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ON THE DOMINATOR COLORINGS IN TREES [1]

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Abstract

In a graph G, a vertex is said to dominate itself and all its neighbors. A dominating set of a graph G is a subset of vertices that dominates every vertex of G. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A proper coloring of a graph G is a function from the set of vertices of the graph to a set of colors such that any two adjacent vertices have different colors. A dominator coloring of a graph G is a proper coloring such that every vertex of V dominates all vertices of at least one color class (possibly its own class). The dominator chromatic number $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of G. Gera showed that every nontrivial tree T satisfies $\gamma(T) + 1 \leq \chi_d(T) \leq \gamma(T) + 2$. In this note we characterize nontrivial trees T attaining each bound.

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1. Introduction

Let $G = (V, E)$ be a simple graph. A vertex in a graph G is said to *dominate* itself and every vertex adjacent to it. A set D of vertices in G is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality among all the dominating sets of G.

A *proper coloring* of a graph $G = (V, E)$ is a function from the set of vertices of the graph to a set of colors such that any two adjacent vertices have different colors. A *dominator coloring* of a graph G is a proper coloring such that every

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vertex of V dominates all vertices of at least one color class (possibly its own class). The *dominator chromatic number* $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of G. A dominator coloring of G with $\chi_d(G)$ colors will be called a $\chi_d(G)$-DC. The concept of dominator coloring was introduced by Gera, Horton and Rasmussen [4] and studied further in [2] and [3], and recently in [1].

It is shown in [2, 3] that for every nontrivial tree T, $\gamma(T)+1 \leq \chi_d(T) \leq \gamma(T)+2$. However, computing the exact value of the dominator coloring number of a tree remains an open problem. Our aim in this note is to characterize all nontrivial trees T attaining each bound. To this end we will focus only on trees T with $\chi_d(T) = \gamma(T)+1$.

Let us introduce some notations and definitions. The *open neighborhood* $N(v)$ of a vertex v consists of the vertices adjacent to v, and $N[v] = N(v) \cup \{v\}$ is the *closed neighborhood* of v. For a vertex set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. The *degree* of a vertex v is the cardinality of its open neighborhood. A *leaf* of a graph G is a vertex of degree 1, and its neighbor is called a *stem*. For a set $S \subseteq V$, the *private neighborhood* $pn(v,S)$ of $v \in S$ is defined by $pn(v,S) = N[v] - N[S - \{v\}]$. If D is a minimum dominating set of G, then let $D_I = \{v \in D : pn(v,D) = \{v\}\}$ and $D_R = D - D_I$. Clearly if $v \in D_I$, then v has no neighbor in D. Also every vertex w of $V - D$ adjacent to v has another neighbor in D besides v (for otherwise $pn(v,D) = \{v,w\}$, contradicting the fact that $v \in D_I$). Moreover, if D is a minimum dominating set of a tree T, then for every pair of vertices $u, v \in D_I$, $N(u) \cap N(v) = \emptyset$. Indeed, let $z \in N(u) \cap N(v)$ and $D' = \{z\} \cup D - \{u,v\}$. If D' does not dominate T, then there is a vertex $w \in V - D$ adjacent to both u and v but then $\{z,u,v,w\}$ induces a cycle C_4 which is excluded since T is a tree. Hence D' is a dominating set of T of size less than D, a contradiction too. Let $V_1, V_2, \ldots, V_{\chi_d(G)}$ be the color classes of a dominator coloring of G. A vertex $v \in V_i$ is called *solitary* if $|V_i| = 1$. We denote by C_P the set of color classes containing solitary vertices, by C_S the set of color classes such that each of them contains at least two vertices and is dominated by some vertex of V, and by C_G the set of color classes such that each of them contains at least two vertices andis not dominated by any vertex of V. Clearly C_P, C_S, C_G are disjoint sets and $C_G \cup C_P \cup C_S = \{V_1, V_2, \ldots, V_{\chi_d(G)}\}$. Let A be the set of all solitary vertices and B be the set of all vertices belonging to color classes in C_G. Clearly $|C_P| = |A|$. We denote by x_S a vertex dominating the color class S and let $DS = \{x_S \in V : S \in C_S\}$. Recall that a subset of vertices $S \subseteq V$ is *independent* if no edge of G has its two endvertices in S.

