

## ON SUPER $(a, d)$ -EDGE ANTIMAGIC TOTAL LABELING OF CERTAIN FAMILIES OF GRAPHS

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### Abstract

A  $(p, q)$ -graph  $G$  is  $(a, d)$ -edge antimagic total if there exists a bijection  $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p + q\}$  such that the edge weights  $\Lambda(uv) = f(u) + f(uv) + f(v)$ ,  $uv \in E(G)$  form an arithmetic progression with first term  $a$  and common difference  $d$ . It is said to be a super  $(a, d)$ -edge antimagic total if the vertex labels are  $\{1, 2, \dots, p\}$  and the edge labels are  $\{p + 1, p + 2, \dots, p + q\}$ . In this paper, we study the super  $(a, d)$ -edge antimagic total labeling of special classes of graphs derived from copies of generalized ladder, fan, generalized prism and web graph.

**Keywords:** edge weight, magic labeling, antimagic labeling, ladder, fan graph, prism and web graph.

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### 1. INTRODUCTION

By a *graph*  $G$  we mean a finite, undirected, connected graph without any loops or multiple edges. Let  $V(G)$  and  $E(G)$  be the set of vertices and edges of a graph  $G$ , respectively. The *order* and *size* of a graph  $G$  is denoted as  $p = |V(G)|$  and  $q = |E(G)|$  respectively. For general graph theoretic notions we refer Harray [6].

By a *labeling* we mean a one-to-one mapping that carries the set of graph elements onto a set of numbers (usually positive or non-negative integers), called *labels*. There are several types of labelings and a detailed survey of many of them can be found in the dynamic survey of graph labeling by Gallian [5].

Kotzig and Rosa [9] introduced the concept of *magic labeling*. They define an *edge magic total labeling* of a  $(p, q)$ -graph  $G$  as a bijection  $f$  from  $V(G) \cup E(G)$  to the set  $\{1, 2, \dots, p + q\}$  such that for each edge  $uv \in E(G)$ , the edge weight  $f(u) + f(uv) + f(v)$  is a constant.

Enomoto *et al.* [3] defined a *super edge magic labeling* as an edge magic total labeling such that the vertex labels are  $\{1, 2, \dots, p\}$  and edge labels are  $\{p + 1, p + 2, \dots, p + q\}$ . They have proved that if a graph with  $p$  vertices and  $q$  edges is super edge magic then,  $q \leq 2p - 3$ . They also conjectured that every tree is super edge magic.

As a natural extension of the notion of edge magic total labeling, Hartsfield and Ringel [7] introduced the concept of an *antimagic labeling* and they defined an *antimagic labeling* of a  $(p, q)$ -graph  $G$  as a bijection  $f$  from  $E(G)$  to the set  $\{1, 2, \dots, q\}$  such that the sums of label of the edges incident with each vertex  $v \in V(G)$  are distinct.

Simanjuntak *et al.* [10] defined an  $(a, d)$ -edge *antimagic total labeling* as a one to one mapping  $f$  from  $V(G) \cup E(G)$  to  $\{1, 2, \dots, p + q\}$  such that the set of edge weight  $\{f(u) + f(uv) + f(v) : uv \in E(G)\}$  is equal to  $\{a, a + d, a + 2d, \dots, a + (q - 1)d\}$  for any two integers  $a > 0$  and  $d \geq 0$ .

An  $(a, d)$ -edge antimagic total labeling of a  $(p, q)$ -graph  $G$  is said to be *super  $(a, d)$ -edge antimagic total* if the vertex labels are  $\{1, 2, \dots, p\}$  and the edge labels are  $\{p + 1, p + 2, \dots, p + q\}$ . The super  $(a, 0)$ -edge antimagic total labeling is usually called as super edge magic in the literature (see [3, 4]).

An  $(a, d)$ -edge *antimagic vertex labeling* of a  $(p, q)$ -graph  $G$  is defined as a one to one mapping  $f$  from  $V(G)$  to the set  $\{1, 2, \dots, p\}$  such that the set of edge weight  $\{f(u) + f(v) : uv \in E(G)\}$  is equal to  $\{a, a + d, a + 2d, \dots, a + (q - 1)d\}$  for any two integers  $a > 0$  and  $d \geq 0$ .

