Discussiones Mathematicae
Graph Theory 32 (2012) 461–471
doi:10.7151/dmgt.1618

THE TOTAL $\{k\}$-DOMATIC NUMBER OF DIGRAPHS

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Abstract

For a positive integer k, a *total* $\{k\}$*-dominating function* of a digraph D is a function f from the vertex set $V(D)$ to the set $\{0, 1, 2, \ldots, k\}$ such that for any vertex $v \in V(D)$, the condition $\sum_{u \in N^-(v)} f(u) \geq k$ is fulfilled, where $N^-(v)$ consists of all vertices of D from which arcs go into v. A set $\{f_1, f_2, \ldots, f_d\}$ of total $\{k\}$-dominating functions of D with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(D)$, is called a *total* $\{k\}$*-dominating family* (of functions) on D. The maximum number of functions in a total $\{k\}$-dominating family on D is the *total* $\{k\}$*-domatic number* of D, denoted by $d_t^{\{k\}}(D)$. Note that $d_t^{\{1\}}(D)$ is the classic total domatic number $d_t(D)$. In this paper we initiate the study of the total $\{k\}$-domatic number in digraphs, and we present some bounds for $d_t^{\{k\}}(D)$. Some of our results are extensions of well-know properties of the total domatic number of digraphs and the total $\{k\}$-domatic number of graphs.

Keywords: digraph, total $\{k\}$-dominating function, total $\{k\}$-domination number, total $\{k\}$-domatic number.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

In this paper, D is a finite and simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The order $|V|$ of D is denoted by $n = n(D)$. We write $d_D^+(v) = d^+(v)$ for the *outdegree* of a vertex v and $d_D^-(v) = d^-(v)$ for its *indegree.* The *minimum* and *maximum indegree* are $\delta^-(D)$ and $\Delta^-(D)$. The sets $N^+(v) = \{x|(v,x) \in A(D)\}$ and $N^-(v) = \{x|(x,v) \in A(D)\}$ are called the *outset* and *inset* of the vertex v. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x,y) \in A(D)$, the vertex y is an *outer neighbor* of x and x is an *inner neighbor* of y. We write K_n^* for the *complete digraph* of order n. Consult [5] for the notation and terminology which are not defined here.

For a positive integer k, a *total* $\{k\}$*-dominating function* (T$\{k\}$DF) of a digraph D with $\delta^-(D) \geq 1$ is a function f from the vertex set $V(D)$ to the set $\{0,1,2,\ldots,k\}$ such that for any vertex $v \in V(D)$, the condition $\sum_{u\in N^-(v)} f(u) \geq k$ is fulfilled. The *weight* of a T$\{k\}$DF f is the value $\omega(f) = \sum_{v\in V(D)} f(v)$. The *total* $\{k\}$*-domination number* of a digraph D, denoted by $\gamma_t^{\{k\}}(D)$, is the minimum weight of a T$\{k\}$DF of D. A $\gamma_t^{\{k\}}(D)$*-function* is a total $\{k\}$-dominating function of D with weight $\gamma_t^{\{k\}}(D)$. Note that $\gamma_t^{\{1\}}(D)$ is the classical total domination number $\gamma_t(D)$. If F is a minimum total dominating set of a digraph D with $\delta^-(D) \geq 1$, then the function f from $V(D)$ to $\{0,1,2,\ldots,k\}$ with $f(v) = k$ for $v \in F$ and $f(x) = 0$ for $x \in V(D) - F$ is a total $\{k\}$-dominating function of D and therefore

$$\gamma_t^{\{k\}}(D) \leq k|F| = k\gamma_t(D).$$

In this paper we always assume that D is a digraph with $\delta^-(D) \geq 1$.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct total $\{k\}$-dominating functions of D with the property that $\sum_{i=1}^d f_i(v) \leq k$ for each $v \in V(D)$, is called a *total* $\{k\}$*-dominating family* (of functions) on D. The maximum number of functions in a total $\{k\}$-dominating family (T$\{k\}$D family) on D is the *total* $\{k\}$*-domatic number* of D, denoted by $d_t^{\{k\}}(D)$. The total $\{k\}$-domatic number is well-defined and

$$d_t^{\{k\}}(D) \geq 1, \text{ for all digraphs } D \text{ with } \delta^-(D) \geq 1, \tag{1}$$

since the set consisting of the function $f : V(D) \to \{0,1,2,\ldots,k\}$ defined by $f(v) = k$ for each $v \in V(D)$, forms a T$\{k\}$D family on D. The total domatic number of a digraph was introduced by Jacob and Arumugam in [6].