We shall prove:

Theorem 1. *Let T be a nontrivial tree. Then $\chi_d(T) = \gamma(T)+1$ if and only if T admits a minimum dominating set $D = D_I \cup D_R$ such that $V(T) - (D_R \cup N[D_I])$ is an independent set.*

2. Proof of Theorem 1

We begin by the following straightforward observation.

Observation 2. *Let T be a nontrivial tree. Then for every $\chi_d(T)$-DC of T, either each stem is solitary or it is adjacent to exactly one leaf and that leaf is solitary.*

Proof. Let u be a stem of T and v its leaf neighbor. From the definition of a dominator coloring either v is alone in its color class and hence v is solitary or v is adjacent to all vertices of at least one other class. Clearly in the later case the color class dominated by v contains only u and so u is solitary. Now suppose that u is adjacent to at least two leaves and consider any $\chi_d(T)$-DC of T in which u is not solitary. Then, as seen above, every leaf neighbor of u is solitary but then we can decrease $\chi_d(T)$ by giving to u the color of v and to every leaf neighbor of u the color initially given to u, a contradiction. Thus if a stem is not solitary in a $\chi_d(T)$-DC of T, then it is adjacent to exactly one leaf and that leaf is solitary. ■

Lemma 3. *Every tree T of order at least three admits a dominator coloring with $\chi_d(T)$ colors such that all leaves of T have the same color.*

Proof. Let c be any dominator coloring of T with $\chi_d(T)$ colors and maximum number of solitary stems. If some stem u is not solitary, then by Observation 2, u is adjacent to exactly one leaf, say v, where v is a solitary vertex. In this case we can swap the colors between the two vertices and so u becomes a solitary vertex. Clearly u now dominates its own color class and every vertex adjacent to u dominates the color class containing u. So c is turned into a dominator coloring c' with the same number of colors $\chi_d(T)$ but c' has more solitary stems than c, a contradiction. Hence c is a $\chi_d(T)$-DC of T in which every stem is solitary. Now if all leaves have the same color, then we are done. In the other case, each leaf uses a color not used by stems and clearly we can decrease or leave unchanged $\chi_d(T)$ by giving the same color to all leaves of T. ■

Lemma 4. *For every $\chi_d(T)$-DC of a tree T, every color class $S \in C_S$ is dominated by exactly one vertex. In that case we have $|C_S| = |DS|$.*

Proof. Consider a dominator coloring of T with $\chi_d(T)$ colors and suppose that a color class $S \in C_S$ is dominated by two vertices x_S and y_S. Then x_S, y_S and any two vertices of S induce a cycle C_4, a contradiction. Now $|C_S| = |DS|$ follows immediately. ■

Lemma 5. *For every $\chi_d(T)$-DC of a tree T, $A \cup DS$ is a dominating set of T.*

Proof. Consider a dominator coloring of T with $\chi_d(T)$ colors. Every vertex x of T dominates at least one color class, say H_x. If $H_x \in C_p$, then x is dominated by A and if $H_x \in C_S$, then x belongs to DS. Hence $A \cup DS$ dominates all vertices of T. ■

Lemma 6. *Let T be a nontrivial tree with $\chi_d(T) = \gamma(T) + 1$. Then for every $\chi_d(T)$-DC of T, there is at most one color class dominated by no vertex.*

Proof. Let T be a nontrivial tree with $\chi_d(T) = \gamma(T)+1$. Consider any dominator coloring of T with $\chi_d(T)$ colors. We have to prove that $|C_G| \leq 1$. By Lemma 5, $A \cup DS$ is a dominating set of T and so $\gamma(T) \leq |A \cup DS|$. It follows that $\gamma(T) + |C_G| \leq |A \cup DS| + |C_G| \leq |A| + |DS| + |C_G|$. Using the fact that $|C_S| = |DS|$ (see Lemma 4) we obtain $\gamma(T)+|C_G| \leq |A|+|C_S|+|C_G| = \chi_d(T) = \gamma(T)+1$ and so $|C_G| \leq 1$. ■

Lemma 7. *Let T be a nontrivial tree different from a star. If $\chi_d(T) = \gamma(T)+1$, then for every dominator coloring with $\chi_d(T)$ colors such that all leaves of T have the same color we have:*

(a) $|C_G| = 1$.

(b) *$A \cup DS$ is a minimum dominating set.*

(c) $A \cap DS = \emptyset$.

