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STRUCTURAL RESULTS ON MAXIMAL k-DEGENERATE GRAPHS

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Abstract

A graph is k-degenerate if its vertices can be successively deleted so that when deleted, each has degree at most k. These graphs were introduced by Lick and White in 1970 and have been studied in several subsequent papers. We present sharp bounds on the diameter of maximal k-degenerate graphs and characterize the extremal graphs for the upper bound. We present a simple characterization of the degree sequences of these graphs and consider related results. Considering edge coloring, we conjecture that a maximal k-degenerate graph is class two if and only if it is overfull, and prove this in some special cases. We present some results on decompositions and arboricity of maximal k-degenerate graphs and provide two characterizations of the subclass of k-trees as maximal k-degenerate graphs. Finally, we define and prove a formula for the Ramsey core numbers.

Keywords: k-degenerate, k-core, k-tree, degree sequence, Ramsey number.

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1. Introduction

One of the basic properties of graphs is the existence of subgraphs with specified degree conditions. (See [3] and [18] for basic terminology.)

Definition. The k-core of a graph G, $C_k(G)$, is the maximal induced subgraph $H \subseteq G$ such that $\delta(H) \geq k$, if it exists.

Cores were introduced by S.B. Seidman [16] and have been studied extensively in [1]. It is easy to show that the k-core is well-defined and that the cores of a graph are nested.

Definition. The core number $C(v)$ of a vertex v is the largest value for k such that $v \in C_k(G)$. The maximum core number of a graph, $\widehat{C}(G)$, is the maximum of the core numbers of the vertices of G.

There is a simple algorithm for determining the k-core of a graph, which we shall call the k-core algorithm.

Algorithm 1 (k-Core Algorithm)**.** Iteratively delete vertices of degree less than k until none remain.

This will produce the k-core if it exists. This suggests the following concept.

Definition. A graph is k-degenerate if its vertices can be successively deleted so that when deleted, each has degree at most k. The degeneracy of a graph is the smallest k such that it is k-degenerate.

As a corollary of the k-core algortithm, we have the following min-max relationship.

Corollary 2. *For any graph, its maximum core number is equal to its degeneracy.*

A graph is k-core-free if it does not contain a k-core. A graph is maximal with respect to some property if no edge can be added without violating this property. The k-core algorithm also implies that a graph G is k-degenerate if and only if G is $(k+1)$-core-free, and maximal k-degenerate graphs are equivalent to maximal $(k+1)$-core-free graphs.

The term k-degenerate was introduced in 1970 by Lick and White [11]; the concept has been introduced under other names both before and since. In particular, the term 'k-dense tree' has been used for 'k-degenerate graph' and 'k-arch graph' has been used for 'maximal k-degenerate graph'.

Our examination of k-degenerate graphs will focus on maximal k-degenerate graphs. Most of the properties given can be generalized with appropriate modification to all k-degenerate graphs. The most basic result is the size of maximal k-degenerate graphs; we include a proof for completeness.

Theorem 3. *The size of a maximal k-degenerate with order $n \geq k$ is $k \cdot n - \binom{k+1}{2}$.*

Proof. If G is k-degenerate, then its vertices can be successively deleted so that when deleted they have degree at most k. Since G is maximal, the degrees of the deleted vertices will be exactly k until the number of vertices remaining is at most k. After that, the $n - j^{th}$ vertex deleted will have degree j. Thus the size of G is $m = \sum_{i=0}^{k-1} i + \sum_{i=k}^{n-1} k = \frac{k(k-1)}{2} + k\,(n-k) = k \cdot n - \binom{k+1}{2}$. ∎

Thus for k-core-free graphs, maximal and maximum are equivalent. Hence a k-degenerate graph is maximal if and only if it has size $k \cdot n - \binom{k+1}{2}$.

Corollary 4. *Every graph with order n, size $m \geq (k-1)\,n - \binom{k}{2} + 1$, $1 \leq k \leq n-1$, has a k-core.*

The basic properties of maximal k-degenerate graphs were established by Lick and White [11] and Mitchem [13]. An early survey of results appears in [17].

Theorem 5. *Let G be a maximal k-degenerate graph of order n, $1 \leq k \leq n-1$. Then*

(a) *G contains a $(k+1)$-clique and for $n \geq k+2$, G contains $K_{k+2} - e$ as a subgraph.*

(b) *For $n \geq k+2$, G has $\delta(G) = k$, and no two vertices of degree k are adjacent.*

(c) *G has connectivity $\kappa\,(G) = k$.*

(d) *For any integer r, $1 \leq r \leq n$, G contains a maximal k-degenerate graph of order r as an induced subgraph. For $n \geq k+2$, if $d\,(v) = k$, then G is maximal k-degenerate if and only if $G - v$ is maximal k-degenerate.*

(e) *G is maximal 1-degenerate if and only if G is a tree.*

In fact, maximal k-degenerate graphs are one type of generalization of trees.

Several corollaries follow immediately from these basic results.

Corollary 6. *Let G be a maximal k-degenerate graph of order n, $1 \leq k \leq n-1$. Then*

(a) *For $k \geq 2$, the number of nonisomorphic maximal k-degenerate graphs of order $k+3$ is 3.*

(b) *The number of nonisomorphic maximal k-degenerate subgraphs of order $n-1$ is equal to the number of vertices of degree k in G that are in distinct automorphism classes.*

Proof. (a) $K_{k+2} - e$ is the unique maximal k-degenerate graph of order $k+2$. It has two automorphism classes of vertices, one with two, one with k. Thus there are three possibilities for order $k+3$.

