FRACTIONAL DISTANCE DOMINATION IN GRAPHS

S. ARUMUGAM\textsuperscript{1,2}, VARUGHSE MATHEW\textsuperscript{3} AND K. KARUPPASAMY\textsuperscript{1}

\textsuperscript{1} National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH)  
Kalasalingam University, Anand Nagar, Krishnankoil-626 126, India  
\textsuperscript{2} School of Electrical Engineering and Computer Science  
The University of Newcastle, NSW 2308, Australia  
\textsuperscript{3} Department of Mathematics, Mar Thoma College, Tiruvalla-689 103, India  
e-mail: s.arumugam.khu@gmail.com  
\{k_karuppasamy,varughse_m1\}@yahoo.co.in

Abstract

Let $G = (V, E)$ be a connected graph and let $k$ be a positive integer with $k \leq \text{rad}(G)$. A subset $D \subseteq V$ is called a distance $k$-dominating set of $G$ if for every $v \in V - D$, there exists a vertex $u \in D$ such that $d(u, v) \leq k$. In this paper we study the fractional version of distance $k$-domination and related parameters.

Keywords: domination, distance $k$-domination, distance $k$-dominating function, $k$-packing, fractional distance $k$-domination.

2010 Mathematics Subject Classification: 05C69, 05C72.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For basic terminology in graphs we refer to Chartrand and Lesniak [3]. For basic terminology in domination related concepts we refer to Haynes et al. [9].

Let $G = (V, E)$ be a graph. A subset $D$ of $V$ is called a dominating set of $G$ if every vertex in $V - D$ is adjacent to at least one vertex in $D$. A dominating set $D$ is called a minimal dominating set if no proper subset of $D$ is a dominating set of $G$. The minimum (maximum) cardinality of a minimal dominating set of $G$ is called the domination number (upper domination number) of $G$ and is denoted by $\gamma(G)$ ($\Gamma(G)$). Let $A$ and $B$ be two subsets of $V$. We say that $B$ dominates $A$ if...
every vertex in $A - B$ is adjacent to at least one vertex in $B$. If $B$ dominates $A$, then we write $B \to A$. Meir and Moon [12] introduced the concept of a $k$-packing and distance $k$-domination in a graph as a natural generalisation of the concept of domination. Let $G = (V, E)$ be a graph and $v \in V$. For any positive integer $k$, let $N_k(v) = \{ u \in V : d(u, v) \leq k \}$ and $N_k[v] = N_k(v) \cup \{v\}$. A set $S \subseteq V$ is a distance $k$-dominating set of $G$ if $N_k[v] \cap S \neq \emptyset$ for every vertex $v \in V - S$. The minimum (maximum) cardinality among all minimal distance $k$-dominating sets of $G$ is called the distance $k$-domination number (upper distance $k$-domination number) of $G$ and is denoted by $\gamma_k(G)$ ($\Gamma_k(G)$). A set $S \subseteq V$ is said to be an efficient distance $k$-dominating set of $G$ if $|N_k[v] \cap S| = 1$ for all $v \in V - S$. Clearly, $\gamma(G) = \gamma_1(G)$. A distance $k$-dominating set of cardinality $\gamma_k(G)$ ($\Gamma_k(G)$) is called a $\gamma_k$ ($\Gamma_k$)-set. Hereafter, we shall use the term $k$-domination for distance $k$-domination.

Note that, $\gamma_k(G) = \gamma(G^k)$, where $G^k$ is the $k^{th}$ power of $G$, which is obtained from $G$ by joining all pairs of distinct vertices $u, v$ with $d(u, v) \leq k$. A subset $S \subseteq V(G)$ of a graph $G = (V, E)$ is said to be a $k$-packing ([12]) of $G$, if $d(u, v) > k$ for all pairs of distinct vertices $u$ and $v$ in $S$. The $k$-packing number $\rho_k(G)$ is defined to be the maximum cardinality of a $k$-packing set in $G$. The corona of a graph $G$, denoted by $G \circ K_1$, is the graph formed from a copy of $G$ by attaching to each vertex $v$ a new vertex $v'$ and an edge $\{v, v'\}$. The Cartesian product of graphs $G$ and $H$, denoted by $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G \Box H$ if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. For a survey of results on distance domination we refer to Chapter 12 of Haynes et al. [10].

