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GENERALIZATIONS OF THE TREE PACKING CONJECTURE

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Abstract

The Gyárfás tree packing conjecture asserts that any set of trees with $2, 3, \ldots, k$ vertices has an (edge-disjoint) packing into the complete graph on k vertices. Gyárfás and Lehel proved that the conjecture holds in some special cases. We address the problem of packing trees into k-chromatic graphs. In particular, we prove that if all but three of the trees are stars then they have a packing into any k-chromatic graph. We also consider several other generalizations of the conjecture.

Keywords: packing, tree packing.

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1. Introduction

A set of (simple) graphs $G_1, G_2, \ldots, G_k$ has a *packing* into a graph H if $G_1, G_2, \ldots, G_k$ appear as edge-disjoint subgraphs of H. In general we are concerned with the case when each G_i is a tree. One of the best-known packing problems is the Tree Packing Conjecture (TPC) posed by Gyárfás [8]:

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Conjecture 1 (TPC). *For $2 \leq i \leq n$, let T_i be a tree on i vertices. Then the set of trees $T_2, \ldots, T_n$ has a packing into the complete graph on n vertices.*

A number of partial results related to the TPC have been found. The first results are by Gyárfás and Lehel [8] who proved that the TPC holds with the additional assumption that all but two of the trees are stars. Gyárfás and Lehel also showed that the TPC is true if each tree is either a path or a star. A second proof is by Zaks and Liu [14]. Bollobás [1] showed that the trees $T_2, \ldots, T_s$ have a packing into K_n if $s \leq n/\sqrt{2}$ and T_i has i vertices. From the other side, Hobbs, Bourgeois and Kasiraj [10] showed that any three trees T_n, T_{n-1}, T_{n-2} have a packing into K_n if T_i has i vertices. A series of papers by Dobson [4, 5, 6] concerns packing trees with some technical conditions.

Instead of packing trees into the complete graph, a number of papers have examined packing trees into complete bipartite graphs. Hobbs *et al.* [10] conjectured that the trees $T_2, \ldots, T_n$ have a packing into the complete bipartite graph $K_{n-1,\lceil n/2 \rceil}$ if T_i has i vertices. The conjecture is true if each of the trees is a star or path. The case when n is even was shown by Zaks and Liu [14] and when n is odd by Hobbs [9]. Yuster [13] showed that $T_2, \ldots, T_s$ have a packing into $K_{n-1,\lceil n/2 \rceil}$ if $s \leq \lfloor \sqrt{5/8}n \rfloor$ and T_i has i vertices (improving the previously best-known bound by Caro and Roditty [2]).

Now we introduce a conjecture that would imply the TPC:

Conjecture 2. *For $2 \leq i \leq k$, let T_i be a tree on i vertices. If G is a k-chromatic graph, then the set of trees $T_2, \ldots, T_k$ has a packing into G.*

The main result of the present paper concerns a special case of Conjecture 2.

Theorem 3. *For $2 \leq i \leq k$, let T_i be a tree on i vertices. If G is a k-chromatic graph and there are at most three non-stars among $T_2, \ldots, T_k$, then they can be packed into G.*

Note that Theorem 3 can be stated in a stronger way as the proof only requires G to have a subgraph that has a Grundy k-coloring (see e.g. [3]) and minimum degree $k-1$. The immediate corollary of Theorem 3 for complete graphs was proved by Roditty [11][3].

Corollary 4. *The TPC is true with the additional assumption that all but three of the trees are stars.*

2. Proof of Theorem 3

Before moving to the proof let us introduce some additional definitions.

[3]This proof contains some errors which have recently been corrected by the author [12].

Let x be a vertex with exactly one neighbor y of degree greater than 1 and at least one neighbor of degree 1. The induced substar R spanned by x and its neighbors of degree 1 is called a *pending star*. The vertex y will be referred to as the *neighbor of* R. A *spider* is a tree that has a vertex whose removal results in isolated vertices and edges (i.e. a spider is a graph with a central vertex and some branches of length 1 or 2).

Proof. The proof will be by induction on k, but the precise form of the induction depends on the structure of the largest trees. For $k \leq 3$ the statement of the theorem is trivial. Now let us assume that the statement of the theorem holds for all values less than k.

Without loss of generality we can assume G is a vertex-critical k-chromatic graph. Thus G has minimum degree at least $k-1$. Let us choose a k-coloring of G with color classes $A_1, A_2, \ldots, A_k$ such that any vertex $x \in A_i$ has a neighbor in each color class $A_1, A_2, \ldots, A_{i-1}$. Let $G_i = G \setminus (A_1 \cup A_2 \cup \cdots \cup A_{k-i})$ be the induced subgraph of G on the color classes $A_k, A_{k-1}, \ldots, A_{k-i+1}$. Note that G_i has chromatic number i.