(b) Deleting any minimum degree vertex yields such a subgraph, and deleting any other vertex destroys maximality. The subgraphs will be distinct unless two minimum degree vertices are in the same automorphism class. ■

A trivial edge cut is an edge cut such that all the edges are incident with one vertex.

Corollary 7. *Let G be a maximal k-degenerate graph of order n, $1 \leq k \leq n-1$. Then G has edge-connectivity $\kappa'\,(G) = k$, and for $k \geq 2$, an edge set is a minimum edge cut if and only if it is a trivial edge cut.*

Proof. First, $k = \kappa(G) \leq \kappa'(G) \leq \delta(G) = k$. Certainly the edges incident with a vertex of minimum degree form a minimum edge cut. The result holds for K_{k+1}. Assume the result holds for all maximal k-degenerate graphs of order r, and let G have order $r+1$, $v \in G$, $d(v) = k$, $H = G - v$. Let F be a minimum edge cut of G. If $F \subset E(H)$, the result holds. If F is a trivial edge cut for v, the result holds. If F contained edges both from H and incident with v, it would not disconnect H and would not disconnect v from H. ■

2. Diameter

We can bound the diameter of a maximal k-degenerate graph.

Theorem 8. *A maximal k-degenerate graph G with $n \geq k+2$ has $2 \leq diam(G) \leq \frac{n-2}{k} + 1$.*

If the upper bound is an equality, then G has exactly two vertices of degree k and every diameter path has them as its endpoints.

Proof. Let G be maximal k-degenerate with $r = diam(G)$. For $n \geq k+2$, G is not complete, so $diam(G) \geq 2$. Now G contains u, v with $d(u,v) = r$. Now G is k-connected, so by Menger's Theorem there are at least k independent paths of length at least r between u and v. Thus $n \geq k(r-1)+2$, so $r \leq \frac{n-2}{k} + 1$.

Let the upper bound be an equality, and $d(u,v) = r$. Then $n = k(r-1)+2$, and since there are k independent paths between u and v, all the vertices are on these paths. Thus $d(u) = d(v) = k$. If another vertex w had degree k, then $G - w$ would be maximal k-degenerate with $\kappa(G-w) = k-1$, which is impossible. Thus any other pair of vertices has distance less than r. ■

The lower bound is sharp. For example, the graph $K_k + \overline{K}_{n-k}$ has diameter 2. The upper bound is sharp for all k. For $k = 1$, the unique extremal graph is P_{d+1}. In general, form a graph as follows. Establish a $k \times (r-1)$ grid of vertices. Add the edges between vertices $v_{i,j}$ and $v_{s,t}$ if $t = j+1$ or $t = j = 1$. (Thus we have a graph that decomposes into $r-2$ copies of $K_{k,k}$ and one K_k.) Finally, add a vertex u adjacent to $v_{i,1}$ for all i and a vertex v adjacent to $v_{i,r-1}$ for all i. It is easily checked that this graph is maximal k-degenerate.

We can provide an operation characterization of graphs that achieve the upper bound of the previous theorem.

Theorem 9. *A maximal k-degenerate graph G has $diam(G) = \frac{n-2}{k} + 1$ if and only if G can be constructed by the following algorithm.*

(1) *Begin with either $K_k + 2K_1$ or a graph formed from any maximal k-degenerate graph of order $2k$ by adding two vertices of degree k with no common neighbors.*

(2) *Iterate the following operation. Let $v = v_0$ be vertex of degree k in G with neighbors $\{u_1, \ldots, u_k\}$. Successively add $k-1$ vertices $\{v_1, \ldots, v_{k-1}\}$ with degree k when added so that the neighbors of v_i are all in $\{u_1, \ldots, u_k, v_0, \ldots, v_{i-1}\}$. Then add a new vertex v' adjacent to $\{v_0, \ldots, v_{k-1}\}$.*

Proof. ($\Leftarrow$) The initial graphs satisfy the equality. One iteration of the operation increases the order by k and the diameter by one, so equality is maintained.

($\Rightarrow$) Let G be a graph with $diam\,(G) = \frac{n-2}{k} + 1$. Then G has two vertices u and v of degree k with $d\,(u, v) = diam\,(G)$. Label the vertices with their distance from u and call the vertices with a common label a column. Then vertices in non-consecutive columns cannot be adjacent and each internal column contains exactly k vertices.

We show that G contains a subgraph as provided in 1. G can be constructed beginning with K_{k+1}, and this clique must be contained in two consecutive columns. If it contains all of one column and one in an adjacent column, then the first vertex added on the other side can only be adjacent to all the vertices in the filled column, creating a copy of $K_k + 2K_1$. If not, then constructing G must add vertices within the two columns that the clique overlaps until one of them is full. As before, the first vertex added on the other side can only be adjacent to all the vertices in the filled column. We can separately add vertices to fill the second column, creating a graph with order $2k$, and adding one more vertex on each side creates the subgraph specified in 1.

Thus G contains a maximal subgraph H that can be constructed using the algorithm. Assume that this is not all of G. Let u' and v' be the opposite vertices of degree k in H. At least one of these is not u or v, respectively; WLOG assume this is u'. Now continue constructing G by adding vertices on this side of the graph. They can only be adjacent to u', its neighbors, or some of the vertices previously added. They must fill the column of u' before a vertex can be added to the next column, which must be adjacent to all the vertices of the column of u'. But this is a larger subgraph constructed using the algorithm. ∎

Note that the second part of step 1 is unnecessary when k is 2 or 3, but may be necessary for larger k.

3. Degree Sequences

We can characterize the degree sequences of maximal k-degenerate graphs. A different characterization with a longer proof was offered in [2].