Hedetniemi et al. [11] introduced the concept of fractional domination in graphs. Grinstead and Slater [6] and Domke et al. [5] have presented several results on fractional domination and related parameters in graphs. Arumugam et al. [1] have investigated the fractional version of global domination in graphs.

Let $G = (V, E)$ be a graph. Let $g : V \to \mathbb{R}$ be any function. For any subset $S$ of $V$, let $g(S) = \sum_{v \in S} g(v)$. The weight of $g$ is defined by $|g| = g(V) = \sum_{v \in V} g(v)$. For a subset $S$ of $V$, the function $\chi_S : V \to \{0, 1\}$ defined by

$$
\chi_S(v) = \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{if } v \notin S,
\end{cases}
$$

is called the characteristic function of $S$.

A function $g : V \to [0, 1]$ is called a dominating function (DF) of the graph $G = (V, E)$ if $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ for all $v \in V$. For functions $f, g$ from $V \to [0, 1]$ we write $f \leq g$ if $f(v) \leq g(v)$ for all $v \in V$. Further, we write $f < g$ if $f \leq g$ and $f(v) < g(v)$ for some $v \in V$. A DF $g$ of $G$ is minimal (MDF) if $f$ is not a DF for all functions $f : V \to [0, 1]$ with $f < g$. 

The fractional domination number $\gamma_f(G)$ and the upper fractional domination number $\Gamma_f(G)$ are defined as follows:

$$\gamma_f(G) = \min \{|g| : g \text{ is a minimal dominating function of } G\},$$

$$\Gamma_f(G) = \max \{|g| : g \text{ is a minimal dominating function of } G\}.$$  

For a dominating function $f$ of $G$, the boundary set $B_f$ and the positive set $P_f$ are defined by

$$B_f = \{u \in V(G) : f(N[u]) = 1\} \quad \text{and} \quad P_f = \{u \in V(G) : f(u) > 0\}.$$  

A function $g : V \to [0, 1]$ is called a packing function (PF) of the graph $G = (V, E)$ if $g(N[v]) = \sum_{u \in N[v]} g(u) \leq 1$ for all $v \in V$. The lower fractional packing number $p_f(G)$ and the fractional packing number $P_f(G)$ are defined as follows:

$$p_f(G) = \min \{|g| : g \text{ is a maximal packing function of } G\},$$

$$P_f(G) = \max \{|g| : g \text{ is a maximal packing function of } G\}.$$  

It was observed in Chapter 3 of [10] that for every graph $G$, $1 \leq \gamma_f(G) = P_f(G) \leq \gamma(G) \leq \Gamma(G) \leq \Gamma_f(G)$. We need the following theorems:

**Theorem 1.1** [5]. For a graph $G$, $p_f(G) \leq \rho_2(G) \leq P_f(G)$.

**Theorem 1.2** [2]. A DF $f$ of $G$ is an MDF if and only if $B_f \to P_f$.

**Theorem 1.3** [2]. If $f$ and $g$ are MDFs of $G$ and $0 < \lambda < 1$ then $h_\lambda = \lambda f + (1 - \lambda)g$ is an MDF of $G$ if and only if $B_f \cap B_g \to P_f \cup P_g$.

**Theorem 1.4** [5]. If $G$ is an $r$-regular graph of order $n$, then $\gamma_f(G) = \frac{n}{r+1}$.

**Theorem 1.5** [4]. Let $G$ be a block graph. Then for any integer $k \geq 1$, we have $\rho_{2k}(G) = \gamma_k(G)$.

For other families of graphs satisfying $\rho_2(G) = \gamma(G)$, we refer to Rubalcaba et al. [13].

**Definition 1.6** [15]. A linear Benzenoid chain $B(h)$ of length $h$ is the graph obtained from $P_2 \square P_{h+1}$ by subdividing exactly once each edge of the two copies of $P_{h+1}$. Hence $B(h)$ is a subgraph of $P_2 \square P_{2h+1}$. The graph $B(4)$ is given in Figure 1.

![Figure 1. B(4).](image)

**Theorem 1.7** [15]. For the linear benzenoid chain $B(h)$, we have

$$\gamma_k(B(h)) = \begin{cases} [\frac{h+1}{k}] & \text{if } k \neq 2, \\ [\frac{h+2}{k}] & \text{if } k = 2. \end{cases}$$
We refer to Scheinerman and Ullman [14] for fractionalization techniques of various graph parameters. Hattingh et al. [8] introduced the distance $k$-dominating function and proved that the problem of computing the upper distance fractional domination number is NP-complete. In this paper we present further results on fractional distance $k$-domination.

