_i \in \{1, 2, 3\} \) for \( 1 \leq i \leq n \) satisfying conditions (1)–(3). We consider two cases.

**Case 1.** \( n \equiv 2(\text{mod } 4) \). So \( n = 4r + 2 \) for \( r \geq 1 \). Consider the cyclic sequence \( s : 1, 3, 1, 3, \ldots, 1, 3 \) of \( 4r + 2 \) terms. Then the sum of the terms of \( s \) is \( 8r + 4 = 2n \). Since the sum of the terms of any subsequence of \( s \) is either odd or a multiple of 4, this sum is not \( n \).

**Case 2.** \( n \equiv 0(\text{mod } 4) \). So \( n = 4r \) for \( r \geq 2 \). Consider the cyclic sequence \( s : 1, 3, 1, 3, \ldots, 1, 3, 2 \) of \( 4r \) terms where there are \( 2r - 1 \) terms between the occurrences of 2 in \( s \). Then the sum of the terms of \( s \) is \( 8r = 2n \). Now observe that the sum of the terms of any subsequence

- (i) containing both terms 2 exceeds \( n \),
- (ii) containing neither term 2 is less than \( n \) and
- (iii) containing exactly one term 2 is either odd or is congruent to 2 modulo 4 and consequently is not \( n \).

We now consider \( \text{hce}(\vec{C}_n) \) where \( n \geq 3 \) is odd. We saw that \( \text{hce}(\vec{C}_3) \) does not exist. In fact, \( \text{hce}(\vec{C}_5) \) does not exist either.

**Proposition 4.3.** The number \( \text{hce}(\vec{C}_5) \) does not exist.

**Proof.** Let \( D = \vec{C}_5 \). Assume, to the contrary, that \( \text{hce}(D) \) exists. By Observation 4.1, \( 3 \leq \text{hce}(D) \leq \text{diam}(D) = 4 \), that is, either \( \text{hce}(D) = 3 \) or \( \text{hce}(D) = 4 \).

If \( \text{hce}(D) = 3 \), then there exists a cyclic sequence \( s : a_1, a_2, a_3, a_4, a_5 \) with \( a_i \in \{1, 2, 3\}, 1 \leq i \leq 5 \), satisfying (1)–(3). Necessarily, some term, say \( a_2 \), is 3. If either \( a_1 = 2 \) or \( a_3 = 2 \), then either \( a_1 + a_2 = 5 \) or \( a_2 + a_3 = 5 \), which is impossible. Thus \( a_1 = a_3 = 1 \). However then, \( a_1 + a_2 + a_3 = 5 \), also impossible.

If \( \text{hce}(D) = 4 \), then there exists a cyclic sequence \( s : a_1, a_2, a_3, a_4, a_5 \) with \( a_i \in \{1, 2, 3, 4\}, 1 \leq i \leq 5 \), satisfying (1)–(3). Necessarily, some term, say \( a_3 \), is 4. Neither \( a_2 \) nor \( a_4 \) is 1, for otherwise, either \( a_2 + a_3 = 5 \) or \( a_3 + a_4 = 5 \), which is impossible. Also, we cannot have \( a_2 = a_4 = 3 \) for then \( a_2 + a_3 + a_4 = 10 \), also impossible. Thus, one of \( a_2 \) and \( a_4 \) is 2 and the other is 2 or 3. First, suppose that \( a_2 = 3 \) and \( a_4 = 2 \). Now \( a_5 \neq 1 \), for otherwise, \( a_2 + a_3 + a_4 + a_5 = 10 \), which is impossible. Also, \( a_5 \neq 3 \), for otherwise, \( a_4 + a_5 = 5 \). Finally, \( a_5 \neq 4 \), for otherwise, \( a_3 + a_4 + a_5 = 10 \). Thus, this case cannot occur. Next suppose that \( a_2 = a_4 = 2 \). Neither \( a_1 = 4 \) nor \( a_5 = 4 \) for otherwise, either \( a_1 + a_2 + a_3 = 10 \)
or \( a_3 + a_4 + a_5 = 10 \). Also, neither \( a_1 = 3 \) nor \( a_5 = 3 \), for otherwise, \( a_1 + a_2 = 5 \) or \( a_4 + a_5 = 5 \). Consequently, \( a_1 = a_5 = 1 \), which contradicts (1). Again, this is impossible.

On the other hand, \( \text{hce}(\vec{C}_n) \) exists for each odd integer \( n \geq 7 \). First, we present a lemma.

**Lemma 4.4.** For every odd integer \( n \geq 7 \), \( \text{hce}(\vec{C}_n) \neq 3 \).

**Proof.** Assume, to the contrary, that there is an odd integer \( n \geq 7 \) such that \( \text{hce}(\vec{C}_n) = 3 \). Let \( D = \vec{C}_n = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1) \). Hence there exists a properly colored Hamiltonian cycle \( C' = (v_1, v_2, \ldots, v_n, u_{n+1} = u_1) \) in \( D^3 \), where \( u_1 = v_1 \) and where \( C' \) proceeds about \( \vec{C}_n \) twice. If \( s : a_1, a_2, \ldots, a_n \) is the corresponding cyclic sequence of colors for \( C' \), then no two consecutive terms in \( s \) are equal, \( \sum_{i=1}^n a_i = 2n \) and no proper subsequence of \( s \) has the property that the sum of its terms is \( n \). Since \( C' \) is an odd cycle, all three colors 1, 2 and 3 must appear in \( s \). Furthermore, since the sum \( \sum_{i=1}^n a_i \) is even and the average term in this sum is 2, the colors 1 and 3 must appear an equal number of times, implying that the color 2 must appear an odd number of times in \( s \).

First, we show that neither 1, 2, 1 nor 3, 2, 3 can occur as a subsequence of \( s \). If 1, 2, 1 occurs as a subsequence of \( s \), then \( C' \) contains the path \( (v_i, v_{i+1}, v_{i+3}, v_{i+4}) \) for some \( i \) with \( 1 \leq i \leq n \) where the subscripts are expressed as integers modulo \( n \). This, however, implies that \( (v_{i-1}, v_{i+2}, v_{i+5}) \) is a path on \( C' \) and that 3, 3 is a subsequence of \( s \), which is impossible. If 3, 2, 3 occurs as a subsequence of \( s \), then \( C' \) contains the path \( (v_i, v_{i+3}, v_{i+5}, v_{i+8}) \) for some \( i \) (\( 1 \leq i \leq n \)). Since \( C' \) proceeds about \( \vec{C}_n \) twice, \( (v_{i+1}, v_{i+2}, v_{i+4}, v_{i+6}, v_{i+7}) \) is also a path on \( C' \) and so 1, 2, 2, 1 is a subsequence of \( s \), which is impossible.