Our purpose in this paper is to initiate the study of the total $\{k\}$-domatic number in digraphs. We first study basic properties and bounds for the total $\{k\}$-domatic number of a digraph. In addition, we determine the total $\{k\}$-domatic number of some classes of digraphs. Some of our results are extensions of well-know properties of the total domatic number of digraphs and the total $\{k\}$-domatic number of graphs (see, for example, [2, 3, 4, 6, 8]).

We start with the following observation.

Observation 1. *Let k be an integer, and let D be a digraph with $\delta^-(D) \geq 1$. Then $\gamma_t^{\{k\}}(D) \geq k+1$, with equality if and only if there exists a subset $S \subseteq V(D)$ of size $k+1$ such that $D[S]$ is a complete digraph, and each vertex $x \in V(D) - S$ has at least k inner neighbors in S.*

Proof. Let f be a $\gamma_t^{\{k\}}(D)$-function, and let $v \in V(D)$ be an arbitrary vertex. The definition implies that $\sum_{x \in N^-(v)} f(x) \geq k$. If $\sum_{x \in N^-(v)} f(x) \geq k+1$, then $\gamma_t^{\{k\}}(D) \geq k+1$. If $\sum_{x \in N^-(v)} f(x) = k$, then let $u \in N^-(v)$ be a vertex such that $f(u) \geq 1$. Since $\sum_{x \in N^-(u)} f(x) \geq k$ and $u \notin N^-(u)$, we deduce that $\omega(f) = \sum_{x \in V(D)} f(v) \geq \sum_{x \in (N^-(u) \cup \{u\})} f(x) \geq k+1$ and therefore $\gamma_t^{\{k\}}(D) \geq k+1$.

Assume that $\gamma_t^{\{k\}}(D) = k+1$. Let f be a $\gamma_t^{\{k\}}(D)$-function. If there exists a vertex v such that $f(v) \geq 2$, then we obtain the contradiction $\sum_{x \in N^-(v)} f(x) \leq k+1-2 = k-1$. Hence $f(x) = 1$ or $f(x) = 0$ for each vertex $x \in V(D)$. Let $S \subseteq V(D)$ such that $f(x) = 1$ for each $x \in S$. Then $|S| = k+1$, $D[S]$ is a complete digraph, and each vertex $x \in V(D) - S$ has at least k inner neighbors in S.

Conversely, assume that there exists a subset $S \subseteq V(D)$ of size $k+1$ such that $D[S]$ is a complete digraph, and each vertex $x \in V(D) - S$ has at least k inner neighbors in S. Define the function f by $f(x) = 1$ for $x \in S$ and $f(x) = 0$ for $x \in V(D) - S$. Then f is a total $\{k\}$-dominating function of D such that $\omega(f) = k+1$. Since $\gamma_t^{\{k\}}(D) \geq k+1$, we deduce that $\gamma_t^{\{k\}}(D) = k+1$. ■

2. PROPERTIES OF THE $\{k\}$-DOMATIC NUMBER

In this section we mainly present basic properties of $d_t^{\{k\}}(D)$ and bounds on the total $\{k\}$-domatic number of a digraph.

Theorem 2. *If D is a digraph of order n, then $\gamma_t^{\{k\}}(D) \cdot d_t^{\{k\}}(D) \leq kn$. Moreover, if $\gamma_t^{\{k\}}(D) \cdot d_t^{\{k\}}(D) = kn$, then for each T$\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d_t^{\{k\}}(D)$, each function f_i is a $\gamma_t^{\{k\}}(D)$-function and $\sum_{i=1}^d f_i(v) = k$ for all $v \in V(D)$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a T$\{k\}$D family on D such that $d = d_t^{\{k\}}(D)$. Then

$$\begin{aligned} d \cdot \gamma_t^{\{k\}}(D) &= \sum_{i=1}^d \gamma_t^{\{k\}}(D) \leq \sum_{i=1}^d \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^d f_i(v) \\ &\leq \sum_{v \in V(D)} k = kn. \end{aligned}$$

If $\gamma_t^{\{k\}}(D) \cdot d_t^{\{k\}}(D) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the T$\{k\}$D family $\{f_1, f_2, \ldots, f_d\}$ on D and for each i,

$\sum_{v \in V(D)} f_i(v) = \gamma_t^{\{k\}}(D)$. Thus each function f_i is a $\gamma_t^{\{k\}}(D)$-function, and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V(D)$. ■

The special case $k = 1$ in Theorem 2 can be found in [6].