Lemma 10. *Let G be maximal k-degenerate with order n and nonincreasing degree sequence $d_1, \ldots, d_n$. Then $d_i \leq k + n - i$.*

Proof. Assume to the contrary that $d_i > k + n - i$ for some i. Let H be the graph formed by deleting the $n - i$ vertices of smallest degree. Then $\delta(H) > k$, so G has a $(k+1)$-core. ■

Lemma 11. *Let $d_1, \ldots, d_n$ be nonincreasing sequence of integers with $\sum_{i=1}^{n} d_i = 2(k \cdot n - \binom{k+1}{2})$ such that $k \leq d_i \leq \min\{n-1, k+n-i\}$. Then at most $k+1$ terms of the sequence achieve the upper bound.*

Proof. Visualize the problem as stacking boxes in adjacent columns so that the height of the ith column is d_i. If all the terms other than d_n that achieve the upper bound are at the beginning of the sequence, then there are at most k, since $\sum_{i=1}^{n} d_i \geq k(n-1) + (n-k)k = 2k \cdot n - k(k+1)$. Filling the row at height $k+1$ would require $n-k-1$ more boxes, which would have to be moved from at least two of the columns. Similarly, filling more rows requires disrupting at least as many columns. Thus there are at most $k+1$ terms that achieve the upper bound when all the columns that achieve the upper bound are at the beginning or end of the sequence. Suppose there is sequence that is a counterexample, and let it maximize the number of columns at the beginning or end that achieve the maximum. There must be a column somewhere in the middle that achieves the upper bound. Then some boxes can be moved to a column or row next to the the run of those at the beginning or end that to achieve the upper bound, producing a contradiction. ■

Similar analysis shows that only n columns at the beginning and one at the end can achieve the upper bound exactly $k+1$ times, in which case the corresponding graph must be $K_k + \overline{K}_{n-k}$.

Theorem 12. *A nonincreasing sequence of integers $d_1, \ldots, d_n$ is the degree sequence of a maximal k-degenerate graph G if and only if $k \leq d_i \leq \min\{n-1, k+n-i\}$ and $\sum_{i=1}^{n} d_i = 2(k \cdot n - \binom{k+1}{2})$ for $0 \leq k \leq n-1$.*

Proof. Let $d_1, \ldots, d_n$ be such a sequence.

($\Rightarrow$) Certainly $\Delta(G) \leq n-1$. The other three conditions have already been shown.

($\Leftarrow$) For $n = k+1$, the result holds for $G = K_{k+1}$. Assume the result holds for order r. Let $d_1, \ldots, d_{r+1}$ be a nonincreasing sequence that satisfies the given properties. Let $d'_1, \ldots, d'_r$ be the sequence formed by deleting d_{r+1} and decreasing k other numbers greater than k by one, including any that achieve the maximum. (There are at most k by the preceding lemma.) Then the new sequence satisfies all the hypotheses and has length r, so it is the degree sequence for some maximal k-degenerate graph H. Add vertex v_{r+1} to H, making it adjacent to the vertices with degrees that were decreased for the new sequence. Then the resulting graph G has the original degree sequence and is maximal k-degenerate. ■

The upper bound in this theorem can be improved.

Corollary 13. *For a nonincreasing degree sequence $d_1, \ldots, d_n$ of a maximal k-degenerate graph, $d_i \leq \frac{k(n-k-1)}{i} + k$. Hence*

$$d_i \leq \min\{n-1, \left\lfloor \frac{k(n-k-1)}{i} \right\rfloor + k, k+n-i\}$$

and for each i, there is some maximal k-degenerate graph that attains this bound.

Proof. Since $d_i \geq k$, $d_i - k \geq 0$. Now $i\,(d_i - k) \leq \Sigma_{i=1}^n (d_i - k) = \Sigma_{i=1}^n d_i - k \cdot n = 2\left(k \cdot n - \binom{k+1}{2}\right) - k \cdot n = k \cdot n - k\,(k+1) = k\,(n-k-1)$. Hence $d_i \leq \frac{k(n-k-1)}{i} + k$. The next upper bound follows immediately.

Consider stacking boxes as in Lemma 11. Then each column has at least k, and there are $k\,(n-k-1)$ left to work with. If $n-1$ is the lowest of the three bounds, add $n-k-1$ boxes to the first k columns. If the second is lowest, add an additional $\left\lfloor \frac{k(n-k-1)}{i} \right\rfloor$ boxes to the first i columns, and distribute any leftovers arbitrarily. If the third is lowest, add $n-i$ boxes to the first i columns, and distribute any leftovers arbitrarily. This is possible since in this case, the third bound is smaller than the second. Then the corresponding sequence attains the upper bound at i and by the previous theorem, there is some maximal k-degenerate graph with this degree sequence. ■

We can bound the maximum degree of a maximal k-degenerate graph. Intuitively, since there are approximately $k \cdot n$ edges in G, its maximum degree should be at least $2k$, provided that G has order large enough to overcome the constant $\binom{k+1}{2}$ subtracted from the size. The following theorem was proven in [7].

Theorem 14. *Let G be maximal k-degenerate of order n.*

(1) *Let $k \geq 2$ and $0 \leq s \leq k-2$. If $n > \frac{k^2+(3+2s)k}{2(1+s)} - \frac{s}{2}$, then $\Delta(G) \geq 2k - s$.*

(2) *In particular, if $n \geq \binom{k+2}{2}$, then $\Delta(G) \geq 2k$.*

(3) *If $n \leq \frac{1+\sqrt{1+8k}}{2} + k$, then $\Delta(G) = n-1$.*

Part 2 is the best possible in two senses. First, no larger lower bound for the minimum degree can be guaranteed, regardless how large the order is. Second, the hypothesis on n is the smallest that guarantees the result. This can be seen by constructing a maximal k-degenerate graph so that when added, each new vertex is made adjacent to the k vertices of smallest degree.