Therefore, each occurrence of the color 2 in \( s \) must occur as 1, 2, 3 or 3, 2, 1. If 1, 2, 3 occurs in \( s \), then \( C' \) contains the path \( (v_i, v_{i+1}, v_{i+3}, v_{i+6}) \) for some \( i \) (\( 1 \leq i \leq n \)), implying that \( C' \) also contains \( (v_{i-1}, v_{i+2}, v_{i+4}, v_{i+5}) \) and so 3, 2, 1 is a subsequence (later) in \( s \). Similarly, if 3, 2, 1 occurs in \( s \), then 1, 2, 3 occurs (later) in \( s \). That is, the subsequences 1, 2, 3 and 3, 2, 1 occur in pairs in \( s \), implying that 2 appears an even number of times in \( s \), which is a contradiction.

We next show that \( \text{hce}(\vec{C}_7) = \text{hce}(\vec{C}_9) = 5 \), beginning with \( \text{hce}(\vec{C}_7) = 5 \).

**Proposition 4.5.** \( \text{hce}(\vec{C}_7) = 5 \).

**Proof.** Let \( D = \vec{C}_7 = (v_1, v_2, \ldots, v_7, v_1) \). Since the cyclic sequence
\[
s : 1, 5, 3, 2, 1, 5, 4
\]
corresponds to the properly colored Hamiltonian cycle
\[
(v_1, v_2, v_7, v_3, v_5, v_6, v_4, v_1)
\]
in \( D^5 \), it follows that \( \text{hce}(\vec{C}_7) \leq 5 \). By Lemma 4.4, \( \text{hce}(\vec{C}_7) \geq 4 \). Thus \( \text{hce}(\vec{C}_7) = 4 \) or \( \text{hce}(\vec{C}_7) = 5 \). We show that \( \text{hce}(\vec{C}_7) = 5 \).
Assume, to the contrary, that $\text{hce}(\vec{C}_7) = 4$. Then $D^4$ contains a properly colored Hamiltonian cycle $C'$. Corresponding to $C'$ is a cyclic sequence of colors $s : a_1, a_2, \ldots, a_7$, where $\sum_{i=1}^7 a_i = 14$ or $\sum_{i=1}^7 a_i = 21$. Necessarily, at least one of the terms in $s$ is the color 4, say $a_4 = 4$. Since the sum of the terms in no proper subsequence of $s$ is a multiple of 7, it follows that (1) neither $a_3$ nor $a_5$ is 3 and (2) $\{a_3, a_5\} \neq \{1, 2\}$. Hence either $a_3 = a_5 = 1$ or $a_3 = a_5 = 2$. First, assume that $a_3 = a_5 = 1$. Thus either $a_1 + a_2 + a_6 + a_7 = 8$ or $a_1 + a_2 + a_6 + a_7 = 15$. Since no two consecutive terms in $s$ are 4, it follows that $a_1 + a_2 + a_6 + a_7 = 8$.

If one of the colors $a_1, a_2, a_6$ and $a_7$ is 4, then two of them are 1, contradicting the assumption of the case. Again, the assumption of the case implies that no two the colors $a_1, a_2, a_6, a_7$ can be 1. Consequently, we may assume that $s : 1, 2, 1, 4, 1, 3, 2$. Since $a_2 + a_3 + a_4 = 7$, a contradiction is produced. Next, assume that $a_3 = a_5 = 2$. First, we observe that neither $a_2$ nor $a_6$ is 1 since the sum of the terms in no proper subsequence of $s$ is 7. Also, since the sum of the terms in no proper subsequence of $s$ is 14, it cannot occur that $a_2 = a_6 = 3$. Therefore, either $a_2 = a_6 = 4$ or we may assume that $a_2 = 3$ and $a_6 = 4$. If $a_2 = a_6 = 4$, then $a_1 \notin \{1, 2, 3, 4\}$, for otherwise, the sum of the terms in a proper subsequence of $s$ is a multiple of 7; if $a_2 = 3$ and $a_6 = 4$, then $a_7 \notin \{1, 2, 3, 4\}$, a contradiction.

**Proposition 4.6.** $\text{hce}(\vec{C}_9) = 5$.

**Proof.** Let $D = \vec{C}_9 = (v_1, v_2, \ldots, v_9, v_1)$. Since the cyclic sequence $s : 1, 4, 3, 4, 3, 5, 2, 3, 2$ corresponds to the properly colored Hamiltonian cycle

$$(v_1, v_2, v_6, v_9, v_4, v_7, v_3, v_5, v_8, v_1)$$

in $D^5$, it follows that $\text{hce}(\vec{C}_9) \leq 5$. By Lemma 4.4, $\text{hce}(\vec{C}_9) \geq 4$. Thus $\text{hce}(\vec{C}_9) = 4$ or $\text{hce}(\vec{C}_9) = 5$. We show that $\text{hce}(\vec{C}_9) = 5$.

Assume, to the contrary, that $\text{hce}(\vec{C}_9) = 4$. Then $D^4$ contains a properly colored Hamiltonian cycle $C'$. Corresponding to $C'$ is a cyclic sequence of colors $s : a_1, a_2, \ldots, a_9$, where $\sum_{i=1}^9 a_i = 18$ or $\sum_{i=1}^9 a_i = 27$. (There is no proper subsequence of $s$, the sum of whose terms is a multiple of 9.) We consider two cases.

**Case 1.** $\sum_{i=1}^9 a_i = 18$. Then the cycle $C'$ proceeds about $\vec{C}_9$ exactly twice. Since at least one of the terms in $s$ is the color 4, we may assume that $(v_1, v_5)$ is a path on $C'$. However then, $(v_2, v_3, v_4)$ is also path on $C'$, implying that $1, 1$ is a subsequence of $s$, which is impossible.