In [2] Bača *et al.* proved that if a  $(p, q)$ -graph  $G$  has an  $(a, d)$ -edge antimagic vertex labeling then  $d(q - 1) \leq 2p - 1 - a \leq 2p - 4$ .

Also in [1] Bača and Barrientos proved the following: if a graph with  $q$  edges and  $q + 1$  vertices has an  $\alpha$ -labeling, then it has an  $(a, 1)$ -edge antimagic vertex labeling. A tree has  $(3, 2)$ -edge antimagic vertex labeling if and only if it has an  $\alpha$ -labeling and the number of vertices in its two partite set differ by at most 1. If a tree with at least two vertices has a super  $(a, d)$ -edge antimagic total labeling, then  $d$  is at most 3. If a graph has an  $(a, 1)$ -edge antimagic vertex labeling, then it also has a super  $(a_1, 0)$ -edge antimagic total labeling and a super  $(a_2, 2)$ -edge antimagic total labeling.

In [12] Sugeng *et al.* studied the super  $(a, d)$ -edge antimagic total properties

of ladders, generalized prisms and antiprisms.

We make use of the following lemmas for our further discussion.

**Lemma 1.** *If a  $(p, q)$ -graph  $G$  is super  $(a, d)$ -edge antimagic total, then  $d \leq \frac{2p+q-5}{q-1}$ .*

**Lemma 2.** *If a  $(p, q)$ -graph  $G$  has an  $(a, 1)$ -edge antimagic vertex labeling and odd number of edges, then it has a super  $(a', 1)$ -edge antimagic total labeling, where  $a' = a + p + \frac{q+1}{2}$ .*

**Lemma 3.** *If a  $(p, q)$ -graph  $G$  has an  $(a, d)$ -edge antimagic vertex labeling, then  $G$  has a super  $(a', d')$ -edge antimagic total labeling, where  $a' = a + p + 1$  and  $d' = d + 1$  or  $a' = a + p + q$  and  $d' = d - 1$ .*

Lemma 2 appeared in [11] and Lemma 3 appeared in [2].

In this paper, we study the super  $(a, d)$ -edge antimagic total labeling of special classes of graphs derived from copies of generalized ladder, fan, generalized prism and web graph.

## 2. A GRAPH DERIVED FROM COPIES OF GENERALIZED LADDER

Let  $(u_{i,1}, u_{i,2}, \dots, u_{i,n}, v_{i,1}, v_{i,2}, \dots, v_{i,n}), 1 \leq i \leq t$ , be a collection of  $t$  disjoint copies of the *generalized ladder*  $\mathcal{L}_n, n \geq 2$ , such that  $u_{i,j}$  is adjacent to  $u_{i,j+1}, v_{i,j+1}$  and  $v_{i,j}$  is adjacent to  $v_{i,j+1}$  for  $1 \leq j \leq n - 1$  and  $u_{i,j}$  is adjacent to  $v_{i,j}$  for  $1 \leq j \leq n$ . We denote the graph obtained by joining  $u_{i,n}$  to  $u_{i+1,1}, u_{i+1,2}, v_{i+1,1}, 1 \leq i \leq t - 1$ , as  $\mathcal{L}_n^{(t)}$ . Clearly, the vertex set  $V$  and the edge set  $E$  of the graph  $\mathcal{L}_n^{(t)}$  are given by

$$V(\mathcal{L}_n^{(t)}) = \{u_{i,j}, v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq n\} \text{ and } E(\mathcal{L}_n^{(t)}) = E_1 \cup E_2 \cup E_3 \text{ where}$$

$$E_1 = \{u_{i,j}u_{i,j+1}, v_{i,j}v_{i,j+1}, u_{i,j}v_{i,j+1} : 1 \leq i \leq t, 1 \leq j \leq n - 1\},$$

$$E_2 = \{u_{i,j}v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq n\},$$

$$E_3 = \{u_{i,n}u_{i+1,1}, u_{i,n}u_{i+1,2}, u_{i,n}v_{i+1,1} : 1 \leq i \leq t - 1\}.$$

It is easy to see that for  $\mathcal{L}_n^{(t)}$ , we have  $p = 2nt$  and  $q = 4nt - 3$ .