Lemma 15. *If G is maximal k-degenerate, then $G+v$ is maximal $(k+1)$-degenerate. If G has a vertex v of degree $n-1$, then $G-v$ is maximal $(k-1)$-degenerate.*

Corollary 16. *Let $d_1, \ldots, d_n$ be the nonincreasing degree sequence of a maximal k-degenerate graph G with $k \geq 2$ and $0 \leq s \leq k-2$. If $n > \frac{k^2+(3+2s)k}{2(1+s)} - \frac{s}{2}$, then $d_i \geq 2k+1-s-i$.*

Proof. The previous theorem shows this for d_1. If $d_1 < n-1$, then by shifting boxes as in Lemma 11, we can find a maximal k-degenerate graph G^* with the same order so that $d_1^* = n-1$ and $d_i^* \leq d_i$. Then $H = G^* - v_1^*$ is maximal $(k-1)$-degenerate with order $n-1 \geq \left(\frac{k^2+(3+2s)k}{2(1+s)} - \frac{s}{2}\right) - 1 \geq \frac{(k-1)^2+(3+2s)(k-1)}{2(1+s)} - \frac{s}{2}$. Thus $\Delta(H) \geq 2(k-1) - s$, so $d_2 \geq d_2^* \geq 2(k-1) - s + 1 = 2k - s - 1$. Iterating this process produces the result for larger i. ■

The following relationship between the numbers of vertices of different degrees was proved in [2].

Proposition 17. *Let G be maximal k-degenerate with $\Delta(G) = r$, $n \geq k+1$, and n_i the number of vertices of degree i, $k \leq i \leq r$. Then*

$$\sum_{i=k}^{r} (i-2k)\, n_i + k(k+1) = 0.$$

4. Edge Coloring

A proper edge coloring of a graph assigns a color to each edge so that adjacent edges are colored differently. The edge chromatic number of a graph, $\chi_1(G)$, is the smallest number of colors that can be used in a proper edge coloring. Clearly the edge chromatic number is at least as large as the maximum degree. Vizing showed that it is never more than $\Delta(G) + 1$. A graph is called class one if $\chi_1(G) = \Delta(G)$, and class two if $\chi_1(G) = \Delta(G) + 1$. Determining which of the two is the case is a difficult problem in general. We consider this problem for maximal k-degenerate graphs. It is easily shown that every tree is class one.

Zhou Goufei [8] proved the following result on edge coloring of k-degenerate graphs. Its proof uses Vizing's adjacency lemma.

Theorem 18. *Every k-degenerate graph with $\Delta \geq 2k$ is class one.*

This theorem and Theorem 14 produce the following corollary.

Corollary 19. *If G is maximal k-degenerate with $n \geq \binom{k+2}{2}$, then G is class one.*

This implies that almost all maximal k-degenerate graphs are class one. In particular, this theorem implies that if G is 2-degenerate and $\Delta(G) \geq 4$, then G is class one.

A graph G is overfull if n is odd and $m > \frac{n-1}{2}\Delta(G)$. It is easily seen that an overfull graph is class two. This result and the preceding theorem imply that the only maximal 2-degenerate graphs of class two are K_3 and K_4 with a subdivided edge.

Conjecture 20. *A maximal k-degenerate graph is class two if and only if it is overfull.*

A related conjecture is the overfull conjecture [5].

Conjecture 21 (Overfull Conjecture)**.** *If a graph G with n vertices has $\Delta(G) > \frac{n}{3}$, then G is Class 1 if and only if G has no overfull subgraph H with $\Delta(G) = \Delta(H)$.*

If a maximal k-degenerate graph has an overfull subgraph with the same maximum degree then it is itself overfull. Also, the largest order than can could violate this conjecture is $\binom{k+2}{2} - 1$. Now $\frac{\binom{k+2}{2}-1}{3} \leq 2k-1$ implies $k \leq \frac{9+\sqrt{57}}{2} < 9$. Thus for small k, $(1 \leq k \leq 8)$ the overfull conjecture implies Conjecture 20.

Conjecture 20 holds for $k = 3$.

Theorem 22. *The maximal 3-degenerate graphs of class two are exactly K_3, $K_5 - e$, and all those of order 9 and maximum degree 5.*

Proof. This is easily checked for orders 1–5. Maximal 3-degenerate graphs of order 9 and maximum degree 5 have $m = 3 \cdot 9 - 6 = 21 > 20 = \frac{9-1}{2}5 = \frac{n-1}{2}\Delta$ and hence are overfull. By the previous theorems, maximal 3-degenerate graphs with $\Delta \geq 6$ and $n \geq 10$ are class one. Thus we need only check graphs with orders 6–8 and maximum degree 5.

There are three maximal 3-degenerate graphs of order 6, $G_1 = K_3 + \overline{K}_3 = K_6 - K_3$, $G_2 = P_4 + K_2 = K_6 - P_4$, and $G_3 = K_6 - (P_3 \cup K_2)$. They can be 5-edge colored as in the first table below. Now the orbits are $\{1,2,3\}, \{4,5,6\}$ for G_1, $\{1,4\}, \{2,3\}, \{5,6\}$ for G_2, and $\{1,2\}, \{3\}, \{4,5\}, \{6\}$ for G_3. Hence up to isomorphism, there are respectively 1, 2, and 5 maximal 3-degenerate graphs of order 7 and maximum degree 5 that can be built off these three graphs (there are seven total since one is repeated among the last two sets). By labeling the vertices with the colors not used on edges incident with them as in the second table, it is easy to check that the 5-edge colorings can be extended in each case.