For simplicity we will use edge-coloring terminology. A *partial edge-coloring* of a graph G is an assignment of colors to some of the edges of G. (We will omit the word "partial".) An edge that receives no color is referred to as *uncolored*.

We will construct an edge-coloring of G such that the subgraph consisting of the edges of color i is isomorphic to the tree T_i. Clearly this edge-coloring problem is equivalent to packing the trees into G.

The proof is divided into several claims and cases according to the structure of the trees in $T_2, \ldots, T_k$. In each case we remove parts from $t \leq 3$ non-stars and delete t stars from $T_2, \ldots, T_k$ such that we are left with a sequence of trees of order $2, \ldots, k-t$ containing at most three non-stars. By induction we have a $(k-t-1)$-edge-coloring of G_{k-t} such that each tree in the new sequence is isomorphic to a subgraph spanned by the edges of a single color. To complete the desired edge-coloring of G we have two steps. First we color a few more edges to finish the non-stars in the original sequence. Second we introduce t new colors and color edges of G to get the deleted stars. Generally the (easy) details of the second step are left to the reader.

Throughout the proof if we remove some vertices of a tree T_i we denote the remaining graph by T_i'. Note that although T_i denotes a tree with i vertices, T_i' will always have fewer than i vertices.

Let x be a vertex in the tree T. After the inductive step we have an isomorphism between T and a (monochromatic) subgraph of G. For simplicity the image of x in G will also be called x.

Claim 5. *If T_k is a star and $k \geq 3$, then $T_2, \ldots, T_k$ have a packing into G.*

Proof. By induction there is a $(k-2)$-edge-coloring of G_{k-1} such that each tree $T_2, \dots, T_{k-1}$ is isomorphic to a subgraph spanned by the edges of a single color. There is at least one vertex a in A_1 and its degree is at least $k-1$ in G. Thus we can color $k-1$ edges incident to a with a new color to complete the edge-coloring of G. □

Claim 6. *If T_{k-1} is a star and $k \geq 3$, then $T_2, \dots, T_k$ have a packing into G.*

Proof. Remove a leaf with neighbor u from T_k and let T'_k be the resulting graph. By induction there is a $(k-2)$-edge-coloring of G_{k-1} such that each tree $T_2, \dots, T_{k-2}, T'_k$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k *blue*. The vertex u has a neighbor $a \in A_1$. We color the edge ua blue to get a blue T_k. The degree of a is at least $k-1$ and ua is the only colored edge incident to a. Thus a has at least $k-2$ uncolored incident edges. We color $k-2$ of these edges with a new color to get a monochromatic T_{k-2}. This completes the edge-coloring of G. □

Note that Claim 6 implies the theorem for $k = 4$.

Claim 7. *If T_k and T_{k-1} are not stars and T_{k-2} and T_{k-3} are both stars and $k \geq 5$, then $T_2, \dots, T_k$ have a packing into G.*

Proof. The trees T_k and T_{k-1} are not stars so we can remove two leaves with neighbors u and v from T_k and two leaves with neighbors x and y from T_{k-1} such that $u \neq v$ and $x \neq y$. Let T'_k and T'_{k-1} be the remaining graphs.

By induction there is a $(k-3)$-edge-coloring of G_{k-2} such that each tree $T_2, \dots, T_{k-4}, T'_{k-1}, T'_k$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k and T'_{k-1} *blue* and *red* respectively.

Without loss of generality we can suppose that $u \neq x$ and $v \neq y$ in G. There is a neighbor $a \in A_1$ of u, a neighbor $b \in A_2$ of v, a neighbor $a' \in A_1$ of x and a neighbor $b' \in A_2$ of y.

We color the edges ua and vb blue to get a blue T_k. We color the edges xa' and yb' red to get a red T_{k-1}. The vertex a is incident to at least $k-3$ uncolored edges. We color $k-3$ of these edges with a new color to get a monochromatic T_{k-2}. Now the vertex b is incident to at least $k-4$ uncolored edges. We color $k-4$ of these edges with another new color to get a monochromatic T_{k-3}. This completes the edge-coloring of G. □

Note that Claim 7 implies the theorem for $k = 5$. Furthermore, the above three claims are essentially the same as the proof of the first theorem in [8].

Claim 8. *If there is a pending star R of order r in T_k and T_{k-r} is a star, then $T_2, \dots, T_k$ have a packing into G.*

Proof. Let u be the neighbor of R. Remove R from T_k and let T'_k be the remaining graph.