**Case 2.** $\sum_{i=1}^9 a_i = 27$. Consider the three subsequences of $s$,

$s_1 : a_1, a_2, a_3, s_2 : a_4, a_5, a_6, s_3 : a_7, a_8, a_9$,

where $\sigma_i$ is the sum of the terms in $s_i$ for $i = 1, 2, 3$. Necessarily, no $\sigma_i$ has the value 9. Since $\sigma_1 + \sigma_2 + \sigma_3 = 27$, two of the numbers $\sigma_1, \sigma_2, \sigma_3$ exceed 9 or two.
are less than 9. First assume that two of the numbers \( \sigma_1, \sigma_2, \sigma_3 \) exceed 9, say \( \sigma_1 \) and \( \sigma_2 \). Thus each of \( \sigma_1 \) and \( \sigma_2 \) is 10 or 11. If \( \sigma_1 = 11 \), then \( s_1 : 4, 3, 4 \). If \( \sigma_1 = 10 \), then \( s_1 : 4, 2, 4 \) or \( s_1 : 3, 4, 3 \). Since \( a_3 \neq a_4 \), we may assume that \( s_1 : 3, 4, 3 \) and either \( s_2 : 4, 2, 4 \) or \( s_2 : 4, 3, 4 \). Since \( a_3 + a_4 + a_5 \neq 9 \), it follows that \( s_1 : 3, 4, 3 \) and \( s_2 : 4, 3, 4 \). Thus \( \sigma_3 = 6 \), which implies that \( a_7 + a_8 + a_9 + a_1 = 9 \), producing a contradiction. Next, assume that two of the numbers \( \sigma_1, \sigma_2, \sigma_3 \) are less than 9, say \( \sigma_1 \) and \( \sigma_3 \). Thus \( \sigma_2 = 11 \), which implies that \( s_2 : 4, 3, 4 \). Hence \( \sigma_1 = \sigma_3 = 8 \). Consequently, \( s_1 \) is one of (1) 4, 3, 1, (2) 4, 1, 3 or (3) 1, 4, 3; while \( s_3 \) is one of (1') 1, 3, 4, (2') 3, 1, 4 or (3') 3, 4, 1. Since \( a_1 \neq a_9, a_9 + a_1 + a_2 \neq 9 \) and \( a_8 + a_9 + a_1 \neq 9 \), none of these are possible.

We now show that \( hce(C_n) = 5 \) for each odd integer \( n \geq 7 \).

Figure 4. Properly colored Hamiltonian cycles in the 5th powers of \( \tilde{C}_{11} \) and \( \tilde{C}_{17} \).

**Theorem 4.7.** For every odd integer \( n \geq 7 \), \( hce(C_n) = 5 \).

**Proof.** Let \( D = \tilde{C}_n = (v_1, v_2, \ldots, v_n, v_1) \). We have seen by Propositions 4.5 and 4.6 that \( hce(C_7) = hce(C_9) = 5 \). Hence we may assume that \( n \geq 11 \). We first show that \( hce(C_n) \leq 5 \). There are three cases, according to whether \( n \) is congruent to 5, 1 or 3 modulo 6.

**Case 1.** \( n \equiv 5 \pmod{6} \). First, observe that the cyclic sequence
\[
s_{11} : 5, 1, 3, 4, 2, 3, 5, 2, 5, 2, 1
\]
corresponds to the properly colored Hamiltonian cycle
\[
C'_{11} = (v_1, v_6, v_7, v_{10}, v_3, v_5, v_8, v_2, v_4, v_9, v_{11}, v_1)
\]
shown in Figure 4(a) in the 5th power of \( \tilde{C}_{11} \); while the cyclic sequence
\[
s_{17} : 5, 1, 5, 1, 3, 4, 2, 4, 2, 3, 5, 2, 4, 2, 5, 2, 1
\]
corresponds to the properly colored Hamiltonian cycle
\[
(1) \quad C'_{17} = (v_1, v_6, v_7, v_{12}, v_{16}, v_3, v_5, v_9, v_{11}, v_{14}, v_2, v_4, v_8, v_{10}, v_{15}, v_{17}, v_1)
\]
shown in Figure 4(b) in the 5th power of $\vec{C}_{17}$. Thus $hce(\vec{C}_{11}) \leq 5$ and $hce(\vec{C}_{17}) \leq 5$. For the cycle $C'_{17}$ (in (1) and in Figure 4(b)), let $n = 17$ and relabel $v_i$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Properly colored Hamiltonian cycles in the 5th powers of $\vec{C}_{13}$ and $\vec{C}_{19}$.}
\end{figure}

(1 $\leq i \leq 17 = n$) as $v_{i+6}$ and delete the arcs $(v_{n+6}, v_7), (v_{n+5}, v_9), (v_{n+3}, v_8)$. We next add vertices $v_1, v_2, \ldots, v_6$ along with all arcs of $C'_{17}$ incident with and directed away from $v_1, v_2, \ldots, v_6$. Finally, we add the arcs $(v_{n+6}, v_1), (v_{n+5}, v_3), (v_{n+3}, v_2)$. This produces a properly colored Hamiltonian cycle $C'$ for the 5th power of $\vec{C}_{23}$. Corresponding to this cycle is the cyclic sequence

$s' : 5, 1, 5, 1, 3, 4, 2, 4, 2, 4, 2, 3, 5, 2, 4, 2, 5, 2, 1$.

By first letting $n = 23$ and then proceeding successively as above, we obtain a properly colored Hamiltonian cycle in the 5th power of $\vec{C}_n$ for each $n \geq 29$ such that $n \equiv 5 \pmod{6}$. Such a cycle also corresponds to the cyclic sequence obtained by inserting in $s'$ (a) the sequence 5, 1 between 5, 1 and 3, 4, (b) the sequence 2, 4 between 2, 4 and 2, 3 and (c) the sequence 2, 4 between 2, 4 and 2, 5.