Label	G_1	G_2	G_3
1	23,16	23,46	23,56
2	13,24	13,24	13,24
3	14,25,36	14,25,36	14,25,36
4	15,34,26	15,34,26	15,34
5	12,35	12,35	12,35,46
absent	45,46,56	45,16,56	45,16,26

Vertex	G_1	G_2	G_3
1	x	1	1
2	x	x	4
3	x	x	x
4	1,5	5	1
5	1,2	1,2	2
6	2,5	2,5	2,4

Now each maximal 3-degenerate graph of order 7 and maximum degree 5 has size 15, so each edge color class contains three edges and misses one of the vertices. Hence when adding a vertex adjacent to three vertices of degree less than five, the 5-edge coloring can be extended to the new vertex. Thus all maximal 3-degenerate graphs of order 8 are class one. ■

The only overfull maximal 4-degenerate graphs are K_3 and K_5. Conjecture 20 is easily verified for maximal 4-degenerate graphs with $\Delta(G) \leq 6$. An argument similar to that used in the last paragraph of the proof of the previous theorem would extend 7-edge colorings of maximal 4-degenerate graphs with order 13 and $\Delta(G) = 7$ to order 14. Hence only these graphs need to be checked to verify the conjecture for $k = 4$.

5. Decompositions and Arboricity

Maximal k-degenerate graphs have some interesting decompositions.

Theorem 23. *Let $t_1, \ldots, t_r$ be r positive integers which sum to t. Then a maximal t-degenerate graph can be decomposed into r graphs with degeneracies at most $t_1, \ldots, t_r$, respectively.*

Proof. Consider sucessively deleting vertices of a maximal t-degenerate graph G so that each vertex has degree at most t when deleted. When a vertex is deleted, the edges incident with it can be allocated to r subgraphs with at most $t_1, \ldots, t_r$ edges going to the respective subgraphs. Thus the subgraphs have at most the stated degeneracies. ■

In particular, a k-degenerate graph decomposes into k forests. These can be almost trees, except for the initial k-clique. The graph G/H is formed by contracting the subgraph H of G to a single vertex.

Corollary 24. *A maximal k-degenerate graph G of order $n \geq k$ can be decomposed into K_k and k trees of order $n - k + 1$, which span G/K_k.*

Proof. If $n = k$, $G = K_k$, so let the k trees be k distinct isolated vertices. Build G by successively adding vertices of degree k. Allocate one edge to each of the k trees in such a way that each is connected. To do this, assign an edge incident with a vertex of the original clique to the unique tree containing that vertex. Any other edges can be assigned to any remaining tree, since every tree contains every vertex not in the original clique. ■

Note that if k is odd, a maximal k-degenerate graph decomposes into k trees of order $n - \frac{k-1}{2}$, since given $k = 2r - 1$, K_{2r} can be decomposed into k trees of order $r + 1$.

The previous corollary also implies that a maximal 2-degenerate graph has two spanning trees that contain all its edges and overlap on exactly one edge. This 'overlap edge' can be any edge that is the last to be deleted by the k-core algorithm.

Definition. The edge-arboricity, or simply arboricity $a_1(G)$ is the minimum number of forests into which G can be decomposed. The a-density of a nontrivial graph G is $\frac{m}{n-1}$.

Nash-Williams [14] (see also [4] and [9]) showed that for every nonempty graph G, $a_1(G) = \max \left\lceil \frac{m(H)}{n(H)-1} \right\rceil$, where the maximum is taken over all induced subgraphs H of G. This maximum may be difficult to calculate in general.

We now determine an explicit formula for the arboricity of maximal k-degenerate graphs. This question has been previously considered in [15]. We provide a much shorter proof. Note that it follows immediately from Theorem 23 that if G is maximal k-degenerate, then $a_1(G) \leq k$. The arboricity may be smaller if n is small relative to k.

Theorem 25. *Let G be maximal k-degenerate. Then* $a_1(G) = \left\lceil k - \binom{k}{2} \frac{1}{n-1} \right\rceil$.

Proof. A maximal k-degenerate graph of order n has size $m = k \cdot n - \binom{k+1}{2}$. Then its a-density is $\frac{m}{n-1} = \left[k \cdot n - \binom{k+1}{2}\right] \frac{1}{n-1} = k + \left[k - \binom{k+1}{2}\right] \frac{1}{n-1} = k - \binom{k}{2} \frac{1}{n-1}$. Note that this function is monotone with respect to n. Now any subgraph of a k-degenerate graph is also k-degenerate, so this implies that any proper subgraph of G has smaller a-density. Then by Nash-Williams' theorem, $a_1(G) = \left\lceil k - \binom{k}{2} \frac{1}{n-1} \right\rceil$. ■

Since any graph with $\widehat{C}(G) = k$ is contained in a maximal k-degenerate graph, this theorem implies that the bound $a_1(G) \leq \widehat{C}(G)$ is sharp for all k. More specifically, for a given k, it is sharp for all $n \geq \binom{k}{2} + 2$. For $n \leq \binom{k}{2} + 1$, it is not sharp. But this theorem implies the following easy-to-calculate upper bound.

Corollary 26. *Let $k = \widehat{C}(G)$. Then* $a_1(G) \leq \left\lceil k - \binom{k}{2} \frac{1}{n-1} \right\rceil$.

In our efforts to reduce the number of subgraphs of a graph G that must be checked to determine its arboricity, we can also bound the orders of the subgraphs. Clearly a very small subgraph has no chance of achieving the maximum.