(d) *Every color class $S \in C_S$ is dominated by a vertex of B.*

Proof. Consider a dominator coloring with $\gamma(T) + 1$ colors such that all leaves of T have the same color. Note that such a dominator coloring exists by Lemma 3.

(a) Since T is not a star, all leaves of T form a color class and this class is dominated by no vertex, that is $|C_G| \geq 1$. Equality follows from Lemma 6.

(b) $\gamma(T) + 1 = \chi_d(T) = |C_P| + |C_S| + |C_G| = |C_P| + |C_S| + 1$, implying that $\gamma(T) = |C_P| + |C_S|$. Since $|C_P| = |A|$ and $|C_S| = |DS|$ it follows that $\gamma(T) = |A| + |DS|$ and so $A \cup DS$ is a minimum dominating set.

(c) Follows from (b).

(d) Let S be any color class of C_S and assume that S is not dominated by a vertex of B. Let x_S be a vertex dominating the color class S. Thus $x_S \in DS$ and $x_S \notin B$. By item (c) $x_S \notin A$ and hence $x_S \in V(T) - (A \cup B)$, that is x_S belongs to some color class in C_S. In this case we shall prove that there is a color class, say $S^* \in C_S$, such that each vertex of S^* dominates a color class in C_P. Let us assume that S^* does not exist. Then since every vertex of T must dominate a color class, there is a vertex of S, say x_{S_1} that dominates a color class $S_1 \in C_S$ (otherwise every vertex of S dominates a color class in C_P, and so $S = S^*$, a contradiction to our assumption). By the same argument, there is a vertex x_{S_2} of S_1 that dominates a color class $S_2 \in C_S$, and so on. Since T is finite, the process stops by providing a cycle in the subgraph induced by the vertices $x_S, x_{S_1}, x_{S_2}, \ldots$ contradicting the fact that T is a tree. Hence S^* exists and so every vertex of S^* is dominated by A. Now since $S^* \in C_S$, let x_{S^*} be the vertex of DS that dominates all vertices of S^*. By item (b), $A \cup DS$ is a minimum dominating set of T, and hence $A \cup DS - \{x_{S^*}\}$ dominates T and has size $\gamma(T) - 1$, a contradiction. Thus S is dominated by a vertex of B and therefore every color class of C_S is dominated by a vertex of B. ∎

Now we are ready to prove Theorem 1.

***Proof of Theorem* 1.** Let T be a nontrivial tree and assume that T admits a minimum dominating set $D = D_I \cup D_R$ such that $V(T) - (D_R \cup N[D_I])$ is an independent set. To see that $\chi_d(T) = \gamma(T) + 1$, we color the vertices of T as follow.

• We give a different color to every vertex of D_R.

• For every vertex $y \in D_I$ we give a new color to the vertices of $N(y)$. (Recall that any two vertices y' and y'' of D_I satisfy $N(y') \cap N(y'') = \emptyset$.)

• We give the same (but new) color to the remaining vertices.

Obviously the previous coloring is a dominator coloring. Hence $\gamma(T) + 1 \leq \chi_d(T) \leq |D_R| + |D_I| + 1 = |D| + 1 = \gamma(T) + 1$, and the equality follows.

Conversely, let T be a nontrivial tree with $\chi_d(T) = \gamma(T) + 1$. Suppose T is a star of center vertex, say x. Then $D = \{x\}$ is a minimum dominating set, where $D_I = \emptyset$ and clearly the set $V(T) - (D_R \cup N(D_I))$ that consists of the set of leaves of the star is independent. Therefore the theorem is valid. Now assume that T is a tree different from a star and let us consider a dominator coloring with $\chi_d(T)$ colors such that all leaves of T have the same color. Note that such a dominator coloring exists by Lemma 3. Also by Lemma 7, $A \cup DS$ is a minimum dominating set of T. Let $D = A \cup DS$, $D_I = \{v \in D : pn(v, D) = \{v\}\}$ and $D_R = D - D_I$. We shall show that every vertex in $V(T) - (A \cup B)$ is adjacent to a vertex of D_I, that is $V(T) - (A \cup B) \subset N(D_I)$. Let x be any vertex of a color class $S \in C_S$. Let x_S be a vertex of DS that dominates S. By Lemma 7(d), $x_S \in B$. Recall that $x_S \in D$ since $x_S \in DS$. It is well known by Ore's theorem (see [7]) that every vertex in a minimum dominating set has a private neighborhood. Suppose that $x^* \neq x_S$ is a private neighbor of x_S with respect to D. Clearly $x^* \notin D$ (for otherwise $D - \{x^*\}$ would be a dominating set smaller than D, a contradiction). Therefore x^* does not dominate a color class of C_S (else $x^* \in DS \subset D$). Also since $x^* \in pn(x_S, D)$, x^* has no neighbor in A but then x^* does not dominate any color class, a contradiction. Consequently x_S has no private neighbor other than itself, that is $x_S \in D_I$. Thus $V(T) - (A \cup B) \subset N(D_I)$. It follows now that all vertices of $V(T) - (D_R \cup N[D_I]) \subseteq B$ and since B is an independent set we are done. ∎