By induction there is a $(k-2)$-edge-coloring of G_{k-1} such that each tree $T_2, \ldots, T_{k-r-1}, T'_k, T_{k-r+1}, \ldots, T_{k-1}$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k *blue*.

There is a neighbor $a \in A_1$ of u and a has at least $k-2$ other neighbors in G. Then there are at least $r-1$ vertices $d_1, \ldots, d_{r-1}$ which are neighbors of a but are not in T'_k i.e. there are no blue edges incident to $d_1, \ldots, d_{r-1}$. We color the edges ua and $ad_1, \ldots, ad_{r-1}$ blue to get a blue T_k. Now the vertex a is incident to at least $k-r-1$ uncolored edges. We color $k-r-1$ of these edges with a new color to get a monochromatic T_{k-r}. This completes the edge-coloring of G.

□

Claim 9. *For $2 \le i \le k \le 6$, let T_i be a tree on i vertices. If G is a k-chromatic graph, then $T_2, \ldots, T_k$ can be packed into G.*

Proof. By the above claims, the only remaining case is when $k = 6$ and none of T_6, T_5, T_4 are stars. It is easy to see that T_2, T_3, T_4 are unique (they are all paths) and T_5 and T_6 each have two possible configurations (either a path or a spider).

Remove a pending star of order 2 from T_6 and a leaf from T_4 and let T'_6 and T'_4 be the remaining graphs. Note that both of these remaining graphs are paths. We can reconstruct T_4 by adding an edge to either endpoint of T'_4. Similarly, we can reconstruct T_6 by adding a pending star of order 2 to either endpoint (if T_6 is a path) or to either interior point (if T_6 is a spider).

Because the statement of the claim holds for $k = 5$ there is a 4-edge-coloring of G_5 such that each tree T_2, T'_4, T'_6, T_5 is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_6 and T'_4 *blue* and *red* respectively.

First we consider the case when T_6 is a spider (if T_6 is a path, then the argument below works if we replace "interior point" with "endpoint" everywhere). We distinguish two subcases.

Case A. The two endpoints u and v of T'_4 are equal to the two interior points of T'_6 in G_5. The vertex u has a neighbor $a \in A_1$ and a has a neighbor $d_1 \in G$ which is not in T'_6. We color ua and ad_1 blue to get a monochromatic T_6. The vertex v has a neighbor $b \in A_1$ (note that b and a can be the same vertex, but still the edge vb is uncolored). We color vb red to get a monochromatic T_4. Now there are at least two uncolored edges incident to a. We can color them with a new color to get a monochromatic T_3 to complete the edge-coloring of G.

Case B. There is an interior point u of T'_6 which is not an endpoint of T'_4. The vertex u has a neighbor $a \in A_1$ and a has a neighbor $d_1 \in G$ which is not in T'_6. We color ua and ad_1 blue to get a monochromatic T_6. One of the endpoints

v of T'_4 is not equal to d_1. The vertex v has a neighbor $b \in A_1$ (note that b and a can be the same vertex, but still the edge vb is uncolored). We color vb red to get a monochromatic T_4. Now there are at least two uncolored edges incident to a. We can color them with a new color to get a monochromatic T_3 to complete the edge-coloring of G. □

From now on we can suppose that none of the conditions of the above five claims hold. In particular, $k > 6$ and T_k, T_{k-1} plus exactly one of T_{k-2} and T_{k-3} are not stars. Thus all other trees are stars. Furthermore, all the pending stars in T_k have order 2 (in the case T_{k-2} is not a star) or order 3 (in the case T_{k-3} is not a star).

We now distinguish two cases and several subcases.

Case 1. Every pending star in T_k is of order 3 (i.e. the case T_{k-3} is not a star). Let R be a pending star of order 3 in T_k with neighbor u and let v be the neighbor of a leaf such that $u \neq v$ and v is not in R (such a leaf can be easily found as $k > 6$). Let x be the neighbor of a leaf in T_{k-1}. Remove R and a leaf which is a neighbor of v from T_k and let T'_k be the remaining graph. Remove a leaf which is a neighbor of x from T_{k-1} and let T'_{k-1} be the remaining graph.

By induction there is a $(k-3)$-edge-coloring of G_{k-2} such that each tree $T_2, \dots, T_{k-5}, T'_k, T_{k-3}, T'_{k-1}$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k and T'_{k-1} *blue* and *red* respectively.

The vertex v has a neighbor $a \in A_1$ and u has a neighbor $b \in A_2$. There are at least $k-2$ neighbors of b which are different from a. There are $k-4$ vertices in T'_k, hence there are at least two vertices d_1 and d_2 adjacent to b that are not in T'_k and not equal to a.