\textbf{Case 2.} $n \equiv 1 \pmod{6}$. First, observe that the cyclic sequence

$s_{13} : 5, 1, 5, 1, 3, 1, 4, 3, 4, 3, 4, 1, 4$

corresponds to the properly colored Hamiltonian cycle

$C'_{13} = (v_1, v_6, v_7, v_12, v_13, v_3, v_4, v_8, v_11, v_2, v_5, v_9, v_{10}, v_1)$

shown in Figure 5(a) in the 5th power of $\vec{C}_{13}$; while the cyclic sequence

$s_{19} : 5, 1, 5, 1, 5, 1, 3, 1, 4, 2, 4, 3, 4, 3, 4, 2, 4, 1, 4$

corresponds to the properly colored Hamiltonian cycle

\begin{equation}
C'_{19} = (v_1, v_6, v_7, v_{12}, v_{13}, v_{18}, v_{19}, v_3, v_4, v_8, v_{10}, v_{14}, v_{17}, v_5, v_9, v_{11}, v_{15}, v_{16}, v_1)
\end{equation}

shown in Figure 5(b) in the 5th power of $\vec{C}_{19}$. Thus $hce(\vec{C}_{13}) \leq 5$ and $hce(\vec{C}_{19}) \leq 5$. For the cycle $C'_{19}$ (in (2) and in Figure 5(b)), let $n = 19$ and relabel $v_i$
We next add vertices \( v_1, v_2, \ldots, v_6 \) along with all arcs of \( C_9 \) incident with and directed away from \( v_1, v_2, \ldots, v_6 \). Finally, we add the arcs \((v_{n+6}, v_1), (v_{n+5}, v_3), (v_{n+4}, v_2)\). This produces a properly colored Hamiltonian cycle \( C' \) for the 5th power of \( \tilde{C}_{25} \). Corresponding to this cycle is the cyclic sequence

\[
s' : 5, 1, 5, 1, 5, 1, 3, 1, 4, 2, 4, 2, 4, 3, 4, 2, 4, 1, 4.
\]

By first letting \( n = 25 \) and then proceeding successively as above, we obtain a properly colored Hamiltonian cycle in the 5th power of \( \tilde{C}_n \) for every integer \( n \geq 31 \) such that \( n \equiv 1 \pmod{6} \). Such a cycle also corresponds to the cyclic sequence obtained by inserting in \( s' \) (a) the sequence 5, 1 between between 5, 1 and 3, 1, 4, (b) the sequence 2, 4 after 3, 1, 4 and (c) the sequence 2, 4 after 3, 4, 3, 4.

**Case 3.** \( n \equiv 3 \pmod{6} \). First, observe that the cyclic sequence

\[
s_{15} : 5, 1, 5, 1, 5, 1, 4, 2, 5, 2, 3, 4, 2, 3, 2
\]

corresponds to the properly colored Hamiltonian cycle

\[
C'_{15} = (v_1, v_6, v_7, v_{12}, v_{13}, v_3, v_4, v_8, v_{10}, v_{15}, v_2, v_5, v_9, v_{11}, v_{14}, v_1)
\]

shown in Figure 6(a) in the 5th power of \( \tilde{C}_{15} \); while the cyclic sequence

\[
s_{21} : 5, 1, 5, 1, 5, 1, 5, 1, 4, 2, 4, 2, 5, 2, 3, 4, 2, 4, 3, 2
\]

corresponds to the properly colored Hamiltonian cycle

\[
(3) \quad C'_{21} = (v_1, v_6, v_7, v_{12}, v_{13}, v_{18}, v_{19}, v_3, v_4, v_8, v_{10}, v_{14}, v_{16}, v_{21}, v_2, v_5, v_9, v_{11}, v_{15}, v_{17}, v_{20}, v_1)
\]

shown in Figure 6(b) in the 5th power of \( \tilde{C}_{21} \). Thus \( \text{hce}(\tilde{C}_{15}) \leq 5 \) and \( \text{hce}(\tilde{C}_{21}) \leq 5 \).

Figure 6. Properly colored Hamiltonian cycles in the 5th powers of \( \tilde{C}_{15} \) and \( \tilde{C}_{21} \).

For the cycle \( C'_{21} \) (in (3) and in Figure 6(b)), let \( n = 21 \) and relabel \( v_i \) (\( 1 \leq i \leq 21 = n \)) as \( v_{i+6} \) and delete the arcs \((v_{n+6}, v_8), (v_{n+5}, v_7), (v_{n+4}, v_9)\). We next
add vertices \( v_1, v_2, \ldots, v_6 \) along with all arcs of \( C'_{21} \) incident with and directed away from \( v_1, v_2, \ldots, v_6 \). Finally, we add the arcs \((v_{n+6}, v_2), (v_{n+5}, v_1), (v_{n+4}, v_3)\).

This produces a properly colored Hamiltonian cycle \( C' \) for the 5th power of \( C_n \). Corresponding to this cycle is the cyclic sequence

\[ s' : 5, 1, 5, 1, 5, 1, 5, 1, 5, 1, 4, 2, 4, 2, 4, 2, 5, 2, 3, 4, 2, 4, 2, 3, 2. \]

By first letting \( n = 27 \) and then proceeding successively as above, we obtain a properly colored Hamiltonian cycle in the 5th power of \( C_n \) for every integer \( n \) such that \( n \geq 33 \) and \( n \equiv 3 \pmod{6} \). Such a cycle also corresponds to the cyclic sequence obtained by inserting in \( s' \) (a) the sequence 5, 1 between 5, 1 and 4, 2, (b) the sequence 2, 4 between 2, 4 and 2, 5 and (c) the sequence 2, 4 between 2, 4 and 2, 3.

Next, we show that \( \text{hce}(\vec{C}_n) \geq 5 \). We have seen by Lemma 4.4 that \( \text{hce}(\vec{C}_n) \geq 4 \) for every odd integer \( n \geq 7 \). Thus it remains only to show that \( \text{hce}(\vec{C}_n) \neq 4 \) for all such integers \( n \). Assume, to the contrary, that the distance-colored digraph \( D^4 \) contains a properly colored Hamiltonian cycle \( C \), which we assume begins and ends at \( v_1 \). Thus, the arcs of \( C \) are colored with elements of the set \( \{1, 2, 3, 4\} \).

Since \( \text{hce}(\vec{C}_n) \geq 4 \), at least one arc of \( C \) is colored 4, say \((v_i, v_{i+4})\) is colored 4 for some \( i \). If the cycle \( C \) proceeds about \( \vec{C}_n \) only twice, then \( C \) must contain the path \((v_{i+1}, v_{i+2}, v_{i+3})\), which implies that two consecutive arcs of \( C \) are colored 1, which is impossible. Consequently, \( C \) proceeds about \( \vec{C}_n \) exactly three times.

We claim that no arc of \( C \) is colored 1. Suppose that this is not the case. Then one or more arcs of \( C \) are colored 1. We may assume that \((v_1, v_2)\) is colored 1 and this is the first arc of \( C \). Thus \((v_2, v_3)\) is not an arc of \( C \). Let \( v_{k+1} \) \((2 \leq k \leq n)\) be the next vertex of \( C \) that is incident with an arc colored 1, where \( v_{n+1} = v_1 \).