Corollary 3. *Let k, n be two positive integers. If $k+1$ is a divisor of n and $\frac{n}{k+1} \geq 2$, then $d_t^{\{k\}}(K_n^*) = \frac{kn}{k+1}$.*

Proof. Applying Observation 1 and Theorem 2, we see that $d_t^{\{k\}}(K_n^*) \leq \frac{kn}{k+1}$. Now we consider a partition of $V(K_n^*)$ into $s = \frac{n}{k+1}$ sets $V_1, V_2, \ldots, V_s$ such that $|V_i| = k+1$ for each i. Let $V_i = \{v_1^i, v_2^i, \ldots, v_{k+1}^i\}$ for $1 \leq i \leq s$. Define, for $1 \leq i \leq s$ and $1 \leq j \leq k$,

$$f_i^j(v_1^i) = \cdots = f_i^j(v_j^i) = 1, \ f_i^j(v_{j+1}^{i+1}) = \cdots = f_i^j(v_{k+1}^{i+1}) = 1 \text{ and}$$
$$f_i^j(x) = 0 \text{ otherwise, where the indices } i+1 \text{ are taken modulo } s.$$

It is easy see that $\{f_i^j \mid 1 \leq i \leq \frac{n}{k+1}, 1 \leq j \leq k,\}$ is a T$\{k\}$D family on K_n^*, and therefore $d_t^{\{k\}}(K_n^*) \geq \frac{kn}{k+1}$. Since $k+1$ is a divisor of n, the proof is complete. ■

A further consequence of Theorem 2 and Observation 1 now follows.

Corollary 4. *If $k \geq 2$ is an integer, and D is a digraph of order $k+1$, then $d_t^{\{k\}}(D) \leq k-1$.*

Proof. Since $\gamma_t^{\{k\}}(D) \geq k+1$, it follows from Theorem 2 that $d_t^{\{k\}}(D) \leq k$. If $\gamma_t^{\{k\}}(D) \geq k+2$, then Theorem 2 implies $d_t^{\{k\}}(D) \leq k-1$ immediately. If $\gamma_t^{\{k\}}(D) = k+1$ and $d_t^{\{k\}}(D) = k$, then for the T$\{k\}$D family $\{f_1, f_2, \ldots, f_k\}$ on D, each function f_i is a $\gamma_t^{\{k\}}(D)$-function, and Observation 1 leads to the contradiction that $f_1 \equiv f_2 \equiv \cdots \equiv f_k$. This completes the proof. ■

Corollary 5. *If k is a positive integer, and D is a digraph of order n, then $d_t^{\{k\}}(D) \leq \frac{kn}{k+1}$, with equality only if $k+1$ is a divisor of n and $\frac{n}{k+1} \geq 2$ when $k \geq 2$.*

Proof. Since $\gamma_t^{\{k\}}(D) \geq k+1$, it follows from Theorem 2 that $d_t^{\{k\}}(D) \leq \frac{kn}{\gamma_t^{\{k\}}(D)} \leq \frac{kn}{k+1}$, and this is the desired inequality.

Assume that $d_t^{\{k\}}(D) = \frac{kn}{k+1}$. Since $(k, k+1) = 1$, $k+1$ must be a divisor of n. If $k \geq 2$, then it follows from Corollary 4 that $\frac{n}{k+1} \geq 2$. ■

Corollary 3 demonstrates that Corollary 5 is sharp.