**Lemma 4.** *The graph  $\mathcal{L}_n^{(t)}, n, t \geq 2$  has an  $(a, 1)$ -edge antimagic vertex labeling.*

**Proof.** Let us define a bijection  $f_1 : V(\mathcal{L}_n^{(t)}) \rightarrow \{1, 2, \dots, 2nt\}$  as follows:

$$f_1(u_{i,j}) = 2(i - 1)n + 2j - 1 \quad \text{if } 1 \leq i \leq t \text{ and } 1 \leq j \leq n,$$

$$f_1(v_{i,j}) = 2(i - 1)n + 2j \quad \text{if } 1 \leq i \leq t \text{ and } 1 \leq j \leq n.$$

By direct computation, we observe that the edge weights of all the edges of  $\mathcal{L}_n^{(t)}$ , constitute an arithmetic sequence  $\{3, 4, \dots, 4nt - 1\}$ . Thus  $f_1$  is an  $(3, 1)$ -edge antimagic vertex labeling of  $\mathcal{L}_n^{(t)}$ . ■

**Theorem 5.** *The graph  $\mathcal{L}_n^{(t)}$ ,  $n, t \geq 2$ , has a super  $(a, d)$ -edge antimagic total labeling if and only if  $d \in \{0, 1, 2\}$ .*

**Proof.** If the graph  $\mathcal{L}_n^{(t)}$ ,  $n, t \geq 2$ , is super  $(a, d)$ -edge antimagic total, then by Lemma 1, we get  $d \leq 2$ .

Conversely, by Lemma 4 and Lemma 3, we see that the graph  $\mathcal{L}_n^{(t)}$ ,  $n, t \geq 2$  has a super  $(6nt, 0)$ -edge antimagic total labeling and a super  $(2nt + 4, 2)$ -edge antimagic total labeling.

Also by Lemma 2, we conclude that the graph  $\mathcal{L}_n^{(t)}$ ,  $n, t \geq 2$ , has a super  $(4nt + 2, 1)$ -edge antimagic total labeling, since  $q = 4nt - 3$ , which is odd for all  $n$  and  $t$ . ■

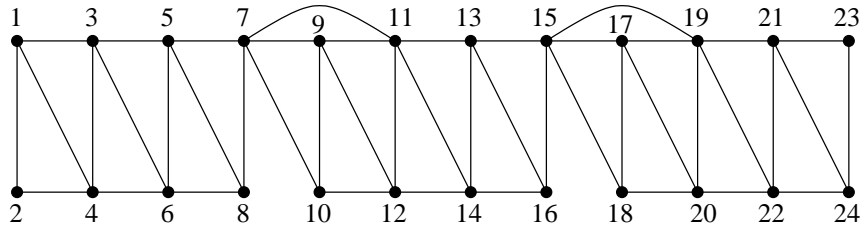


Figure 1.  $(a, 1)$ -edge antimagic vertex labeling of  $\mathcal{L}_4^{(3)}$ .

### 3. A GRAPH DERIVED FROM COPIES OF FAN GRAPH

Let  $(u_i, w_i, v_{i,1}, v_{i,2}, \dots, v_{i,m})$ ,  $1 \leq i \leq t$ , be a collection of  $t$  disjoint copies of the fan graph  $\mathcal{F}_{m,2}$ ,  $m \geq 2$ , such that  $u_i$  is adjacent to  $w_i$  and  $v_{i,j}$  is adjacent to both  $u_i$  and  $w_i$  for  $1 \leq j \leq m$ . We denote the graph [8] obtained by joining  $v_{i,m}$  to  $u_{i+1}, v_{i+1,1}, v_{i+1,2}$ ,  $1 \leq i \leq t - 1$ , as  $\mathcal{F}_{m,2}^{(t)}$ . Clearly, the vertex set  $V$  and the edge set  $E$  of the graph  $\mathcal{F}_{m,2}^{(t)}$  are given by

$$V(\mathcal{F}_{m,2}^{(t)}) = \{u_i, w_i, v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq m\} \text{ and}$$

$$E(\mathcal{F}_{m,2}^{(t)}) = \{u_i w_i, u_i v_{i,j}, w_i v_{i,j} : 1 \leq i \leq t, 1 \leq j \leq m\}$$

$$\cup \{v_{i,m} u_{i+1}, v_{i,m} v_{i+1,1}, v_{i,m} v_{i+1,2} : 1 \leq i \leq t - 1\}.$$

It is easy to see that for  $\mathcal{F}_{m,2}^{(t)}$ , we have  $p = (m + 2)t$  and  $q = (m + 2)t - 3$ .