The edges va, ub, bd_1 and bd_2 are colored blue to get a blue T_k. There are at least $k-2$ uncolored edges incident to a and $k-4$ uncolored edges incident to b. Although x could coincide with u or v, in any case there is at least one uncolored edge between x and a vertex in A_1 or A_2. We color this edge red to get a red T_{k-1}. Then a and b have either at least $k-3$ and $k-4$ or at least $k-2$ and $k-5$ uncolored incident edges. In either case, it is easy to see that we can color edges incident to a or b with two new colors to get T_{k-2} and T_{k-4} to complete the edge-coloring of G.

Case 2. Every pending star in T_k is of order 2 (i.e. the case T_{k-2} is not a star).

Case 2.1. T_k is not a spider. Let R_1 and R_2 be pending stars in T_k of order 2 with neighbors u and v such that $u \neq v$. Let $x \neq y$ be neighbors of leaves in T_{k-1}. Remove R_1 and R_2 from T_k and let T'_k be the remaining graph. Remove a leaf with neighbor x and a leaf with neighbor y from T_{k-1} and let T'_{k-1} be the remaining graph.

By induction there is a $(k-3)$-edge-coloring of G_{k-2} such that each tree $T_2, \dots, T_{k-5}, T'_k, T'_{k-1}, T_{k-2}$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k and T'_{k-1} *blue* and *red* respectively.

Without loss of generality we can suppose $u \neq x$ and $v \neq y$. There is a neighbor $a \in A_1$ of u and a neighbor $b \in A_2$ of v. There are at least two vertices adjacent to a and at least two vertices adjacent to b which are not in T'_k and are different from a and b. Thus we can find two vertices $d_1, d_2 \notin T'_k$ such that d_1 is adjacent to a and d_2 is adjacent to b and either $d_1 \neq x$ and $d_2 \neq y$ or $d_1 = x$ and $d_2 = y$.

Then the edges ua, ad_1, vb, bd_2 are colored blue to get a blue T_k. Now there is an uncolored edge between x and $A_1 \cup A_2$ and an uncolored edge between y and $A_1 \cup A_2$. Color these two edges red to get a red T_{k-1} Now a is incident to at least $k-4$ uncolored edges. We color $k-4$ of these edges with a new color to get a monochromatic T_{k-3}. Now b is incident to at least $k-5$ uncolored edges. We color $k-5$ of these edges with another new color to get a monochromatic T_{k-4}. This completes the edge-coloring of G.

Case 2.2. T_k is a spider. As $k > 6$, we can suppose that there exist three distinct vertices u_1, u_2, u_3 in T_k each with at least one neighbor that is a leaf.

Case 2.2.1. T_{k-1} has a pending star R of order $r \geq 4$. Let x be the neighbor of R. Let $w \neq z$ be neighbors of leaves in T_{k-2}. Remove a neighboring leaf from each vertex u_1, u_2, u_3 in T_k and let T'_k be the remaining graph. Remove the pending star R from T_{k-1} and let T'_{k-1} be the remaining graph. Remove a leaf with neighbor w and a leaf with neighbor z from T_{k-2} and let T'_{k-2} be the remaining graph.

By induction there is a $(k-4)$-edge-coloring of G_{k-3} such that each tree $T_2, \dots, T_{k-r-2}, T'_{k-1}, T_{k-r}, \dots, T_{k-5}, T'_{k-2}, T'_k$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k, T'_{k-1} and T'_{k-2} *blue*, *red* and *green* respectively.

The vertex x has a neighbor $a \in A_1$. There are at least $k-1$ neighbors of a and at least $k-1-(k-1-r) = r$ of them, say $d_1, \dots, d_r$ are not in T'_{k-1}. If any of them is equal to u_1, u_2, u_3 then without loss of generality we can assume that d_r is equal to u_3 (other equalities are also possible). If not, we still can suppose without loss of generality that $u_3 \neq x$. We color the edges $xa, ad_1, \dots, ad_{r-1}$ red to get a red T_{k-1}.

There is a neighbor $b \in A_2$ of u_1 and a neighbor $c \in A_3$ of u_2 and there is an uncolored edge between u_3 and A_1. Now we color the edges u_1b, u_2c and the uncolored edge between u_3 and A_1 with color blue to get a blue T_k. It is easy to see that we can color either an edge between w and A_2 and an edge between z and A_3, or an edge between w and A_3 and an edge between z and A_2 with color green to get a green T_{k-2}.

Now a is incident to at least $k-r-2$ uncolored edges, so we can color edges incident to a with a new color to get T_{k-r-1}. After this, b and c both are still incident to at least $k-4$ uncolored edges (note that the edge ba and ca may be colored red or with the new color corresponding to T_{k-1}). It is easy to see that we can color edges incident to b and c with two new colors to get T_{k-3} and T_{k-4} to complete the edge-coloring of G.