Theorem 6. *If D is a digraph of order n and k a positive integer, then*

$$\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \leq nk + 1.$$

Proof. Applying Theorem 2, we obtain $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \leq \frac{kn}{d_t^{\{k\}}(D)} + d_t^{\{k\}}(D)$. Note that $d_t^{\{k\}}(G) \geq 1$, by inequality (1), and that Corollary 5 implies that $d_t^{\{k\}}(D) \leq n$. Using these inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $1 \leq x \leq \sqrt{kn}$ and increasing for $\sqrt{kn} \leq x \leq n$, we obtain $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \leq \max\left\{kn+1, \frac{kn}{n} + n\right\} = nk + 1$, and this is the desired bound. ∎

If C_n denotes a directed cycle on n vertices, then the function $f : V(C_n) \to \{0, 1, \ldots, k\}$ defined by $f(x) = k$ for each $x \in V(C_n)$ is the unique total $\{k\}$-dominating function of C_n and hence $\gamma_t^{\{k\}}(C_n) = nk$ and $d_t^{\{k\}}(C_n) = 1$. This demonstrates that Theorem 6 is sharp.

Theorem 7. *Let D be a digraph of order $n \geq 3$, and let $k \geq 1$ be an integer. If $d_t^{\{k\}}(D) \geq 2$, then $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \leq \frac{kn}{2} + 2$.*

Proof. Theorem 2 implies that $\gamma_t^{\{k\}}(D)+d_t^{\{k\}}(D) \leq \gamma_t^{\{k\}}(D)+\frac{kn}{\gamma_t^{\{k\}}(D)}$. It follows from Observation 1 and Theorem 2 that $k+1 \leq \gamma_t^{\{k\}}(D) \leq kn/2$. Using these inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k+1 \leq x \leq \sqrt{kn}$ and increasing for $\sqrt{kn} \leq x \leq kn/2$, we obtain

$$\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \leq \max\left\{k+1+\frac{kn}{k+1}, \frac{kn}{2}+2\right\} = \frac{kn}{2} + 2,$$

and this is the desired bound. ∎

Theorem 8. *If D is a digraph and $k \geq 1$ an integer, then $d_t^{\{k\}}(D) \leq \delta^-(D)$. Moreover, if $d_t^{\{k\}}(D) = \delta^-(D)$, then for each function of any $T\{k\}D$ family $\{f_1, f_2, \ldots, f_d\}$ and for all vertices v of indegree $\delta^-(D)$, $\sum_{u \in N^-(v)} f_i(u) = k$ and $\sum_{i=1}^d f_i(u) = k$ for every $u \in N^-(v)$.*

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a T$\{k\}$D family on D such that $d = d_t^{\{k\}}(D)$, and let v be a vertex of minimum indegree $\delta^-(D)$. Since $\sum_{u \in N^-(v)} f_i(u) \geq k$ for all $i \in \{1, 2, \ldots, d\}$, we obtain $kd \leq \sum_{i=1}^d \sum_{u \in N^-(v)} f_i(u) = \sum_{u \in N^-(v)} \sum_{i=1}^d f_i(u) \leq \sum_{u \in N^-(v)} k = k\delta^-(D)$, and this leads to the desired bound.

If $d_t^{\{k\}}(D) = \delta^-(D)$, then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement. ∎

The special case $k = 1$ in Theorem 8 can be found in [6].

Observation 9. *Let D be a digraph with the property that the underlying graph is connected and bipartite. If $k \geq 1$ is an integer, then $\gamma_t^{\{k\}}(D) \geq 2k$.*

Proof. Let f be a $\gamma_t^{\{k\}}(D)$-function, and let V_1 and V_2 be the partite sets of the underlying graph. If $w_i \in V_i$, then the definition implies that $\sum_{x\in N^-(w_i)} f(x) \geq k$ for $i = 1, 2$. It follows that $w(f) = \sum_{x\in V(D)} f(x) = \sum_{x\in V(D)-V_1} f(x) + \sum_{x\in V(D)-V_2} f(x) \geq \sum_{x\in N^-(w_2)} f(x) + \sum_{x\in N^-(w_1)} f(x) \geq 2k$, thus $\gamma_t^{\{k\}}(D) \geq 2k$. ■

Corollary 10. *If $K^*_{p,p}$ is the complete bipartite digraph and $k \geq 1$ an integer, then $d_t^{\{k\}}(K^*_{p,p}) = p$.*

Proof. Theorem 2 and Observation 9 show that $d_t^{\{k\}}(K^*_{p,p}) \leq p$.