**Lemma 6.** *The graph  $\mathcal{F}_{m,2}^{(t)}$ ,  $m, t \geq 2$ , has an  $(a, 1)$ -edge antimagic vertex labeling.*

**Proof.** Let us define a bijection  $f_2 : V(\mathcal{F}_{m,2}^{(t)}) \rightarrow \{1, 2, \dots, (m + 2)t\}$  as follows:

$$f_2(u_i) = (i - 1)(m + 2) + 1 \quad \text{if } 1 \leq i \leq t,$$

$$f_2(w_i) = (m + 2)i \quad \text{if } 1 \leq i \leq t,$$

$$f_2(v_{i,j}) = f_2(u_i) + j \quad \text{if } 1 \leq i \leq t \text{ and } 1 \leq j \leq m.$$

By direct computation, we observe that the edge weights of all the edges of  $\mathcal{F}_{m,2}^{(t)}$  constitute an arithmetic sequence  $\{3, 4, \dots, 2t(m + 2) - 1\}$ . Thus  $f_2$  is an  $(3, 1)$ -edge antimagic vertex labeling of  $\mathcal{F}_{m,2}^{(t)}$ . ■

**Theorem 7.** *The graph  $\mathcal{F}_{m,2}^{(t)}$ ,  $m, t \geq 2$ , has a super  $(a, d)$ -edge antimagic total labeling if and only if  $d \in \{0, 1, 2\}$ .*

**Proof.** If the graph  $\mathcal{F}_{m,2}^{(t)}$ ,  $m, t \geq 2$ , is super  $(a, d)$ -edge antimagic total, then by Lemma 1, we get  $d \leq 2$ .

Conversely, by Lemmas 3 and 6, we see that the graph  $\mathcal{F}_{m,2}^{(t)}$ ,  $m, t \geq 2$ , has a super  $((m + 2)3t, 0)$ -edge antimagic total labeling and a super  $((m + 2)t + 4, 2)$ -edge antimagic total labeling.

Also by Lemma 2, we conclude that the graph  $\mathcal{F}_{m,2}^{(t)}$ ,  $m, t \geq 2$ , has a super  $((m + 2)2t + 2, 1)$ -edge antimagic total labeling, since  $q = (m + 2)2t - 3$ , which is odd for all  $m$  and  $t$ . ■

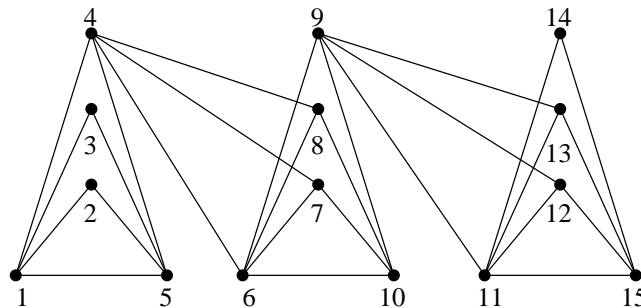


Figure 2.  $(a, 1)$ -edge antimagic vertex labeling of  $\mathcal{F}_{3,2}^{(3)}$ .

#### 4. A GRAPH DERIVED FROM COPIES OF GENERALIZED PRISM

Let  $(v_{i,j}^{(k)}, 1 \leq i \leq m, 1 \leq j \leq n), 1 \leq k \leq t$ , be a collection of  $t$  disjoint copies of the *generalized prism*  $C_m \times P_n$ ,  $m \geq 3, n \geq 2$ , such that  $v_{i,j}^{(k)}$  is adjacent to  $v_{i+1,j}^{(k)}$  for  $1 \leq i \leq m - 1, 1 \leq j \leq n, v_{m,j}^{(k)}$  is adjacent to  $v_{1,j}^{(k)}$  for  $1 \leq j \leq n$  and  $v_{i,j}^{(k)}$  is adjacent to  $v_{i,j+1}^{(k)}$  for  $1 \leq i \leq m, 1 \leq j \leq n - 1$ . We denote the graph obtained by joining  $v_{m,n}^{(k)}$  to  $v_{i,1}^{(k+1)}$  if  $n$  is odd or  $v_{1,n}^{(k)}$  to  $v_{i,1}^{(k+1)}$  if  $n$  is even for  $1 \leq i \leq m, 1 \leq k \leq t - 1$  as  $(C_m \times P_n)^{(t)}$ . Clearly, the vertex set  $V$  and the edge set  $E$  of the graph  $(C_m \times P_n)^{(t)}$  are given by  $V((C_m \times P_n)^{(t)}) = \{v_{i,j}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq$