Case 2.2.2. T_{k-1} has a pending star R of order 3.

Case 2.2.2.1. T_{k-2} has a pending star R' of order $r \geq 3$. Let x be the neighbor of R. Let w be the neighbor of R'. Remove a neighboring leaf from each vertex u_1, u_2, u_3 in T_k and let T'_k be the remaining graph. Remove the pending star R from T_{k-1} and let T'_{k-1} be the remaining graph. Remove the pending star R' from T_{k-2} and let T'_{k-2} be the remaining graph.

By induction there is a $(k-4)$-edge-coloring of G_{k-3} such that each tree $T_2, \dots, T_{k-r-3}, T'_{k-2}, T_{k-r-1}, \dots, T_{k-5}, T'_{k-1}, T'_k$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k, T'_{k-1} and T_{k-2} *blue*, *red* and *green* respectively.

Without loss of generality we can suppose $x \neq u_3$ and $w \neq u_2$. Then x has a neighbor c in A_3. We color the edge xc, an edge between c and A_1 and an edge between c and A_2 with color red to get a red T_{k-1}. There is a neighbor $a \in A_1$ of w. The vertex a has at least $k-2$ neighbors different from c, at least r of them, say $d_1, \dots, d_r$ are not in T'_{k-2}. If any d_i is equal to u_1, u_2, u_3 then let d_r be equal to u_2 (other equalities are also possible). We color the edges wa, $ad_1, \dots, ad_{r-1}$ green to get a green T_{k-2}.

There is an uncolored edge between u_3 and A_3, an uncolored edge between u_2 and A_1 and an uncolored edge between u_1 and A_2. We color these edges blue to get a blue T_k. It is easy to see that there are enough uncolored edges incident to a, b and c such that we can complete the edge-coloring of G with three new colors to get T_{k-r-2}, T_{k-4} and T_{k-3}.

Case 2.2.2.2. All pending stars in T_{k-2} are of order 2. Let x be the neighbor of R. Let R' be a pending star of order 2 in T_{k-2}. Let w be the neighbor of R'. As $k > 6$, there is a leaf in T_{k-2} with neighbor $z \neq w$. Remove a neighboring leaf from each vertex u_1, u_2, u_3 in T_k and let T'_k be the remaining graph. Remove the pending star R from T_{k-1} and let T'_{k-1} be the remaining graph. Remove the pending star R' and a leaf with neighbor z from T_{k-2} and let T'_{k-2} be the remaining graph.

By induction there is a $(k-4)$-edge-coloring of G_{k-3} such that each tree $T_2, \dots, T_{k-6}, T'_{k-2}, T'_{k-1}, T'_k$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k, T'_{k-1} and T_{k-2} *blue*, *red* and *green* respectively.

Without loss of generality we can suppose that $x \neq u_3$, $w \neq u_2$ and $z \neq u_1$.

Then x has a neighbor $c \in A_3$. We color the edge xc, an edge between c and A_1 and an edge between c and A_2 with color red to get a red T_{k-1}. There is a neighbor $a \in A_1$ of w. There is a neighbor $b \in A_2$ of z. The vertex a has at least $k-3$ neighbors different from c and b, at least two of them, d_1 and d_2, are not in T'_{k-2}. Without loss of generality we can suppose that $u_2 \neq d_1$. We color the edges wa, ad_1 and zb green to get a green T_{k-2}. There is an edge between u_3 and A_3, an edge between u_2 and A_1 and an edge between u_1 and A_2. These edges are uncolored. We color these edges blue to get a blue T_k.

It is easy to see that there are enough uncolored edges incident to a, b and c such that we can complete the edge-coloring of G with three new colors to get T_{k-5}, T_{k-4} and T_{k-3}.

Case 2.2.3. Every pending star in T_{k-1} is of order 2.

Case 2.2.3.1. T_{k-1} is not a spider. Let R and R' be pending stars in T_{k-1} of order 2 with neighbors x and y such that $x \neq y$. Let $w \neq z$ be neighbors of leaves in T_{k-2}. Remove a neighboring leaf from each vertex u_1, u_2, u_3 in T_k and let T'_k be the remaining graph. Remove R and R' from T_{k-1} and let T'_{k-1} be the remaining graph. Remove a leaf with neighbor z and a leaf with neighbor w from T_{k-2} and let T'_{k-2} be the remaining graph.

By induction there is a $(k-4)$-edge-coloring of G_{k-3} such that each tree $T_2, \dots, T_{k-6}, T'_{k-1}, T'_{k-2}, T'_k$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k, T'_{k-1} and T'_{k-2} *blue*, *red* and *green* respectively.