Therefore, no arc of \( C \) that is incident with any of \( v_3, v_4, \ldots, v_k \) is colored 1. We refer to the set \( \{v_1, v_2, \ldots, v_k\} \) of vertices as a block of \( C \), where the block is even or odd according to whether \( k \) is even or odd. We show that this block is even.

First, we show that \((v_n, v_4)\) and \((v_n-1, v_3)\) are arcs of \( C \). Certainly, \( v_n \) is adjacent to either \( v_3 \) or \( v_4 \). If \((v_n, v_3)\) is an arc of \( C \), then \((v_1, v_2)\) and \((v_n, v_3)\) belong to two of the three distinct paths that pass by \( v_1 \) as we proceed about \( \vec{C}_n \) on \( C \). However then, the third path that passes by \( v_1 \) must contain an arc \((v_j, v_1)\), where \( j < n \) and \( \ell > 3 \), which is impossible. Hence \((v_n, v_4)\) is an arc on \( C \), which implies that \((v_{n-1}, v_3)\) is an arc on \( C \).

In summary, the cycle \( C \) contains the arc \((v_1, v_2)\) colored 1 and the arcs \((v_n, v_4)\) and \((v_{n-1}, v_3)\), both colored 4. The vertex \( v_2 \) is adjacent to either \( v_5 \) or \( v_4 \). We consider these two cases.

**Case 1.** \((v_2, v_5)\) is an arc on \( C \). In this case, \((v_3, v_6)\) and \((v_4, v_7)\) are arcs of \( C \). This implies that \((v_5, v_9)\) is an arc of \( C \) (see Figure 7). The vertex \( v_{n-2} \) is adjacent to either \( v_{n-1}, v_n \) or \( v_1 \). We consider these three subcases.

**Subcase 1.1.** \((v_{n-2}, v_{n-1})\) is an arc of \( C \). Since \((v_{n-2}, v_{n-1})\) is an arc of \( C \) colored 1, it follows by the previous discussion that \((v_{n-3}, v_n)\) is not an arc of \( C \).
and so \((v_{n-3}, v_1)\) is an arc of \(C\). This, however, implies that \((v_{n-4}, v_n)\) is an arc of \(C\) colored 4, which is impossible since \((v_n, v_4)\) is also an arc of \(C\) colored 4.

**Subcase 1.2.** \((v_{n-2}, v_n)\) is an arc of \(C\). If \((v_7, v_8)\) is an arc of \(C\) colored 1, then the block is even. Thus, we may assume that \((v_7, v_8)\) is not an arc of \(C\). This implies that \((v_7, v_{11})\) is an arc of \(C\). If \(n = 11\), then a contradiction is produced since \((v_n, v_4) = (v_{11}, v_4)\) is also an arc of \(C\). Thus, \(n \geq 13\) and then \((v_6, v_8)\) must be an arc of \(C\). This implies that \((v_8, v_{10})\) cannot be an arc of \(C\). Thus \((v_9, v_{10})\) is an arc of \(C\) colored 1 and the block is even.

**Subcase 1.3.** \((v_{n-2}, v_1)\) is an arc of \(C\). If \((v_7, v_8)\) is an arc of \(C\), then the block is even; otherwise, \((v_7, v_{11})\) is an arc of \(C\). As we saw in Subcase 1.2, a contradiction is produced if \(n = 11\). Thus, \(n \geq 13\). In this case, \((v_6, v_8)\) and \((v_8, v_{12})\) are arcs of \(C\). From this, it follows that \((v_9, v_{10})\) is an arc of \(C\) and the block is even.

**Case 2.** \((v_2, v_6)\) is an arc of \(C\). In this case, \((v_3, v_5)\) and \((v_4, v_7)\) are also arcs of \(C\). See Figure 8. Then \(v_5\) is adjacent to either \(v_8\) or \(v_9\). We consider these two subcases.

**Subcase 2.1.** \((v_5, v_8)\) is an arc of \(C\). Here, both \((v_6, v_9)\) and \((v_7, v_{11})\) are arcs of \(C\). Again, if \(n = 11\), then a contradiction is produced since \((v_n, v_4) = (v_{11}, v_4)\) is an arc of \(C\). Thus, \(n \geq 13\). If \((v_9, v_{10})\) is an arc of \(C\), then the block is even; otherwise, \((v_9, v_{13})\) is an arc of \(C\) as is \((v_8, v_{10})\), which implies that \((v_{11}, v_{12})\) is an arc of \(C\) and once again the block is even.

**Subcase 2.2.** \((v_5, v_9)\) is an arc of \(C\). If \((v_7, v_8)\) is an arc of \(C\), then the block is even; otherwise, \((v_8, v_9)\) is an arc of \(C\), which implies that \((v_7, v_{11})\) is an arc of

![Diagram](image.png)
$C$. As we have seen that $n \neq 11$. Thus $n \geq 13$ and then $(v_8, v_{12})$ is an arc of $C$. From this, it follows that $(v_9, v_{10})$ is an arc of $C$ and so the block is even.

Therefore, each arc of $C$ colored 1 belongs to an even block. Since the distinct blocks produce a partition of $V(\vec{C}_n)$, it follows that $n$ is even, which is a contradiction. Hence no arc of $C$ is colored 1. Consequently, each arc of a properly colored Hamiltonian cycle $C$ of the distance-colored digraph $D^4$ is colored 2, 3 or 4.

Let $s : a_1, a_2, \ldots, a_n$ be the corresponding cyclic sequence of colors of $C$, where, as we noted, $a_i \in \{2, 3, 4\}$ for each $i$ ($1 \leq i \leq n$). Also $\sum_{i=1}^{n} a_i = 3n$. Since $(\sum_{i=1}^{n} a_i)/n = 3$ and $n$ is odd, the color 3 appears an odd number of times in $s$ and the colors 2 and 4 occur an equal number of times.

First, we show that 2, 3 is not a subsequence of $s$, for suppose that it is. We may assume that $(v_3, v_5)$ and $(v_5, v_8)$ are arcs of $C$. Observe that $(v_2, v_6)$ and $(v_4, v_7)$ are arcs of $C$. Then $v_1$ is adjacent to no vertex of $D$ on $C$, a contradiction.