3. Caterpillars

A caterpillar is a tree in which every vertex of degree at least three has at most two non-leaf neighbors. As it was already noted in the introduction, there is no polynomial time algorithm that computes the dominator chromatic number for the class of trees. It was even mentioned by Gera et al. in [4] that an efficient algorithm for computing χ_d of an arbitrary caterpillar would be a worthwhile contribution. Our aim in this section is to give a descriptive characterization of caterpillars T with $\chi_d(T) = \gamma(T) + 1$. Using a result of Volkmann [8] (see Theorem 8), one can check easily whether a caterpillar satisfies $\chi_d(T) = \gamma(T)+1$ or $\chi_d(T) = \gamma(T) + 2$.

A *vertex cover* in a graph G is a set of vertices that covers all edges of G. The minimum cardinality of a vertex cover in a graph G is called the *covering number* of G and is denoted by $\alpha_0(G)$. It is well known that a set D of vertices of G is a vertex cover if and only if $V(G) - D$ is independent. Also every vertex cover set is a dominating set. The following result of Volkmann gives a characterization of nontrivial trees T with equal domination and covering numbers.

Theorem 8 (Volkmann [8]). *A nontrivial tree satisfies $\gamma(T) = \alpha_0(T)$ if and only if each component in the graph resulting from G by removing the set of leaves and their stems is an isolated vertex or a star, where the centers of these stars are not adjacent to any stem in T.*

Now we are ready to state the following result.

Proposition 9. *Let T be a nontrivial caterpillar. Then $\chi_d(T) = \gamma(T)+1$ if and only if $\gamma(T) = \alpha_0(T)$.*

Proof. Let T be a caterpillar with $\gamma(T) = \alpha_0(T)$. Let D be any minimum vertex cover set of G. Then color the vertices of D so that each vertex has a unique color and the remaining vertices of T by a new color. Then $\gamma(T) + 1 \leq \chi_d(T) \leq |D| + 1 = \gamma(T) + 1$, and the equality follows.

Now assume that T is a caterpillar with $\chi_d(T) = \gamma(T) + 1$. By Theorem 1, T admits a minimum dominating set D such that $V(T)\backslash(D_R \cup N[D_I])$ is independent. Assume that $V(T) - D$ is not independent and let u, v be any two adjacent vertices in $V(T) - D$. Clearly since D contains either a leaf or its stem, neither u nor v is a leaf. First, assume that u and v are not stems. Let d_1 and d_2 be two vertices in D such that d_1, u, v, d_2 induce a path P_4. Then d_1 is the unique neighbor of u in D for otherwise u has degree at least three and so u is a stem. Likewise d_2 is the unique neighbor of v in D. Hence d_1 and d_2 belong to D_R and so $V(T)\backslash(D_R \cup N[D_I])$ is not an independent set, contradicting Theorem 1. Hence at least u or v is a stem. Without loss of generality, assume that u is a stem and let f be its leaf. Since f belongs to D, f is the unique leaf adjacent to u. Let us modify D as follows: $D' = \{u\} \cup D\backslash\{f\}$. Clearly D' remains a minimum dominating set for T with less edges in $V(T) - D'$. This procedure can be repeated for every two adjacent vertices not in the current $\gamma(T)$-set until we obtain a $\gamma(T)$-set S for which $V(T) - S$ has no two adjacent vertices. Therefore $\gamma(T) = \alpha_0(T)$. ∎

According to Theorem 8, Proposition 9 can be also stated as follows.

Proposition 10. *Let T be a nontrivial caterpillar. Then $\chi_d(T) = \gamma(T) + 1$ if and only if T is a star or the distance between any two consecutive stems is $1, 2$ or 4.*

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