$m, 1 \leq j \leq n\}$  and  $E((C_m \times P_n)^{(t)}) = E_1 \cup E_2 \cup E_3$  where

$$\begin{aligned} E_1 &= \{v_{i,j}^{(k)} v_{i+1,j}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq m-1, 1 \leq j \leq n\} \\ &\cup \{v_{m,j}^{(k)} v_{1,j}^{(k)} : 1 \leq k \leq t, 1 \leq j \leq n\}, \\ E_2 &= \{v_{i,j}^{(k)} v_{i,j+1}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n-1\}, \\ E_3 &= \{v_{m,n}^{(k)} v_{i,1}^{(k+1)} : \text{if } n \text{ is odd and } 1 \leq k \leq t-1, 1 \leq i \leq m\} \\ &\cup \{v_{1,n}^{(k)} v_{i,1}^{(k+1)} : \text{if } n \text{ is even and } 1 \leq k \leq t-1, 1 \leq i \leq m\}. \end{aligned}$$

It is easy to see that for  $(C_m \times P_n)^{(t)}$ , we have  $p = mnt$  and  $q = m(2nt - 1)$ .

**Lemma 8.** *For odd  $m$ ,  $m \geq 3$  and  $n, t \geq 2$ , the graph  $(C_m \times P_n)^{(t)}$  has an  $(a, 1)$ -edge antimagic vertex labeling.*

*Proof.* Let us define a bijection  $f_3 : V((C_m \times P_n)^{(t)}) \rightarrow \{1, 2, \dots, mnt\}$  as follows.

If  $j$  is odd and  $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq t$ , then

$$f_3(v_{i,j}^{(k)}) = \begin{cases} (k-1)mn + (j-1)m + \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ (k-1)mn + (j-1)m + \frac{m+i+1}{2} & \text{if } i \text{ is even.} \end{cases}$$

If  $j$  is even and  $1 \leq i \leq m, 2 \leq j \leq n, 1 \leq k \leq t$ , then

$$f_3(v_{i,j}^{(k)}) = \begin{cases} (k-1)mn + (j-1)m + \frac{m+i}{2} & \text{if } i \text{ is odd,} \\ (k-1)mn + (j-1)m + \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

By direct computation, we observe that the edge weights of all the edges of  $(C_m \times P_n)^{(t)}$  constitute an arithmetic sequence  $\{\frac{m+3}{2}, \frac{m+5}{2}, \dots, \frac{m+4mnt-3}{2}\}$ . Clearly  $\frac{m+3}{2}$  is an integer only when  $m$  is odd. Thus  $f_3$  is an  $(\frac{m+3}{2}, 1)$ -edge antimagic vertex labeling of  $(C_m \times P_n)^{(t)}$ , for odd  $m$ . ■

**Theorem 9.** *For odd  $m$ ,  $m \geq 3$  and  $n, t \geq 2$ , the graph  $(C_m \times P_n)^{(t)}$  has a super  $(a, d)$ -edge antimagic total labeling if and only if  $d \in \{0, 1, 2\}$ .*

*Proof.* If the graph  $(C_m \times P_n)^{(t)}$ ,  $m \geq 3$  and  $n, t \geq 2$ , is super  $(a, d)$ -edge antimagic total, then by Lemma 1 we get

$$d \leq \frac{2p+q-5}{q-1} = \frac{2mnt+m(2nt-1)-5}{m(2nt-1)-1} = 2 + \frac{m-3}{2mnt-m-1}.$$

Since  $2mnt - m - 1 > 0$ , for  $m \geq 3, n, t \geq 2$ , it follows that  $\frac{m-3}{2mnt-m-1} < 1$  and hence  $d < 3$ .