Consequently, each term 3 in $s$ is immediately preceded by 4 in $s$. Since the number of terms 2 and the number of terms 4 are equal, each subsequence of $s$ between consecutive occurrences of 3 must alternate 2 and 4, beginning with 2 and ending with 4. In particular, each occurrence of 3 in $s$ is immediately followed by 2, 4, that is, 3, 2, 4 is a subsequence of $s$. We may assume therefore that $C$ contains the arcs $(v_1, v_4)$, $(v_4, v_6)$ and $(v_6, v_{10})$. Note that $(v_2, v_5)$ and $(v_3, v_7)$ must be arcs on $C$. However then, $v_5$ is adjacent to no vertex of $D$ on $C$, a contradiction.

Hence, $D^4$ contains no properly colored Hamiltonian cycle. Therefore, $hce(\vec{C}_n) \geq 5$ and so $hce(\vec{C}_n) = 5$ for each odd integer $n \geq 7$.

In summary, $hce(\vec{C}_3)$ and $hce(\vec{C}_5)$ do not exist and

$$hce(\vec{C}_n) = \begin{cases} 3 & \text{if } n \geq 4 \text{ is even}, \\ 5 & \text{if } n \geq 7 \text{ is odd}. \end{cases}$$

5. Distance-colored Digraphs with Prescribed Hamiltonian Coloring Exponent

We saw that there are strong oriented graphs $D$ for which $hce(D)$ does not exist. On the other hand, for each integer $k \geq 2$, there exists a strong oriented graph $D$ such that $hce(D) = k$. In fact, more can be said. We now present a result that is analogous to Theorem 1.1.

**Theorem 5.1.** For each integer $k \geq 2$, there exists a strong oriented graph $D_k$ such that $hce(D_k) = k$. Furthermore, every properly colored Hamiltonian cycle in the $k$th power of $D_k$ must use all $k$ colors.
Proof. By Theorem 3.1, we may assume that \( k \geq 3 \). We consider two cases, according to whether \( k \) is even or \( k \) is odd.

Case 1. \( k \) is even. First, we define four oriented graphs \( H_1, H_2, H_3 \) and \( H_4 \) as follows:
- \( H_1 \) is a transitive tournament of order \( 2k \) with the Hamiltonian path \( (u_1, u_2, \ldots, u_{2k}) \),
- \( H_2 = (v_1, v_2, \ldots, v_k) \) is a directed path of order \( k \),
- \( H_3 \) is a transitive tournament of order \( 2k \) with the Hamiltonian path \( (w_1, w_2, \ldots, w_{2k}) \),
- \( H_4 = (x_1, x_2, \ldots, x_k) \) is a directed path of order \( k \).

The oriented graph \( D_k \) is then constructed from \( H_1, H_2, H_3 \) and \( H_4 \) by adding the arcs \( (w_{2k}, v_1), (v_k, w_1), (w_2k, x_1) \) \( \) and \( (x_k, u_1) \) (see Figure 9). Since \( (u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_{2k}, x_1, x_2, \ldots, x_k, u_1) \) is a Hamiltonian cycle in \( D_k \), it follows that \( D_k \) is a strong oriented graph.

We first show that \( \text{hec}(D_k) \geq k \). Assume, to the contrary, that the distance-colored digraph \( D_k^{k-1} \) contains a properly colored Hamiltonian cycle \( C^* \). Since, for each pair \( i, j \) with \( 1 \leq i, j \leq 2k \) and \( i < j \), we have \( d_{D_k}(w_i, w_j) = 1 \) and \( d_{D_k}(w_j, w_i) > k \), at most two vertices of \( H_3 \) can appear consecutively on \( C^* \). On the other hand, \( v_2, v_3, \ldots, v_k \) are the only vertices of \( D_k \) that are adjacent to vertices of \( H_3 \) in \( D_k^{k-1} \). This implies that \( C^* \) encounters \( H_3 \) at most \( k - 1 \) times and so \( C^* \) contains at most \( 2(k-1) \) vertices of \( H_3 \), which is a contradiction. Next, we show that \( \text{hec}(D_k) \leq k \) by constructing a properly colored Hamiltonian cycle in \( D_k^k \). Consider the \( k \) directed paths \( P_i = (u_{k+i}, v_i, w_i), \) \( 1 \leq i \leq k \), of order 3 in \( D_k^k \). Observe that \( d_{D_k}(u_{k+i}, v_i) = 1 + i \) for \( 1 \leq i \leq k - 1 \), \( d_{D_k}(u_{2k}, v_k) = k \), \( d_{D_k}(v_1, w_1) = k \) and \( d_{D_k}(v_i, w_i) = k + 2 - i \) for \( 2 \leq i \leq k \). Also, \( k \geq 4 \) is even and so \( k + 1 \) is odd. These observations imply that

1. \( 2 \leq d_{D_k}(u_{k+i}, v_i) \leq k \) and \( 2 \leq d_{D_k}(v_i, w_i) \leq k \) for \( 1 \leq i \leq k \),
(2) \(d_{D^k}(u_{k+i}, v_i) \neq d_{D^k}(v_i, w_i)\) for \(1 \leq i \leq k\).

Similarly, consider the \(k\) directed paths \(Q_i = (w_{k+i}, x_i), 1 \leq i \leq k\), of order 3 in \(D^k\). By symmetry, we have

(3) \(2 \leq d_{D^k}(w_{k+i}, x_i) \leq k\) and \(2 \leq d_{D^k}(x_i, u_i) \leq k\) for \(1 \leq i \leq k\),

(4) \(d_{D^k}(w_{k+i}, x_i) \neq d_{D^k}(x_i, u_i)\) for \(1 \leq i \leq k\).

Since \(d_{D^k}(u_i, u_{k+i}) = 1\) for \(1 \leq i \leq k - 1\), \(d_{D^k}(w_i, w_{k+i}) = 1\) for \(1 \leq i \leq k\) and \(d_{D^k}(u_k, u_{k+1}) = 1\), it follows by (1)–(4) that \((P_1, Q_1, P_2, Q_2, \ldots, P_k, Q_k, u_{k+1})\) is a properly colored Hamiltonian cycle in \(D^k\).