Now let $\{u_1, u_2, \ldots, u_p\}$ and $\{v_1, v_2, \ldots, v_p\}$ be the partite sets of the complete bipartite digraph. Define $f_i(u_i) = f_i(v_i) = k$ and $f_i(x) = 0$ for each vertex $x \in V(D) - \{u_i, v_i\}$ and each $i \in \{1, 2, \ldots, p\}$. Then we observe that f_i is a T$\{k\}$DF of $K^*_{p,p}$ for each $i \in \{1, 2, \ldots, p\}$. Therefore $\{f_1, f_2, \ldots, f_p\}$ is a T$\{k\}$D family on $K^*_{p,p}$. Consequently, $d_t^{\{k\}}(K^*_{p,p}) \geq p$ and so $d_t^{\{k\}}(K^*_{p,p}) = p$. ■

Corollary 10 demonstrates that Theorem 8 is sharp.

Theorem 11. *Let $k \geq 1$ be an integer and D a digraph of order n with $\delta^-(D) \geq 1$. If $\delta^-(D) \mid k$, then $d_t^{\{k\}}(D) \geq \delta^-(D) - 1$.*

Proof. If $\delta^-(D) = 1$, then the result is immediate.

Let $\delta^-(D) \geq 2$ and let $V(D) = \{v_1, v_2, \ldots, v_n\}$. Define $f_i : V(D) \to \{0, 1, \ldots, k\}$ by

$$f_i(v_j) = \begin{cases} \frac{k}{\delta^-(D)} + 1 & \text{if} \quad j = i, \\ \frac{k}{\delta^-(D)} & \text{if} \quad j \neq i, \end{cases} \quad \text{for every } 1 \leq i \leq \delta^-(D) - 1 \text{ and } 1 \leq j \leq n.$$

Then for each $v \in V(D)$ and each $1 \leq i \leq \delta^-(D) - 1$,

$$\sum_{u\in N^-(v)} f_i(u) \geq \sum_{u\in N^-(v)} \frac{k}{\delta^-(D)} \geq \frac{k}{\delta^-(D)}\delta^-(D) = k.$$

Hence f_i is a T$\{k\}$DF of D for each $1 \leq i \leq \delta^-(D) - 1$. Now, since $\delta^-(D) \mid k$, we have

$$\sum_{i=1}^{\delta(D)-1} f_i(v) \leq \frac{k}{\delta^-(D)}(\delta^-(D) - 2) + \left(\frac{k}{\delta^-(D)} + 1\right) = k + \left(1 - \frac{k}{\delta^-(D)}\right) \leq k$$

for each $v \in V(D)$. Thus $\{f_1, f_2, \ldots, f_{\delta^-(D)-1}\}$ is a T$\{k\}$D family on D, and the proof is complete. ■

Theorem 12. *Let $k \geq 1$ be an integer and D a digraph of order n. If $\delta^-(D) \nmid k$, then $d_t^{\{k\}}(D) \geq \left\lfloor \frac{k}{\lceil k/\delta^-(D)\rceil} \right\rfloor$.*

Proof. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. Define $f_i : V(D) \to \{0, 1, \ldots, k\}$ by

$$f_i(v_j) = \begin{cases} \lfloor \frac{k}{\delta^-(D)} \rfloor & \text{if} \quad j = i, \\ \lceil \frac{k}{\delta^-(D)} \rceil & \text{if} \quad j \neq i, \end{cases} \quad \text{for every } 1 \le i \le \left\lfloor \frac{k}{\lceil k/\delta^-(D) \rceil} \right\rfloor \text{ and } 1 \le j \le n.$$

Then for each $v \in V(D)$ and each $1 \le i \le \left\lfloor \frac{k}{\lceil k/\delta^-(D) \rceil} \right\rfloor$,

$$\sum_{u \in N^-(v)} f_i(u) \ge \left\lfloor \frac{k}{\delta^-(D)} \right\rfloor + \left\lceil \frac{k}{\delta^-(D)} \right\rceil (\delta^-(D) - 1) \ge \left\lceil \frac{k}{\delta^-(D)} \right\rceil \delta^-(D) - 1 \ge k.$$

Hence f_i is a T$\{k\}$DF of D for each i. Since $\delta^-(D) \nmid k$, we have

$$\sum_{i=1}^{\left\lfloor \frac{k}{\lceil k/\delta^-(D) \rceil} \right\rfloor} f_i(v) \le \left\lceil \frac{k}{\delta^-(D)} \right\rceil \cdot \left\lfloor \frac{k}{\lceil \frac{k}{\delta^-(D)} \rceil} \right\rfloor \le \left\lceil \frac{k}{\delta^-(D)} \right\rceil \cdot \frac{k}{\lceil \frac{k}{\delta^-(D)} \rceil} = k$$

for each $v \in V(D)$. Thus $\{f_1, f_2, \ldots, f_{\left\lfloor \frac{k}{\lceil k/\delta^-(D) \rceil} \right\rfloor}\}$ is a T$\{k\}$D family on D, and the proof is complete. ■

Using Theorems 2, 8, 11 and 12, we will improve Theorem 6 considerably for some cases.