There is a neighbor $a \in A_1$ of x and a neighbor $b \in A_2$ of y. There are at least three neighbors d_1, d_2, d_3 of a not in T'_{k-1} and other than b and at least three neighbors f_1, f_2, f_3 of b not in T'_{k-1} and other than a. If there is a $u_i = d_j$ and/or $u_i = f_l$, we can suppose $j = 3$ and/or $l = 3$. Also we can suppose $d_1 \neq f_1$. We color the edges xa, ad_1, yb and bf_1 red to get a red T_{k-1}.

There are at most two of u_1, u_2 and u_3 equal to some of x, y, d_1 and f_1, moreover, we can suppose that they are not u_1 and u_2. Then there is a neighbor $c \in A_3$ of u_3. We color the edge u_3c blue. At most one of u_1 and u_2, say u_1 is connected by a red edge to a vertex in A_1 or A_2, say A_1, then we color an edge between u_1 and A_2 and an edge between u_2 and A_1 with color blue to get a blue T_k. Any vertex in G_{k-3} is connected by a colored edge to at most two of A_1, A_2 or A_3 and there are no two distinct vertices in G_{k-3} connected by colored edges to the same two of A_1, A_2 or A_3. Thus we can find an uncolored edge from w to one class A_1, A_2, A_3 and an uncolored edge from z to a different class A_1, A_2, A_3. We color these two edges green to get a green T_{k-2}.

There are at least $k-5$, $k-4$ and $k-3$ or at least $k-5$, $k-5$ and $k-2$ uncolored edges incident to a, b and c respectively. It is easy to see that we can complete the edge-coloring of G with three new colors to get T_{k-5}, T_{k-4} and T_{k-3}.

Case 2.2.3.2. T_{k-1} is a spider. As $k > 6$, there exist three distinct vertices x_1, x_2, x_3 in T_{k-1} each with at least one neighbor that is a leaf.

Case 2.2.3.2.1. T_{k-2} has a pending star R of order $r \geq 3$. Let w be the neighbor of R. Remove a neighboring leaf from each vertex u_1, u_2, u_3 in T_k and let T'_k be the remaining graph. Remove a neighboring leaf from each vertex x_1, x_2, x_3 in T_{k-1} and let T'_{k-1} be the remaining graph. Remove the pending star R from T_{k-2} and let T'_{k-2} be the remaining graph.

By induction there is a $(k-4)$-edge-coloring of G_{k-3} such that each tree $T_2, \ldots, T_{k-r-3}, T'_{k-2}, T_{k-r-1}, \ldots, T_{k-5}, T'_{k-1}, T'_k$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k, T'_{k-1} and T'_{k-2} *blue*, *red* and *green* respectively.

Without loss of generality we can suppose that u_3, w, x_2 are pairwise distinct. There is a neighbor a of w in A_1. There are at least $r+1$ neighbors $d_1, d_2 \ldots, d_{r+1}$ of a not in T'_{k-2}. If there is $u_i = d_j$, then we can suppose $d_{r+1} = u_3$. If $x_l = d_{r+1}$ also, then we can suppose $l = 3$. If one of $d_1, d_2, \ldots, d_r$ is equal to some x_p, then we can suppose $d_r = x_2$.

Now we color the edges $wa, ad_1, ad_2, \ldots, d_{r-1}$ green to get a green T_{k-2}. There is a neighbor $b \in A_2$ of u_1 and a neighbor $c \in A_3$ of u_2. We color the edges u_3a, u_1b and u_2c blue to get a blue T_k. We color the edge x_2a red. Now there is only one colored edge in G incident to b and only one colored edge in G incident to c. Furthermore, these two colored edges are not adjacent. Hence we can color either the edges x_1b and x_3c or the edges x_1c and x_3b with color red to get a red T_{k-1}.

Now there are at least $k-3$ uncolored edges incident to b, $k-3$ uncolored edges incident to c and $k-r-3$ uncolored edges incident to a. It is easy to see that we can complete the edge-coloring of G with three new colors to get T_{k-r-2}, T_{k-4} and T_{k-3}.

Case 2.2.3.2.2. Every pending star in T_{k-2} is of order 2. Let w be the neighbor of a pending star R in T_{k-2} of order 2. Let $z \neq w$ be a neighbor of a leaf in $V(T_{k-2}) \setminus R$. Remove a neighboring leaf from each vertex u_1, u_2, u_3 in T_k and let T'_k be the remaining graph. Remove a neighboring leaf from each vertex x_1, x_2, x_3 in T_{k-1} and let T'_{k-1} be the remaining graph. Remove the pending star R and a leaf with neighbor z from T_{k-2} and let T'_{k-2} be the remaining graph.