Corollary 27. *Let G be a graph with $k = \widehat{C}(G)$ containing a subgraph with a-density $d < k$. Then any subgraph of maximum a-density has order at least* $n \geq \binom{k}{2} \frac{1}{k-d} + 1$.

Proof. A subgraph H with maximum core number k has maximum a-density when it is maximal k-degenerate. Thus the order n of H must satisfy $k - \binom{k}{2} \frac{1}{n-1} \geq d$. This is equivalent to $k - d \geq \binom{k}{2} \frac{1}{n-1}$, and $n \geq \binom{k}{2} \frac{1}{k-d} + 1$, so the result follows. ■

Thus determining the arboricity can be simplified by checking subgraphs of relatively large order.

6. k-trees

There is one particular subclass of maximal k-degenerate graphs that is of interest.

Definition. A k-tree is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k-clique of the existing graph. The clique used to start the construction is called the root of the k-tree.

It is easy to see that a k-tree is maximal k-degenerate. A 1-tree is just a tree. However, k-trees and maximal k-degenerate graphs are not equivalent for $k \geq 2$.

In fact, every maximal k-degenerate graph of order $n \geq k+1$ contains an induced k-tree. For $n \geq k+2$, $K_{k+2} - e$ must occur. No larger k-tree can be guaranteed. For example, let U be a k-element set of vertices of $K_{k+2} - e$ containing both vertices of degree k, and let V be the partite set of order k of $K_{k,r}$. Then $(K_{k+2} - e) \underset{U=V}{\cup} K_{k,r}$, where the union identifies the sets U and V, has order $n \geq k+3$ and no larger induced k-tree.

Theorem 28. *Every maximal k-degenerate graph G of order $n \geq k+1$ contains a unique k-tree of largest possible order containing a $(k+1)$-clique that can be used to begin the construction of G.*

Proof. It is obvious that every maximal k-degenerate graph can be constructed beginning with a maximal k-tree. We prove uniqueness. Suppose to the contrary that there is a maximal k-degenerate graph containing two distinct maximal k-trees either of which can be used to begin its construction. Let G be a counterexample of minimum order $n \geq k+3$ containing k-trees T_1 and T_2. Divide the vertices of G into $V(T_1)$, $V(T_2)$, and $S = V(G) - V(T_1) - V(T_2)$. Now G has at least one vertex v of degree k. If $v \in S$, then $G - v$ can be constructed starting with either k-tree, so there is a smaller counterexample. If $v \in V(T_1)$ and $n(T_1) \geq k+2$, then $G - v$ can be still be constructed starting with some other vertex of T_1, so there is a smaller counterexample. If $v \in V(T_i)$, $i \in \{1, 2\}$, and $T_i = K_{k+1}$, then G cannot be constructed starting with T_i since any maximal k-tree that can be used to begin construction of G must contain $K_{k+2} - e$. Thus in any case we have a contradiction. ∎

We offer two characterizations of k-trees as maximal k-degenerate graphs. A graph is chordal if every cycle of length more than three has a chord, that is, it contains no induced cycle other than C_3.

Theorem 29. *A graph G is a k-tree if and only if G is maximal k-degenerate and G is chordal with $n \geq k+1$.*

Proof. ($\Rightarrow$) Let G be a k-tree. Then G is clearly maximal k-degenerate. If G is not chordal, then there is a minimal counterexample H which must contain a chordless cycle C that by minimality contains a vertex v of degree k. But since all the neighbors of v are adjacent, C must have a chord.

($\Leftarrow$) Assume G is maximal k-degenerate and chordal. If $k=1$, it is immediate that G is a k-tree, so assume that $k \geq 2$. If $n = k+1$, G is certainly a k-tree. Assume the result holds for order r, and let G have order $r+1$. Then G has a vertex v of degree k. If the neighbors of v do not induce a clique, then v has two nonadjacent neighbors x and y. Since G is k-connected, the graph formed by deleting all the neighbors of v except x and y is 2-connected. Thus an $x-y$ path of shortest length in $G-v$ together with yv and vx would produce a cycle with no chord. Since $G-v$ is a k-tree, so is G. ∎

The second characterization of k-trees as maximal k-degenerate graphs involves subdivisions.

Theorem 30. *A maximal k-degenerate graph is a k-tree if and only if it contains no subdivision of K_{k+2}.*

Proof. ($\Rightarrow$) Let G be a k-tree. Certainly K_{k+1} contains no subdivision of K_{k+2}. Suppose G is a counterexample of minimum order with a vertex v of degree k. Then $G-v$ is a k-tree with no subdivision of K_{k+2}, so the subdivision in G contains v. But then v is not one of the $k+2$ vertices of degree $k+1$ in the subdivision, so it is on a path P between two such vertices. Let its neighbors on P be u and w. But since the neighbors of v form a clique, $uw \in G-v$, so P can avoid v, implying $G-v$ has a subdivision of K_{k+2}. This is a contradiction.

($\Leftarrow$) Let G be maximal k-degenerate and not a k-tree. Since G is constructed beginning with a k-tree, for a given construction sequence there is a first vertex in the sequence that makes G not a k-tree. Let v be this vertex, and H be the maximal k-degenerate subgraph induced by the vertices of the construction sequence up to v. Then $n(H) \geq k+3$, $d_H(v) = k$, v has nonadjacent neighbors u and w, and $H-v$ is a k-tree. Now there is a sequence of at least two $(k+1)$-cliques starting with one containing u and ending with one containing w, such that each pair of consecutive $(k+1)$-cliques in the sequence overlap on a k-clique. Then two of these cliques and a path through v produces a subdivision of K_{k+2}. ∎

Dirac [6] determined the minimum size of a graph G of order n that will guarantee that G contains a subdivision of K_4. We can prove this simply and determine the extremal graphs.