Corollary 13. *Let $k \ge 1$ be an integer, and let D be a digraph of order n. If $\delta^-(D) > k$, then $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \le n + k$.*

Proof. Since $\delta^-(D) > k$, it follows from Theorem 12 that

$$d_t^{\{k\}}(D) \ge \left\lfloor \frac{k}{\lceil \frac{k}{\delta^-(D)} \rceil} \right\rfloor = k.$$

In addition, Theorem 8 implies that $d_t^{\{k\}}(D) \le \delta^-(D) \le n$. Using these two inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n$, Theorem 2 leads to

$$\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \le \frac{kn}{d_t\{k\}(D)} + d_t^{\{k\}}(D) \le \max\left\{\frac{kn}{k} + k, \frac{kn}{n} + n\right\} = n + k.$$

This is the desired bound, and the proof is complete. ■

Corollary 14. *Let $k \ge 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \ge 2$. If $\delta^-(D) \mid k$, then $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \le \frac{kn}{\delta^-(D)-1} + \delta^-(D) - 1$.*

Proof. Since $\delta^-(D) \mid k$, Theorem 11 shows that $d_t^{\{k\}}(D) \ge \delta^-(D) - 1$, and Theorem 8 implies that $d_t^{\{k\}}(D) \le \delta^-(D)$. Using these two inequalities and Theorem 2, we obtain the desired bound as follows $\gamma_t^{\{k\}}(D) + d_t^{\{k\}}(D) \le \frac{kn}{d_t^{\{k\}}(D)} + d_t^{\{k\}}(D) \le \max\left\{\frac{kn}{\delta^-(D)-1} + \delta^-(D) - 1, \frac{kn}{\delta^-(D)} + \delta^-(D)\right\} = \frac{kn}{\delta^-(D)-1} + \delta^-(D) - 1.$ ■

Let D be a digraph. By D^{-1} we denote the digraph obtained by reversing all arcs of D. A digraph without directed cycles of length 2 is called an *oriented graph.* An oriented graph D is a *tournament* when either $(x, y) \in A(D)$ or $(y, x) \in A(D)$ for each pair of distinct vertices $x, y \in V(D)$.

Theorem 15. *For every oriented graph D with $\delta^-(D) \geq 1$ and $\delta^-(D^{-1}) \geq 1$, $d_t^{\{k\}}(D) + d_t^{\{k\}}(D^{-1}) \leq n-1$. If $d_t^{\{k\}}(D) + d_t^{\{k\}}(D^{-1}) = n-1$, then D is a regular tournament.*

Proof. Since $\delta^-(D) + \delta^-(D^{-1}) \leq n-1$, Theorem 8 leads to

$$d_t^{\{k\}}(D) + d_t^{\{k\}}(D^{-1}) \leq \delta^-(D) + \delta^-(D^{-1}) \leq n-1.$$

If D is not a tournament or D is a non-regular tournament, then $\delta^-(D) + \delta^-(D^{-1}) \leq n-2$, and hence we deduce from Theorem 8 that

$$d_t^{\{k\}}(D) + d_t^{\{k\}}(D^{-1}) \leq \delta^-(D) + \delta^-(D^{-1}) \leq n-2.$$

■

Now we present further lower bounds on the total $\{k\}$-domatic number.

Theorem 16. *Let $k \geq 1$ be an integer, and D a digraph with $\delta^-(D) = \delta^- \geq 1$.*

(i) *If $k < \delta^-$, then $d_t^{\{k\}}(D) \geq k$.*

(ii) *If $k = p\delta^-$ with an integer $p \geq 1$, then $d_t^{\{k\}}(D) \geq \delta^- - 1$.*

(iii) *If $k = p\delta^- + r$ with integers $p, r \geq 1$ and $r \leq \delta^- - 1$, then $d_t^{\{k\}}(D) \geq \left\lceil \frac{p(\delta^- -1)+1}{p+1} \right\rceil$.*

Proof. (i) If $k < \delta^-$, then Theorem 12 implies immediately $d_t^{\{k\}}(D) \geq k$.