Conversely, by Lemma 8 and Lemma 3, we obtain that for odd  $m$ , the graph  $(C_m \times P_n)^{(t)}$ ,  $m \geq 3, n, t \geq 2$ , is both super  $(\frac{m+3}{2} + p + q, 0)$ -edge antimagic total and super  $(\frac{m+3}{2} + p + 1, 2)$ -edge antimagic total.

Also by Lemma 2, we conclude that the graph  $(C_m \times P_n)^{(t)}$ ,  $m \geq 3, n, t \geq 2$ , has a super  $(\frac{m+3}{2} + p + \frac{q+1}{2}, 1)$ -edge antimagic total labeling, since  $q = m(2nt - 1)$ , which is odd for odd  $m$ . ■

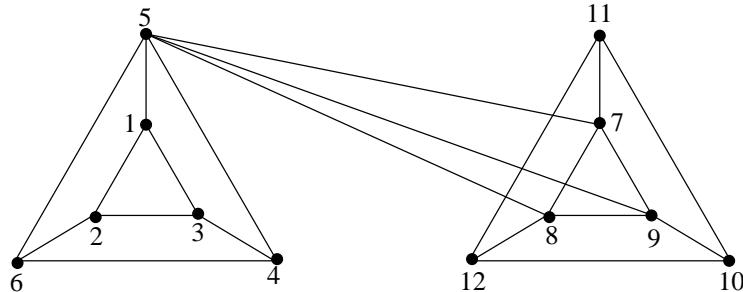


Figure 3.  $(a, 1)$ -edge antimagic vertex labeling of  $(C_3 \times P_2)^{(2)}$ .

5. A GRAPH DERIVED FROM COPIES OF GENERALIZED WEB GRAPH

Let  $(v_{i,j}^{(k)}, 1 \leq i \leq m, 1 \leq j \leq n + 1), 1 \leq k \leq t$ , be a collection of  $t$  disjoint copies of the *generalized web graph*  $W(m, n), m \geq 3, n \geq 2$ , such that  $v_{i,j}^{(k)}$  is adjacent to  $v_{i+1,j}^{(k)}$  for  $1 \leq i \leq m - 1, 1 \leq j \leq n, v_{m,j}^{(k)}$  is adjacent to  $v_{1,j}^{(k)}$  for  $1 \leq j \leq n$  and  $v_{i,j}^{(k)}$  is adjacent to  $v_{i,j+1}^{(k)}$  for  $1 \leq i \leq m, 1 \leq j \leq n$ . We denote the graph obtained by joining  $v_{1,n}^{(k)}$  to  $v_{i,1}^{(k+1)}$  and  $v_{i,2}^{(k+1)}$  for  $1 \leq i \leq m, 1 \leq k \leq t - 1$  as  $(W(m, n))^{(t)}$ . Clearly, the vertex set  $V$  and the edge set  $E$  of the graph  $(W(m, n))^{(t)}$  are given by  $V((W(m, n))^{(t)}) = \{v_{i,j}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n + 1\}$  and  $E((W(m, n))^{(t)}) = E_1 \cup E_2 \cup E_3$  where

$$\begin{aligned}
 E_1 &= \{v_{i,j}^{(k)}v_{i+1,j}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq m - 1, 1 \leq j \leq n\} \\
 &\cup \{v_{m,j}^{(k)}v_{1,j}^{(k)} : 1 \leq k \leq t, 1 \leq j \leq n\}, \\
 E_2 &= \{v_{i,j}^{(k)}v_{i,j+1}^{(k)} : 1 \leq k \leq t, 1 \leq i \leq m, 1 \leq j \leq n\}, \\
 E_3 &= \{v_{1,n}^{(k)}v_{i,1}^{(k+1)}, v_{1,n}^{(k)}v_{i,2}^{(k+1)} : 1 \leq k \leq t - 1, 1 \leq i \leq m\}.
 \end{aligned}$$

It is easy to see that for  $(W(m, n))^{(t)}$ , we have  $p = mt(n + 1)$  and  $q = 2m(nt + t - 1)$ .