It remains to show that every properly colored Hamiltonian cycle in the \(k\)th power of \(D^k\) must use all colors 1, 2, \ldots, \(k\). Let \(C\) be any properly colored Hamiltonian cycle in \(D^k\). As we saw, at most two vertices of \(H_3\) can appear consecutively on \(C\). Thus \(C\) must encounter \(H_3\) at least \(k\) times. On the other hand, since \(v_1, v_2, \ldots, v_k\) are the only vertices that are adjacent to vertices of \(H_3\) in \(D^k\), it follows that \(C\) encounters \(H_3\) exactly \(k\) times. Moreover, \(C\) enters \(H_3\) immediately after encountering a vertex \(v_i\) for some \(i\) with \(1 \leq i \leq k\). Hence, \(C\) contains an arc \((v_i, w)\) for each \(i\) with \(1 \leq i \leq k\) and for some \(w \in V(H_3)\). Since \(d_{D^k}(v_1, w_j) > k\) for \(2 \leq j \leq k\), it follows that \((v_1, w_1)\) is an arc of \(C\). Also, we saw that \(d_{D^k}(v_i, w_j) = k + 2 - i\) for all \(i, j\) with \(2 \leq i \leq k\) and \(2 \leq j \leq k\). This implies that \(C\) contains at least one arc colored by each of the colors 2, 3, \ldots, \(k\). Furthermore, the order of \(H_3\) is 2\(k\) and so two vertices of \(H_3\) must appear consecutively on \(C\), which implies that \(C\) contains at least one arc colored 1.

**Case 2. \(k\) is odd.** We construct a strong oriented graph \(D_k\) in the same fashion as the one in Case 1. First, we define four oriented graphs \(H_1, H_2, H_3\) and \(H_4\) as follows:

- \(H_1\) is a transitive tournament of order 2\(k\) with the Hamiltonian path \((u_1, u_2, \ldots, u_{2k})\),
- \(H_2 = (v_1, v_2, \ldots, v_{k-1})\) is a directed path of order \(k - 1\),
- \(H_3\) is a transitive tournament of order 2\(k\) with the Hamiltonian path \((w_1, w_2, \ldots, w_{2k})\),
- \(H_4 = (x_1, x_2, \ldots, x_{k-1})\) is a directed path of order \(k - 1\).

The oriented graph \(D_k\) is then constructed from \(H_1, H_2, H_3\) and \(H_4\) by adding the arcs \((u_{2k}, v_1), (v_{k-1}, w_1), (w_{2k}, u_1),\) and \((x_{k-1}, u_1)\). (See Figure 9, where we replace \(v_k\) by \(v_{k-1}\) and replace \(x_k\) by \(x_{k-1}\).) Since \((u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_{k-1}, w_1, w_2, \ldots, w_{2k}, x_1, x_2, \ldots, x_{k-1}, u_1)\) is a Hamiltonian cycle in \(D^k\), it follows that \(D_k\) is a strong oriented graph.

We first show that \(hcc(D_k) \geq k\). Assume, to the contrary, that the distance-colored digraph \(D^{k-1}_k\) contains a properly colored Hamiltonian cycle \(C^*\). Since
\( v_1, v_3, \ldots, v_{k-1} \) are the only vertices of \( D_k \) that are adjacent to vertices of \( H_3 \) in \( D_k^{k-1} \), it follows that that \( C^* \) encounters \( H_3 \) at most \( k-1 \) times and so \( C^* \) contains at most \( 2(k-1) \) vertices of \( H_3 \), which is a contradiction. Next, we show that \( \text{hce}(D_k) \leq k \) by constructing a properly colored Hamiltonian cycle in \( D_k^k \). Consider the \( k \) directed paths \( P_i = (u_{k+i}, v_i, w_i) \), \( 1 \leq i \leq k-1 \), and \( P_k = (u_{2k}, w_1) \) of order 3 in \( D_k^k \). Observe that \( d_{D_k}(u_{k+i}, v_i) = 1 + i \) for \( 1 \leq i \leq k-1 \), \( d_{D_k}(v_i, w_i) = k + 1 - i \) for \( 1 \leq i \leq k-1 \) and \( d_{D_k}(u_{2k}, w_1) = k \). Furthermore, \( k \geq 3 \) is odd and \( k+1 \) is even. Thus

\[
(1) \quad 2 \leq d_{D_k}(u_{k+i}, v_i) \leq k \quad \text{and} \quad 2 \leq d_{D_k}(v_i, w_i) \leq k \quad \text{for} \quad 1 \leq i \leq k,
\]

\[
(2) \quad d_{D_k}(u_{k+i}, v_i) \neq d_{D_k}(v_i, w_i) \quad \text{for} \quad 1 \leq i \leq k-1.
\]

Similarly, consider the \( k \) directed paths \( Q_i = (w_{k+i}, u_i, i) \) \((1 \leq i \leq k-1)\) and \( Q_k = (u_{2k}, u_1) \) of order 3 in \( D_k^k \). By symmetry, we have

\[
(3) \quad 2 \leq d_{D_k}(w_{k+i}, u_i) \leq k \quad \text{and} \quad 2 \leq d_{D_k}(x_i, u_i) \leq k-1 \quad \text{for} \quad 1 \leq i \leq k-1,
\]

\[
(4) \quad d_{D_k}(w_{k+i}, u_i) \neq d_{D_k}(x_i, u_i) \quad \text{for} \quad 1 \leq i \leq k-1.
\]

Since \( d_{D_k}(u_i, u_{k+i+1}) = 1 \) for \( 1 \leq i \leq k-1 \), \( d_{D_k}(w_i, w_{k+i}) = 1 \) for \( 1 \leq i \leq k \) and \( d_{D_k}(u_k, u_{k+1}) = 1 \), it follows by (1)–(4) that \( (P_1, Q_1, P_2, Q_2, \ldots, P_k, Q_k, u_{k+1}) \) is a properly colored Hamiltonian cycle in \( D_k^k \).