By induction there is a $(k-4)$-edge-coloring of G_{k-3} such that each tree $T_2, \ldots, T_{k-6}, T'_{k-2}, T'_{k-1}, T'_k$ is isomorphic to a subgraph spanned by the edges of a single color. We will call the color of T'_k, T'_{k-1} and T'_{k-2} *blue*, *red* and *green* respectively.

Without loss of generality we can suppose that $x_1 \neq w$. There is a neighbor $a \in A_1$ of z and a neighbor $b \in A_2$ of w. There are at least three neighbors d_1, d_2, d_3 of b which are not in T'_{k-2} and are different from a. We can suppose that if z is equal to some u_i and/or some x_j, then $z = u_1$ and/or $z = x_2$. We also can suppose that u_1, x_1 and d_1 are pairwise distinct.

We color the edges za, wb and bd_1 green to get a green T_{k-2}. Then we color an edge between u_1 and A_2 blue. We color an edge between x_1 and A_2 red. Now none of u_2, u_3, x_3 has a colored edge incident to A_1 or A_3, but it is possible that there is a colored edge between x_2 and a. We can color an edge from x_2 to A_3 and an edge from x_3 to A_1 red to get a red T_{k-1}. Now we can color either the edges from u_2 to A_1 and from u_3 to A_3 or the edges from u_2 to A_3 and from u_3 to A_1 with color blue to get a blue T_k.

Now there are at least $k-4$ uncolored edges incident to a, $k-5$ uncolored edges incident to b and $k-3$ uncolored edges incident to some $c \in A_3$.

It is easy to see that we can complete the edge-coloring of G with three new colors to get T_{k-5}, T_{k-4} and T_{k-3}. ■

3. Additional Conjectures and Results

In this section we prove simple propositions for tree packings into graphs with minimum or average degree conditions. We also introduce some additional conjectures.

In the case of k-chromatic graphs, we could assume that the minimum degree is at least $k-1$. This suggests the following generalization of Conjecture 2.

Conjecture 10. *For $2 \le i \le k$, let T_i be a tree on i vertices. If a graph G has minimum degree $\delta(G) \ge k-1$, then the set of trees $T_2, \ldots, T_k$ has a packing into G.*

When the number of vertices of G is large with respect to the minimum degree, then Conjecture 10 is true:

Proposition 11. *For $2 \le i \le k$, let T_i be a tree on i vertices. There is a constant $n_0(k)$ such that if G is a graph on $n > n_0(k)$ vertices and minimum degree $\delta(G) \ge k-1$, then $T_2, \ldots, T_k$ can be packed into G.*

This proposition is an easy corollary of the following lemma. Indeed, by the lemma we can find and remove one by one all the required trees.

Lemma 12. *There is a constant $n_0(k)$ such that if G is a graph on $n > n_0(k)$ vertices and minimum degree $\delta(G) \ge k-1$, and G' is the graph remaining after removing an arbitrary set of $\binom{k}{2}$ edges from G, then any tree on k vertices is a subgraph of G'.*

Proof. Let B_1 be the set of vertices with degree less than $k-1$ in G'. Let $B_2 \subset V(G') \setminus B_1$ be the neighbors of B_1 adjacent to less than $k-1$ vertices in

$V(G') \setminus B_1$. For $2 < i \leq k$, let $B_i \subset V(G') \setminus \cup_{j<i} B_j$ be the neighbors of $\cup_{j<i} B_j$ adjacent to less than $k-1$ vertices in $V(G') \setminus \cup_{j<i} B_j$. Finally let $B = \cup_{i \leq k} B_i$. Note that in each step i, each vertex in $\cup_{j<i} B_j$ is adjacent to less than $k-1$ vertices of $V(G') \setminus \cup_{j<i} B_j$.

Clearly $|B_1| \leq 2\binom{k}{2} = k^2 - k$. Then $|B_2| \leq (k-1)|B_1|$ as each vertex in B_2 is a neighbor of some vertex in B_1. For $2 < i \leq k$, by the same argument we have $|B_i| \leq (k-1)|\cup_{j<i} B_j|$, thus $|\cup_{j \leq i} B_j| \leq k|\cup_{j<i} B_j|$. So $|B| = |\cup_{j \leq k} B_j| \leq k^{k-1}(k^2 - k)$ and thus the cardinality of B does not depend on n. Choose $n_0(k)$ to be bigger then this constant, this way there is a vertex not in B.