Corollary 31. *If G has $m \geq 2n-2$, then G contains a subdivision of K_4, and the graphs of size $2n-3$ that fail to contain a subdivision of K_4 are exactly the 2-trees.*

Proof. Let G have $m \geq 2n-2 = (3-1)\,n - \binom{3}{2} + 1$. By Corollary 4, G contains a 3-core. It is known that every 3-core contains a subdivision of K_4. If a graph of size $2n-3$ has no 3-core, it is maximal 2-degenerate. By the previous theorem, exactly the 2-trees do not contain a subdivision of K_4. ∎

A natural generalization of this result is that if $m \geq 3n-5$, G contains a subdivision of K_5. This was conjectured by Dirac and proved by Mader [12] using a much more intricate argument.

7. Ramsey Core Numbers

The problem of Ramsey numbers is one of the major problems of extremal graph theory. Given positive integers $t_1, t_2, \ldots, t_k$, the classical Ramsey number $r\,(t_1, \ldots, t_k)$ is the smallest integer n such that for any decomposition of K_n into k factors, for some i, the i^{th} factor has a t_i-clique. This problem can be modified to require the existence of other classes of graphs. Since classical Ramsey numbers are defined, which is not trivial to show, such modifications are also defined, since every finite graph is a subgraph of some clique. When considering cores, the following modified problem arises naturally.

Definition. Given nonnegative integers $t_1, t_2, \ldots, t_k$, the Ramsey core number $rc\,(t_1, t_2, \ldots, t_k)$ is the smallest n such that for all edge colorings of K_n with k colors, there exists an index i such that the subgraph induced by the i^{th} color, H_i, has a t_i-core.

Several basic results can be obtained immediately.

Proposition 32. (1) $rc\,(t_1, t_2, \ldots, t_k) \leq r\,(t_1 + 1, \ldots, t_k + 1)$.

(2) *For any permutation σ of $[k]$, $rc\,(t_1, t_2, \ldots, t_k) = rc\left(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(k)}\right)$.*

(3) $rc\,(0, t_2, \ldots, t_k) = 1$.

(4) $rc\,(1, t_2, \ldots, t_k) = rc\,(t_2, \ldots, t_k)$.

We can easily determine some classes of multidimensional Ramsey core numbers.

Proposition 33. *Let $t_1 = t_2 = \cdots = t_k = 2$. Then $rc\,(t_1, t_2, \ldots, t_k) = 2k+1$.*

Proof. It is well known that the complete graph K_{2k} can be decomposed into k spanning paths, each of which has no 2-core. Thus $rc\,(2, 2, \ldots, 2) \geq 2k+1$. K_{2k+1} has size $\binom{2k+1}{2} = k\,(2k+1)$, so if it decomposes into k graphs, one of them has at least $2k+1$ edges, and hence contains a cycle. Thus $rc\,(2, 2, \ldots, 2) = 2k+1$. ∎

The technique of this proof suggests a general upper bound for Ramsey core numbers.

Definition. The multidimensional upper bound for the Ramsey core number $rc(t_1, t_2, \ldots, t_k)$ is the function $B(t_1, t_2, \ldots, t_k)$, where $T = \sum t_i$ and

$$B(t_1, \ldots, t_k) = \left\lceil \tfrac{1}{2} - k + T + \sqrt{T^2 - \sum t_i^2 + (2-2k)\,T + k^2 - k + \tfrac{9}{4}} \right\rceil .$$

With a definition like that, this had better actually be an upper bound.

Theorem 34 (The Upper Bound)**.** $rc(t_1, t_2, \ldots, t_k) \le B(t_1, \ldots, t_k)$.

Proof. The size of a maximal k-core-free graph of order n is $(k-1)\,n - \binom{k}{2}$. Now by the Pigeonhole Principle, some H_i has a t_i-core when $\binom{n}{2} \ge \sum_{i=1}^{k}((t_i-1)n - \binom{t_i}{2}) + 1$. This is equivalent to $n^2 - n \ge 2n\sum_{i=1}^{k}(t_i - 1) - \sum_{i=1}^{k}\left(t_i^2 - t_i\right) + 2$. This is a quadratic inequality $n^2 - bn + c \ge 0$ with $b = 1 + 2\sum t_i - 2k$ and $c = \sum\left(t_i^2 - t_i\right) - 2$. By the quadratic formula, $n \ge \frac{1}{2}\left(b + \sqrt{b^2 - 4c}\right)$ and $b^2 - 4c = (1 + 4T - 4k + 4T^2 - 8kT + 4k^2) - (4\sum t_i^2 - 4T - 8) = 4(T^2 - \sum t_i^2 + (2-2k)T + k^2 - k + \frac{9}{4})$.
Thus $n \ge \left\lceil \frac{1}{2} - k + T + \sqrt{T^2 - \sum t_i^2 + (2-2k)\,T + k^2 - k + \frac{9}{4}} \right\rceil = B(t_1, \ldots, t_k)$.
Now $rc(t_1, \ldots, t_k) \le \min\{n \mid n \ge B(t_1, \ldots, t_k)\} = B(t_1, \ldots, t_k)$. ■

Thus to show that a Ramsey core number achieves the upper bound, we must find a decomposition of the complete graph of order $B(t_1, \ldots, t_k) - 1$ for which none of the factors contain the stated cores. To prove this, we state the following theorem due to R. Klien and J. Schonheim [10].

Theorem 35. *Any complete graph with order* $n < B(t_1, \ldots, t_k)$ *has a decomposition into* k *subgraphs with degeneracies at most* $t_1 - 1, \ldots,\ t_k - 1$.