2. Distance $k$-dominating Function

Hattingh et al. [8] introduced the following concept of fractional distance $k$-domination.

**Definition 2.1.** A function $g : V \rightarrow [0, 1]$ is called a distance $k$-dominating function or simply a $k$-dominating function ($kDF$) of a graph $G = (V, E)$, if for every $v \in V$, $g(N_k[v]) = \sum_{u \in N_k[v]} g(u) \geq 1$. A $k$-dominating function ($kDF$) $g$ of a graph $G$ is called a minimal $k$-dominating function ($MkDF$) if $f$ is not a $k$-dominating function of $G$ for all functions $f : V \rightarrow [0, 1]$ with $f < g$. The fractional $k$-domination number $\gamma_{kf}(G)$ and the upper fractional $k$-domination number $\Gamma_{kf}(G)$ are defined as follows:

$$\gamma_{kf}(G) = \min \{|g| : g \text{ is an } MkDF \text{ of } G\},$$

$$\Gamma_{kf}(G) = \max \{|g| : g \text{ is an } MkDF \text{ of } G\}.$$

We observe that if $k \geq rad(G)$, then $\Delta(G^k) = n - 1$ and $\gamma_{kf}(G) = 1$. Hence throughout this paper, we assume that $k < rad(G)$.

**Lemma 2.2** [8]. Let $f$ be a $k$-dominating function of a graph $G = (V, E)$. Then $f$ is minimal $k$-dominating if and only if whenever $f(v) > 0$ there exists some $u \in N_k[v]$ such that $f(N_k[u]) = 1$.

**Remark 2.3.** The characteristic function of a $\gamma_k$-set and that of a $\Gamma_k$-set of a graph $G$ are $MkDF$s of $G$. Hence it follows that $1 \leq \gamma_{kf}(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \Gamma_{kf}(G)$.

**Definition 2.4.** A function $g : V \rightarrow [0, 1]$ is called a distance $k$-packing function or simply a $k$-packing function of a graph $G = (V, E)$, if for every $v \in V$, $g(N_k[v]) \leq 1$. A $k$-packing function $g$ of a graph $G$ is maximal if $f$ is not a $k$-packing function of $G$ for all functions $f : V \rightarrow [0, 1]$ with $f < g$. The fractional $k$-packing number $p_{kf}(G)$ and the upper fractional $k$-packing number $P_{kf}(G)$ are defined as follows:

$$p_{kf}(G) = \min \{|g| : g \text{ is a maximal } k\text{-packing function of } G\},$$

$$P_{kf}(G) = \max \{|g| : g \text{ is a maximal } k\text{-packing function of } G\}.$$

**Observation 2.5.** The fractional $k$-domination number $\gamma_{kf}(G)$ is the optimal solution of the following linear programming problem (LPP).
Lemma 2.6. For any graph $G$ defined on function $h$ with $V(G)$ the graph $H$, we observe that the convex combination of $G$ with $P$ for all $G$ is a minimal 2-dominating function of $G$. Clearly $\gamma_k(G)$ is a minimal $\gamma_k(G)$ of a graph $G$. Let $\gamma_k(G)$ be a graph and let $P_k = \{u \in V(G) : f(u) > 0\}$ and $B_f = \{u \in V(G) : f(N_k[u]) = 1\}$. Then $f$ is an $M_kDF$ of $G$ if and only if $B_f \rightarrow_k P_f$.

Observation 2.8. If $f$ and $g$ are $kDFs$ of a graph $G = (V, E)$ and $\gamma_f(G)$, then the convex combination of $f$ and $g$ defined by $h_\lambda(v) = \lambda f(v) + (1 - \lambda)g(v)$ for all $v \in V$ is a $kDF$ of $G$. However, the convex combination of two $M_kDFs$ of a graph $G$ need not be minimal, as shown in the following example.