**Lemma 10.** *For odd  $m, m \geq 3, n, t \geq 2$ , the graph  $(W(m, n))^{(t)}$  has an  $(a, 1)$ -edge antimagic vertex labeling.*

**Proof.** Let us define a bijection  $f_4 : V(W(m, n))^{(t)} \rightarrow \{1, 2, \dots, mt(n + 1)\}$  as follows:

*Case (i):  $n$  is even.*

If  $j$  is odd and  $1 \leq i \leq m, 1 \leq j \leq n + 1, 1 \leq k \leq t$ , then

$$f_4(v_{i,j}^{(k)}) = \begin{cases} (k - 1)(mn + m) + (j - 1)m + \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ (k - 1)(mn + m) + (j - 1)m + \frac{m+i+1}{2} & \text{if } i \text{ is even.} \end{cases}$$

If  $j$  is even and  $1 \leq i \leq m, 2 \leq j \leq n, 1 \leq k \leq t$ , then

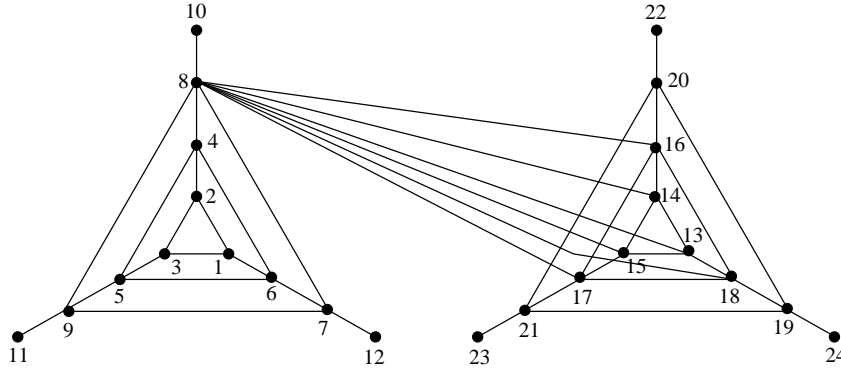


Figure 4.  $(a, 1)$ -edge antimagic vertex labeling of  $(W(3, 3))^{(2)}$ .

$$f_4(v_{i,j}^{(k)}) = \begin{cases} (k - 1)(mn + m) + (j - 1)m + \frac{m+i}{2} & \text{if } i \text{ is odd,} \\ (k - 1)(mn + m) + (j - 1)m + \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

Case (ii):  $n$  is odd.

If  $j$  is odd and  $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq t$ , then

$$f_4(v_{i,j}^{(k)}) = \begin{cases} (k - 1)(mn + m) + (j - 1)m + \frac{m+i}{2} & \text{if } i \text{ is odd,} \\ (k - 1)(mn + m) + (j - 1)m + \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

If  $j$  is even and  $1 \leq i \leq m, 2 \leq j \leq n + 1, 1 \leq k \leq t$ , then

$$f_4(v_{i,j}^{(k)}) = \begin{cases} (k - 1)(mn + m) + (j - 1)m + \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ (k - 1)(mn + m) + (j - 1)m + \frac{m+i+1}{2} & \text{if } i \text{ is even.} \end{cases}$$

In both the cases, we observe that under the bijection  $f_4$ , the edge weights of all the edges of  $(W(m, n))^{(t)}$  constitute an arithmetic sequence  $\{\frac{m+3}{2}, \frac{m+5}{2}, \dots, \frac{m+4mnt+4m(t-1)+1}{2}\}$ . Clearly  $\frac{m+3}{2}$  is an integer only when  $m$  is odd. Hence the vertex labeling  $f_4$  is an  $(\frac{m+3}{2}, 1)$ -edge antimagic vertex labeling of  $(W(m, n))^{(t)}$ , for odd  $m$ . ■

**Theorem 11.** For odd  $m, m \geq 3, n, t \geq 2$  and  $d \in \{0, 2\}$ , the graph  $(W(m, n))^{(t)}$ , has a super  $(a, d)$ -edge antimagic total labeling.

**Proof.** By Lemmas 3 and 10, we see that for odd  $m$ , the graph  $(W(m, n))^{(t)}$ ,  $m \geq 3, n, t \geq 2$  has a super  $(\frac{m+3}{2} + p + q, 0)$ -edge antimagic total labeling and a super  $(\frac{m+3}{2} + p + 1, 2)$ -edge antimagic total labeling. ■

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