The proof of this theorem is difficult. It uses a complicated algorithm to construct a decomposition of a complete graph with order satisfying the inequality into k subgraphs given a decomposition of a smaller complete graph into $k-1$ subgraphs without the first $k-1$ cores, a copy of K_{t_k}, and some extra vertices. Thus the proof that the algorithm works uses induction on the number of subgraphs.

Using this theorem, proving the conjecture is not hard.

Theorem 36. *We have* $rc(t_1, t_2, \ldots, t_k) = B(t_1, \ldots, t_k)$.

Proof. We know that $B(t_1, \ldots, t_k)$ is an upper bound. By the previous theorem, there exists a decomposition of the complete graph of order $B(t_1, \ldots, t_k) - 1$ such that subgraph H_i has degeneracy $t_i - 1$, and hence has no t_i-core. Thus $rc(t_1, t_2, \ldots, t_k) > B(t_1, \ldots, t_k) - 1$, so $rc(t_1, t_2, \ldots, t_k) = B(t_1, \ldots, t_k)$. ■

s\t	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
2	3	5	6	8	9	10	12	13	14	15	17
3	4	6	8	10	11	13	14	15	17	18	19
4	5	8	10	11	13	15	16	18	19	20	22
5	6	9	11	13	15	16	18	20	21	23	24
6	7	10	13	15	16	18	20	21	23	25	26
7	8	12	14	16	18	20	22	23	25	26	28

Table 1. Values of some 2-dimensional Ramsey core numbers.

Since the exact answer depends on a complicated construction, some simpler constructions remain of interest.

Lemma 37. *We have* $rc(t_1+1, t_2, \ldots, t_k) \geq rc(t_1, \ldots, t_k) + 1$.

Proof. Let $n = rc(t_1+1, t_2, \ldots, t_k)$. Then there exists a decomposition of K_{n-1} with each factor having no t_i-core for all i. Consider the decomposition of K_n formed from the previous decomposition by joining a vertex to the first factor. Then the first factor has no t_1+1-core. Thus $rc(t_1+1, t_2, \ldots, t_k) \geq rc(t_1, \ldots, t_k) + 1$. ■

The formula for $rc(2,t)$ can be expressed in another form, and proven using a simple construction.

Theorem 38. *Let* $t = \binom{r}{2} + q$, $1 \leq q \leq r$. *Then* $rc(2,t) = \binom{r}{2} + r + q + 1 = t + r + 1 = B(2,t)$.

Proof. We first show that the Upper Bound for $rc(2,t)$ can be expressed as a piecewise linear function with each piece having slope one and breaks at the triangular numbers. Let $t = \binom{r}{2}$. Let $B'(s,t) = s + t - \frac{3}{2} + \sqrt{2(s-1)(t-1) + \frac{9}{4}}$. Then $B(s,t) = \lceil B'(s,t) \rceil$. Now $B'(2,t) = 2 + t - \frac{3}{2} + \sqrt{2 \cdot 1 (t-1) + \frac{9}{4}} = t + \frac{1}{2} + \sqrt{2\frac{r(r-1)}{2} + \frac{1}{4}} = t + \frac{1}{2} + \sqrt{\left(r - \frac{1}{2}\right)^2} = t + r$, which is an integer. Now $B'(2, t+1) > t+r+1$, so $B(2,t+1) \geq t+r+2$. Then $B(2,t+q) \geq t+r+1+q$ for $q \geq 1$ by the Lower Bound. Now $B'(2, t+r) = B'\left(2, \binom{r+1}{2}\right) = t + r + r + 1$, an integer. Thus $B(2, t+r) = t+r+r+1$, so $B(2, t+q) \leq t+r+1+q$ for $1 \leq q \leq r$ by the previous lemma. Thus $B(2, t+q) = t+r+1+q$, $1 \leq q \leq r$, so $rc(2, t) \leq t+r+1$ for $t = \binom{r}{2} + q$.

We next show that the upper bound is attained with an explicit construction. Let T be a caterpillar whose spine with length r is

$$r - r - (r-1) - (r-2) - \cdots - 4 - 3 - 2,$$

where a number is the degree of a vertex and end-vertices are not shown. Now T has $[(r-1) + (r-2) + (r-3) + \cdots + 2 + 1] + 1 = \binom{r}{2} + 1$ leaves, so it has order $n = \binom{r}{2} + r + 1$. The degrees of corresponding vertices in T and $\overline{T}$ must add up to $n - 1 = \binom{r}{2} + r$. Then the degrees of corresponding vertices in $\overline{T}$ are

$$\binom{r}{2},\ \binom{r}{2},\ \binom{r}{2} + 1,\ \binom{r}{2} + 2, \ldots,\ \binom{r}{2} + r - 3,\ \binom{r}{2} + r - 2.$$

Take the $\left(\binom{r}{2} + 1\right)$-core of $\overline{T}$. The first two vertices will be deleted by the k-core algorithm. The p^{th} vertex will be deleted because it has degree $\binom{r}{2} + p - 2$ and is adjacent to the first $p-2$ vertices, which were already deleted. Thus all the spine vertices will be deleted, leaving $\binom{r}{2} + 1$ vertices, which must also be deleted. Thus $\overline{T}$ has no $\left(\binom{r}{2} + 1\right)$-core, and T has no 2-core. Thus $rc\left(2, \binom{r}{2} + 1\right) \geq \binom{r}{2} + r + 1 + 1$. Thus $rc\left(2, \binom{r}{2} + q\right) \geq \binom{r}{2} + r + 1 + q$ by the Lower Bound.

Thus $rc(2, t) = t + r + 1$ for $t = \binom{r}{2} + q$, $1 \leq q \leq r$. ■

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