Consider the cycle $G = C_7 = (u_1u_2 \ldots u_7u_1)$ with $k = 2$. The function $f : V(G) \rightarrow [0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \{u_1, u_5\}, \\ 0 & \text{otherwise}, \end{cases}$$

is a minimal 2-dominating function of $G$ with $P_f = \{u_1, u_5\}$, $B_f = \{u_1, u_2, u_4, u_5\}$. Also, the function $g : V(G) \rightarrow [0, 1]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \in \{u_3, u_6\}, \\ 0 & \text{otherwise}, \end{cases}$$

Minimize $z = \sum_{i=1}^{n} f(v_i)$, subject to

$$\sum_{u \in N_k[v]} f(u) \geq 1 \text{ and } 0 \leq f(v) \leq 1 \text{ for all } v \in V.$$
is a minimal 2-dominating function of $G$ with $\mathcal{P}_g = \{u_3, u_6\}$, $\mathcal{B}_g = \{u_2, u_3, u_6, u_7\}$.

Let $h = \frac{f}{2} + \frac{g}{2}$. Then $h(u_1) = h(u_2) = h(u_5) = h(u_6) = \frac{f}{2}$, $h(u_2) = h(u_4) = h(u_7) = 0$, $h(N_2[u_1]) = \frac{f}{2}$ for $i \neq 2$ and $h(N_2[u_2]) = 1$. Hence $\mathcal{P}_h = \{u_1, u_3, u_5, u_6\}$ and $\mathcal{B}_h = \{u_2\}$. Since $u_5, u_6 \notin N_2[u_2]$ we have $\mathcal{B}_h$ does not 2-dominate $\mathcal{P}_h$ and hence the kDF $h$ is not minimal.

**Observation 2.9.** If $f$ and $g$ are MkDFs of $G$ and $0 < \lambda < 1$, then $h_\lambda = \lambda f + (1 - \lambda)g$ is an MkDF of $G$ if and only if $\mathcal{B}_f \cap \mathcal{B}_g \rightarrow_k \mathcal{P}_f \cup \mathcal{P}_g$.

**Observation 2.10.** For the cycle $C_n$, the graph $G = C_n^k$ is $2k$-regular and hence it follows from Theorem 1.4 that $\gamma_{k_f}(C_n) = \frac{n}{2k+1}$.

We now proceed to determine the fractional $k$-domination number of several families of graphs.

**Proposition 2.11.** For the hypercube $Q_n$, $\gamma_{k_f}(Q_n) = \frac{2^n}{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}}$.

**Proof.** For any two vertices $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $Q_n$, $d(x, y) \leq k$ if and only if $x$ and $y$ differ in at most $k$ coordinates and hence $Q_n^k$ is $r$-regular where $r = \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{k}$. Hence by Theorem 1.4, we have $\gamma_{k_f}(Q_n) = \frac{2^n}{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}}$. $\square$

**Proposition 2.12.** For the graph $G = P_2 \square C_n$, we have

$$\gamma_{k_f}(G) = \begin{cases} \frac{8}{7} & \text{if } n = 4 \text{ and } k = 2, \\ \frac{n}{2k} & \text{if } n \geq 5. \end{cases}$$

**Proof.** If $n = 4$ and $k = 2$, then $G^2$ is a 6-regular graph and hence $\gamma_{2_f}(G) = \frac{8}{7}$. If $n \geq 5$, $G^k$ is a $(4k - 1)$-regular graph and hence $\gamma_{k_f}(G) = \frac{2^n}{4k - 1 + 1} = \frac{n}{2k}$. $\square$

**Theorem 2.13.** Let $G = C_n \circ K_1$. Then $\gamma_{k_f}(G) = \frac{n}{2k-1}$.

**Proof.** Let $C_n = (v_1v_2 \ldots v_nv_1)$. Let $u_i$ be the pendant vertex adjacent to $v_i$.

Clearly, $|N_k[u_i] \cap V(C_n)| = 2k - 1$ and $N_k[u_i] \subset N_k[v_i], 1 \leq i \leq n$. Hence the function $g : V(G) \rightarrow [0,1]$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x = u_i, \\ \frac{1}{2k-1} & \text{if } x = v_i \end{cases}$$

is a minimal $k$-dominating function of $G$ with $|g| = \frac{n}{2k-1}$. Also we have $|N_k[v_i] \cap \{u_j : 1 \leq j \leq n\}| = 2k - 1$, $1 \leq i \leq n$. Hence the function $h : V(G) \rightarrow [0,1]$ defined by