Choose an arbitrary vertex of the tree as a root and note that each vertex has a fixed distance in the tree from the root, which is at most $k-1$. We denote by *level* i the set of vertices of the tree of distance i from the root. Identify the root with a vertex in $V(G') \setminus B$. There are at most $k-1$ vertices in level 1 and at least $k-1$ neighbors of the root in $V(G') \setminus (B_1 \cup B_2 \cup \cdots \cup B_{k-1})$ so we can identify the vertices in level 1 with the neighbors of the root in $V(G') \setminus (B_1 \cup B_2 \cup \cdots \cup B_{k-1})$. Similarly by induction we can identify vertices in level i with vertices of distance i from the root in $V(G') \setminus (B_1 \cup B_2 \cup \cdots \cup B_{k-i})$. Indeed, suppose we have identified levels 1 through i with vertices in G'. Denote by V_i the vertices of G' that are identified with vertices of level i of the tree. Each vertex of V_i has at least $k-1$ adjacent vertices in $V(G') \setminus (B_1 \cup B_2 \cup \cdots \cup B_{k-i-1})$. Since the order of the tree is k, and since at least one vertex of the tree is already identified with vertices in G', we can easily identify the vertices of level $i+1$ with vertices that have not yet been used in the previous steps. ■

The bound on n given by the proof of Proposition 11 can probably be improved. However, it seems unlikely that Conjecture 2 can be proved with this type of argument.

We can weaken the minimum degree condition in Conjecture 10 to get an even stronger conjecture.

Conjecture 13. *For $2 \leq i \leq k$, let T_i be a tree on i vertices. If the graph G has average degree at least $k-1$, i.e. G has at least $\frac{k-1}{2}n$ edges, then the set of trees $T_2, \ldots, T_k$ has a packing into G.*

In this setting it is easy to prove an analogue of the previously-mentioned result of Bollobás [1].

Proposition 14. *Given a fixed $s \leq k/2$, for $2 \leq i \leq s$, let T_i be a tree on i vertices. If G is a graph with n vertices and at least $\frac{k-1}{2}n$ edges where $k \leq n$, then the set of trees $T_2, \ldots, T_s$ has a packing into G.*

Proof. We proceed by induction on k. For $k = 1$ the statement of the proposition obviously holds. Now let us assume that $k > 1$ and the statement of the proposition holds for all values less than k.

Let G be the graph in the statement of the proposition. Remove all vertices of G with degree less than $\frac{k-1}{2}$. Let us continue to remove all vertices with degree less than $\frac{k-1}{2}$ from the resulting graphs until the procedure stops. In each round the average degree cannot decrease, so when the procedure stops we are left with a graph with minimum degree at least $\frac{k-1}{2}$. Thus G contains a subgraph with minimum degree at least $\frac{k-1}{2} \geq \frac{k}{2} - 1$.

It is easy to see that any tree T_i is a subgraph (i.e. has a packing) into a graph with minimum degree $i-1$. Thus T_s has a packing into G as $s \leq \frac{k}{2}$ and the minimum degree of G is at least $\frac{k}{2} - 1$. If we remove the edges of T_s from G we are left with a graph with at least $\frac{k-1}{2}n - (\frac{k}{2} - 1) \geq \frac{k-2}{2}n$ edges as $k \leq n$ and we are done by induction. ■

Moreover, from the proof it is easy to see that if we have a graph as in the statement of Proposition 14, then any packing of $T_i, \ldots, T_s$ into G can be extended to a packing of $T_2, \ldots, T_s$ using the remaining edges of G.

Conjecture 13 is strongly related to the following conjecture of Erdős and Sós [7].

Conjecture 15 (Erdős and Sós [7]). *Let T_k be a tree with k vertices. If G is a graph with n vertices and more than $\frac{k-2}{2}n$ edges, then T_k is a subgraph of G.*

At first glance, Conjecture 13 seems to ask for much more as we have only a few more edges but we want to pack many more trees. However, for graphs G where $n \geq 2k$, if true, the Erdős-Sós Conjecture easily implies Conjecture 13.

In particular, let G be a graph given in Conjecture 13 and let us assume that the Erdős-Sós Conjecture is true. Then T_k is a subgraph of G as G has more than $\frac{k-2}{2}n$ edges. Removing T_k from G yields a graph G' with $e(G') \geq \frac{k-1}{2}n - (k-1) > \frac{k-2}{2}n$ as $n \geq 2k$. Thus by the Erdős-Sós Conjecture, G' has T_{k-1} as a subgraph. This argument can be continued to find all the trees required by Conjecture 13.

In this paper we proved most of the known results concerning the TPC in the more general setting where we pack trees into any k-chromatic graph. However, missing from the more general setting is the analogue of the result of Gyárfás and Lehel [8] that states that the trees can be packed into K_n if each tree is a path or a star.

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