(ii) If $k = p\delta^-$, then Theorem 11 implies immediately $d_t^{\{k\}}(D) \geq \delta^- - 1$.

(iii) If $k = p\delta^- + r$ with integers $p, r \geq 1$ and $r \leq \delta^- - 1$, then $\lceil \frac{k}{\delta^-} \rceil = p+1$ and therefore we deduce from Theorem 12 that

$$d_t^{\{k\}}(D) \geq \left\lfloor \frac{k}{\lceil k/\delta^- \rceil} \right\rfloor = \left\lfloor \frac{k}{p+1} \right\rfloor = \left\lfloor \frac{p\delta^- + r}{p+1} \right\rfloor \geq \frac{p\delta^- + r}{p+1} - \frac{p}{p+1} \geq \frac{p(\delta^- -1)+1}{p+1}.$$

This leads to the desired bound, and the proof is complete. ■

Corollary 17. *If $k \geq 1$ is an integer and D a digraph with $\delta^-(D) \geq 1$, then $d_t^{\{k\}}(D) \geq \min\left\{k, \frac{\delta^-(D)}{2}\right\}$.*

The *complement* $\overline{D}$ of a digraph D is that digraph with vertex set $V(D)$ such that for two arbitrary different vertices u and v the arc (u, v) belongs to $\overline{D}$ if and only if (u, v) does not belong to D.

Theorem 18. *Let $k \geq 1$ be an integer, and let D be an r-diregular digraph of order $n \geq 3$ with $1 \leq r \leq n-2$. Then* $d_t^{\{k\}}(D) + d_t^{\{k\}}(\overline{D}) \geq \min\left\{k+1, \lceil \frac{n-1}{2} \rceil\right\}$.

Proof. Assume first that $k < \delta^-(D)$. Then it follows from Theorem 16 (i) that $d_t^{\{k\}}(D) \geq k$ and thus $d_t^{\{k\}}(D) + d_t^{\{k\}}(\overline{D}) \geq k+1$.

Assume next that $k \geq \delta^-(D)$ and $k < \delta^-(\overline{D})$. Then Theorem 16 (i) implies $d_t^{\{k\}}(\overline{D}) \geq k$ and so $d_t^{\{k\}}(D) + d_t^{\{k\}}(\overline{D}) \geq k+1$.

Finally assume that $k \geq \delta^-(D)$ and $k \geq \delta^-(\overline{D})$. Applying Theorem 16 (ii) and (iii), we observe that $d_t^{\{k\}}(D) \geq \delta^-(D)/2$ and $d_t^{\{k\}}(\overline{D}) \geq \delta^-(\overline{D})/2$, and hence we deduce that

$$d_t^{\{k\}}(D) + d_t^{\{k\}}(\overline{D}) \geq \frac{\delta^-(D)}{2} + \frac{\delta^-(\overline{D})}{2} = \frac{n-1}{2}.$$

Combining these inequalities, we obtain the desired bound. ■

Theorem 19. *For every digraph D of order n,* $d_t^{\{k\}}(D) \geq \left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor$.

Proof. Let S be any subset of $V(D)$ with $|S| \geq n - \delta^-(D)$. If $v \in V(D) - S$, then there exists at least one vertex $u \in S$ such that $(u,v) \in A(D)$. Let $S_1, S_2, \ldots, S_{\left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor}$ be disjoint subsets of $V(D)$ each of cardinality $n - \delta^-(D)$. Define $f_i : V(G) \rightarrow \{0,1,\ldots,k\}$ by

$$f_i(v) = \begin{cases} k & \text{if } v \in S_i, \\ 0 & \text{otherwise,} \end{cases}$$

for each $1 \leq i \leq \left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor$.

Since $|S_i| = n - \delta^-(D)$, it is clear that f_i is a total $\{k\}$-dominating function of D for each i. Since also S_i are disjoint subsets of $V(D)$, then for every $v \in V(D)$ $\sum_{i=1}^{\left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor} f_i(v) \leq k$. Thus $\{f_1, f_2, \ldots, f_{\left\lfloor \frac{n}{n-\delta^-(D)} \right\rfloor}\}$ is a T$\{k\}$D family on D, and the proof is complete. ■

The special case $k = 1$ in Theorems 15 and 19 can be found in [6].