$$h(x) = \begin{cases} \frac{1}{2k-1} & \text{if } x = u_i, \\ 0 & \text{if } x = v_i \end{cases}$$

is a maximal $k$-packing function of $G$ with $|h| = \frac{n}{2k-1}$. Hence by Observation 2.5, we have $\gamma_{k_f}(G) = \frac{n}{2k-1}$. $\square$
Theorem 2.14. For the grid $G = P_2 \Box P_n$, we have

$$\gamma_{kf}(G) = \begin{cases} \frac{n(n+2k)}{2k(n+k)} & \text{if } n \equiv 0 \pmod{2k}, \\ \lfloor \frac{n}{2k} \rfloor & \text{otherwise.} \end{cases}$$

Proof. Let $P_2 = (u_0, u_1)$ and $P_n = (v_0, v_1, \ldots, v_{n-1})$, so that $V(G) = \{(u_i, v_j) : i = 0, 1, 0 \leq j \leq n-1\}$.

Case 1. $n \equiv 0 \pmod{2k}$. Let $n = 2kp$, $p > 1$. Define $f : V(G) \to [0, 1]$ by

$$f((u_i, v_j)) = \begin{cases} \left(\frac{1}{2p+1}\right)(p - \left\lfloor \frac{j}{2k} \right\rfloor) & \text{if } j \equiv (k-1) \pmod{2k}, \\ \left(\frac{1}{2p+1}\right)(\left\lfloor \frac{j}{2k} \right\rfloor + 1) & \text{if } j \equiv k \pmod{2k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ is a $k$-dominating function of $G$. Also, since $f((u_0, v_j)) = f((u_1, v_j))$ for all $j$, we have $|f| = 2\sum_{j=0}^{n-1} f((u_0, v_j)) = \frac{2}{2p+1}[(p+(p-1)+\cdots+3+2+1)+(1+2+3+\cdots+p)] = \frac{n(n+2k)}{2k(n+k)}$. Now consider the function $h : V(G) \to [0, 1]$ defined by

$$h((u_i, v_j)) = \begin{cases} \left(\frac{1}{2p+1}\right)(p - \left\lfloor \frac{j}{2k} \right\rfloor) & \text{if } j \equiv 0 \pmod{2k}, \\ \left(\frac{1}{2p+1}\right)(\left\lfloor \frac{j}{2k} \right\rfloor + 1) & \text{if } j \equiv (2k-1) \pmod{2k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h$ is a $k$-packing function of $G$ with $|h| = \frac{2p(p+1)}{2p+1} = \frac{n(n+2k)}{2k(n+k)}$. Hence $\gamma_{kf}(G) = \frac{n(n+2k)}{2k(n+k)}$.

Case 2. $n \not\equiv 0 \pmod{2k}$. Let $n = 2kq + r$, $1 \leq r \leq 2k-1$. Let $S = S_1 \cup S_2$ and

$$S_1 = \begin{cases} \{(u_0, v_j) : j \equiv 0 \pmod{4k}\} & \text{if } 1 \leq r \leq k, \\ \{(u_0, v_j) : j \equiv (k-1) \pmod{4k}\} & \text{if } k+1 \leq r \leq 2k-1. \end{cases}$$

$$S_2 = \begin{cases} \{(u_1, v_j) : j \equiv 2k \pmod{4k}\} & \text{if } 1 \leq r \leq k, \\ \{(u_1, v_j) : j \equiv (3k-1) \pmod{4k}\} & \text{if } k+1 \leq r \leq 2k-1. \end{cases}$$

Let $f$ be the characteristic function of $S$. Since $d(x, y) \geq 2k+1$ for all $x, y \in S$, it follows that $f(N_k[u]) = 1$ for all $u \in V(G)$. Thus $f$ is both a minimal $k$-dominating function and a maximal $k$-packing function of $G$ and hence $\gamma_{kf}(G) = |f| = |S| = \left\lfloor \frac{n}{2k} \right\rfloor$.

A special case of the above theorem gives the following result of Hare [7].

Corollary 2.15. For the grid graph $G = P_2 \Box P_n$, we have

$$\gamma_f(G) = \begin{cases} \frac{n(n+2)}{4(n+1)} & \text{if } n \text{ is even,} \\ \lfloor \frac{n}{2} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$
3. Graphs with $\gamma_{ kf}(G) = \gamma_k(G)$

In this section we obtain several families of graphs for which the fractional $k$-domination number and the $k$-domination number are equal.