It remains to show that every properly colored Hamiltonian cycle in the \( k \)th power of \( D_k \) must use all colors \( 1, 2, \ldots, k \). Let \( C \) be any properly colored Hamiltonian cycle in \( D_k^k \). An argument similar to the one in Case 1 shows that \( C \) must enter \( H_3 \) exactly \( k \) times. Since \( u_{2k}, v_1, v_3, \ldots, v_{k-1} \) are the only vertices of \( D_k \) that are adjacent to vertices of \( H_3 \) in \( D_k^k \), each of the vertices \( u_{2k}, v_1, v_3, \ldots, v_{k-1} \) is immediately followed by a vertex of \( H_3 \) on \( C \). This, however, requires that \( C \) contains \((u_{2k}, w_1)\) and an arc \((v_i, w)\) for each \( i \) with \( 1 \leq i \leq k-1 \) and for some \( w \in V(H_3) \). Since \( d_{D_k}(u_{2k}, w_1) = k \) and \( d_{D_k}(v_i, w) = k + 1 - i \) for \( 1 \leq i \leq k-1 \) and \( 2 \leq j \leq k \), it follows that \( C \) contains at least one arc colored by each of the colors \( 2, 3, \ldots, k \). Furthermore, the order of \( H_3 \) is \( 2k \) and so two vertices of \( H_3 \) must appear consecutively on \( C \). Hence \( C \) contains an arc colored 1.

6. On the Existence of Graphs Having Distinct Strong Orientations with Different Hamiltonian Coloring Exponents

By Theorem 5.1, there exists for each integer \( k \geq 2 \) a strong oriented graph \( D \) such that \( \text{hce}(D) = k \). Equivalently, there exists a connected graph \( G \) possessing a strong orientation \( D \) such that \( \text{hce}(D) = k \). It is possible, however, that there may be another strong orientation of \( G \), resulting in a digraph \( D' \) whose Hamiltonian coloring exponent is far differ from that of \( D \). In fact, for two different strong
orientations $D$ and $D'$ of a connected graph, the difference between $\text{hce}(D)$ and $\text{hce}(D')$ can be arbitrarily large.

**Theorem 6.1.** For every positive integer $p$ there exists a connected graph $G$ with strong orientations $D$ and $D'$ such that $\text{hce}(D) - \text{hce}(D') \geq p$.

**Proof.** For a positive integer $p$, let $k$ be an integer such that $k \geq p + 3$ and $k \equiv 0 \pmod{4}$. Now let $G$ be the underlying graph of the strong oriented graph $D_k$ in the proof of Theorem 5.1 when $k$ is even. Following the same vertex labeling for $D_k$ and the same notation for the subdigraphs $H_1$, $H_2$, $H_3$ and $H_4$ in $D_k$ (as described in the proof of Theorem 5.1), let $D'_k$ be the orientation of $G$ obtained from $D$ by replacing the two arcs $(u_1, u_2k)$ and $(u_1, u_2k)$ by $(u_2k, u_1)$ and $(u_2k, u_1)$. Now let $D = D_k$ and $D' = D'_k$. By Theorem 5.1, $\text{hce}(D) = k$. In fact, $\text{hce}(D') = 3$ as we show next.

First, we show that the cube of $D'$ is Hamiltonian-colored. To construct a properly colored Hamiltonian cycle in the cube of $D'$, we first define eight vertex-disjoint properly colored subpaths $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ in the cubes of the subdigraphs $H_1$, $H_2$, $H_3$ and $H_4$ of $D'$, respectively, as follows:

- In the cube of $H_1$, define two vertex-disjoint properly colored paths $P_{u_1}$ and $P_{u_2}$ of order $k - 2$ as $P_{u_1} = (u_k, u_{k-1}, u_4, \ldots, u_{k-2}, u_{k-3}), P_{u_2} = (u_2k-2, u_{k+1}, u_{2k-3}, u_{k+2}, \ldots, u_{k+2}, u_{k+1}).$ Let $A_1 = (u_2, P_{u_1}, u_2k-1)$ and $A_2 = (u_1, P_{u_2}, u_2k)$ be the subpaths of order $k$ in the cube of $H_1$. Then $V(A_1) \cup V(A_2) = V(H_1)$, each of the initial and terminal arcs of $A_1$ and $A_2$ is colored 1 and $A_1$ and $A_2$ are properly colored.

- In the cube of $H_2$, define two vertex-disjoint paths $B_1$ and $B_2$ of order $k/2$ as $B_1 = (v_1, v_2, v_5, v_6, v_9, v_{10}, v_{13}, \ldots, v_{k-6}, v_{k-3}, v_{k-2}), B_2 = (v_3, v_4, v_7, v_8, v_{11}, v_{12}, v_{15}, \ldots, v_{k-4}, v_{k-1}, v_k).$ Observe that $V(B_1) \cup V(B_2) = V(H_2)$ and each of the initial and terminal arcs of $B_1$ and $B_2$ is colored 1. The arcs of $B_1$ and $B_2$ are colored 1 and 3 alternatively.

- In the cube of $H_3$, define two vertex-disjoint properly colored paths $P_{w_1}$ and $P_{w_2}$ of order $k - 2$ as $P_{w_1} = (w_k, w_{k-1}, w_4, \ldots, w_{k-2}, w_{k-3}), P_{w_2} = (w_2k-2, w_{k+1}, w_{2k-3}, w_{k+2}, \ldots, w_{k+2}, w_{k+1}).$ Let $C_1 = (w_1, P_{w_1}, w_{2k-1})$ and $C_2 = (w_2, P_{w_2}, w_{2k})$ be the subpaths of order $k$ in the cube of $H_3$. Then $V(C_1) \cup V(C_2) = V(H_3)$, each of the initial and terminal arcs of $C_1$ and $C_2$ is colored 1 and $C_1$ and $C_2$ are properly colored.

- In the cube of $H_4$, define two vertex-disjoint paths $D_1$ and $D_2$ of order $k/2$ as $D_1 = (x_1, x_2, x_5, x_6, x_9, x_{10}, x_{13}, \ldots, x_{k-6}, x_{k-3}, x_{k-2}), D_2 = (x_3, x_4, x_7, x_8, x_{11}, x_{12}, x_{15}, \ldots, x_{k-4}, x_{k-1}, x_k).$ Observe that $V(D_1) \cup V(D_2) = V(H_3)$ and each of the initial and terminal arcs of $D_1$ and $D_2$ is colored 1. The arcs of $D_1$ and $D_2$ are colored 1 and 3 alternatively.
Then \((A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, u_2)\) is a properly colored Hamiltonian cycle in the cube of \(D'\) and so \(hce(D') \leq 3\). On the other hand, \(D'\) contains an induced path \(P_3\) and so it can be shown that the square of \(D'\) is not Hamiltonian-colored. Thus \(hce(D') = 3\).

Consequently, \(hce(D) - hce(D') = k - 3 \geq p\) as desired.

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\[\text{References}\]


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