3. THE TOTAL $\{k\}$-DOMATIC NUMBER OF GRAPHS

The *total $\{k\}$-dominating function* of a graph G is defined in [7] as a function $f : V(G) \longrightarrow \{0,1,2,\ldots,k\}$ such that $\sum_{x \in N_G(v)} f(x) \geq k$ for all $v \in V(G)$. The sum $\sum_{x \in V(G)} f(x)$ is the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all total $\{k\}$-dominating functions f on G is called the *total $\{k\}$-domination number* of G, denoted by $\gamma_t^{\{k\}}(G)$. In the special case $k = 1$, $\gamma_t^{\{k\}}(G)$ is the classical total domination number $\gamma_t(G)$.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct total $\{k\}$-dominating functions on G such that $\sum_{i=1}^{d} f_i(v) \le k$ for each $v \in V(G)$, is called a *total* $\{k\}$*-dominating family* on G. The maximum number of functions in a total $\{k\}$-dominating family on G is the *total* $\{k\}$*-domatic number* of G, denoted by $d_t^{\{k\}}(G)$. This parameter was introduced by Sheikholeslami and Volkmann in [8] and has been studied in [1]. In the case $k = 1$, we write $d_t(G)$ instead of $d_t^{\{1\}}(G)$ which was introduced by Cockayne, Dawes and Hedetniemi [3], and has been studied in many articles.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 20. *If $D(G)$ is the associated digraph of a graph G, then*
$$\gamma_t^{\{k\}}(D(G)) = \gamma_t^{\{k\}}(D) \text{ and } d_t^{\{k\}}(D(G)) = d_t^{\{k\}}(D).$$

There are a lot of interesting applications of Observation 20. Using Theorems 2 and 6, we obtain the next results immediately.

Corollary 21 [8]**.** *If $k \ge 1$ is an integer and G a graph of order n without isolated vertices, then $\gamma_t^{\{k\}}(G) \cdot d_t^{\{k\}}(G) \le kn$.*

The case $k = 1$ in Corollary 21 leads to the well-known inequality $\gamma_t(G) \cdot d_t(G) \le n$, given by Cockayne, Dawes and Hedetniemi [3] in 1980.

Corollary 22 [8]**.** *If $k \ge 1$ is an integer and G a graph of order n without isolated vertices, then $\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \le nk + 1$.*

Corollary 23 [3]**.** *If G is graph of order n without isolated vertices, then $\gamma_t(G) + d_t(G) \le n + 1$.*

Theorem 7 and Observation 20 lead to the following bound.

Corollary 24 [8]**.** *Let $k \ge 1$ be an integer and G a graph of order n without isolated vertices. If $d_t^{\{k\}}(G) \ge 2$, then $\gamma_t^{\{k\}}(G) + d_t^{\{k\}}(G) \le \frac{kn}{2} + 2$.*

Corollary 25 [4]**.** *If G is a graph of order n without isolated vertices and if $d_t(G) \ge 2$, then $\gamma_t(G) + d_t(G) \le \frac{n}{2} + 2$.*

Since $\delta^-(D(G)) = \delta(G)$, the next result follows from Observation 20 and Theorem 8.

Corollary 26 [8]**.** *If $k \ge 1$ is an integer and G a graph without isolated vertices, then $d_t^{\{k\}}(G) \le \delta(G)$.*

The case $k = 1$ in Corollary 26 can be found in [3]. Theorem 11 and Observation 20 imply the next result.

Corollary 27 [2]. *Let $k \geq 1$ be an integer and G a graph of order n without isolated vertices. If $\delta(G) \mid k$, then $d_t^{\{k\}}(G) \geq \delta(G) - 1$.*

Finally, the next theorem follows from Theorem 18 and Observation 20.

Corollary 28 [1]. *For every δ-regular graph of order $n \geq 5$ in which neither G nor $\overline{G}$ have isolated vertices, $d_t^{\{k\}}(G) + d_t^{\{k\}}(\overline{G}) \geq \min\left\{k+1, \left\lceil \frac{n-2}{2} \right\rceil\right\}$.*

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Received 31 March 2011
Revised 29 August 2011
Accepted 30 August 2011