Lemma 3.1. If a graph $G$ has an efficient $k$-dominating set, then $\gamma_{ kf}(G) = \gamma_k(G)$.

Proof. Let $D$ be an efficient $k$-dominating set of $G$. Then $|N_k[u] \cap D| = 1$ for all $u \in V(G)$. Hence the characteristic function of $D$ is both a minimal $k$-dominating function and a maximal $k$-packing function of $G$ and so $\gamma_{ kf}(G) = \gamma_k(G)$.

Lemma 3.2. For any graph $G$, $\gamma_{ kf}(G) = 1$ if and only if $\gamma_k(G) = 1$.

Proof. Suppose $\gamma_k(G) = 1$. Since $\gamma_{ kf}(G) \leq \gamma_k(G)$, it follows that $\gamma_{ kf}(G) = 1$. Conversely, let $\gamma_{ kf}(G) = 1$. Then $\gamma_{ f}(G^k) = 1$ and hence $\gamma(G^k) = 1$. Since $\gamma(G^k) = \gamma_k(G)$ the result follows.

Lemma 3.3. For any graph $G$, $p_{ kf}(G) \leq \rho_{ kf}(G) \leq P_{ kf}(G)$.

Proof. Let $u \in V(G)$. Since $N_k[u] = N_{G^k}[u]$, we have $p_{ kf}(G) = p_{ f}(G^k)$, $P_{ kf}(G) = P_{ f}(G^k)$ and $\rho_{ kf}(G) = \rho_2(G^k)$.

Hence the result follows from Theorem 1.1.

Corollary 3.4. For any graph $G$, $1 \leq p_{ kf}(G) \leq \rho_{ kf}(G) \leq P_{ kf}(G) = \gamma_{ kf}(G) \leq \gamma_k(G) \leq \Gamma_k(G) \leq \Gamma_{ kf}(G)$.

Corollary 3.5. If $G$ is any graph with $\rho_{kf}(G) = \gamma_k(G)$, then $\gamma_{ kf}(G) = \gamma_k(G)$.

Corollary 3.6. If $G$ is a block graph, then $\gamma_{ kf}(G) = \gamma_k(G)$.

Proof. It follows from Theorem 1.5 that $\rho_{kf}(G) = \gamma_k(G)$ and hence the result follows.

Corollary 3.7. For any tree $T$, we have $\gamma_{ kf}(T) = \gamma_k(T)$.

Theorem 3.8. For the graph $G = P^t_{ k+1} \Box P_n$ where $n \equiv 1 \pmod{(k+1)}$, $k \geq 1$, we have $\gamma_{ kf}(G) = \gamma_k(G) = \lfloor \frac{n}{k+1} \rfloor$.

Proof. Let $n = (k+1)q + 1$, $q \geq 1$. Clearly $|V(G)| = n(k + 1) = (k + 1)^2 + (k + 1)$. Let $P^t_{ k+1} = (u_0, u_1, u_2, \ldots, u_k)$ and $P_n = (v_0, v_1, \ldots, v_n)$ so that $V(G) = \{(u_i, v_j) : 0 \leq i \leq k, 0 \leq j \leq n - 1\}$.

Now let $S_1 = \{(u_0, v_i) : i \equiv 0 \pmod{(2(k+1))}\}$, $S_2 = \{(u_k, v_i) : i \equiv (k+1) \pmod{(2(k+1))}\}$ and $S = S_1 \cup S_2$. Clearly, $d(x, y) = (2k + 1)r$, $r \geq 1$, for all $x, y \in S$ and $|S| = \left\lfloor \frac{n}{k+1} \right\rfloor = q + 1$. Also, $(u_0, v_0)$ and exactly one of
the vertices \((u_0, v_{n-1})\) or \((u_k, v_{n-1})\) are in \(S\) and each of these two vertices \(k\)-dominates \(\frac{(k+1)(k+2)}{2}\) vertices of \(G\). Also, if \(u \in N_k[x] \cap N_k[y]\), where \(x, y \in S\), then \(d(u, x) \leq k\), \(d(u, y) \leq k\) and so \(d(x, y) \leq d(x, u) + d(y, u) \leq 2k\), which is a contradiction. Thus \(N_k[x] \cap N_k[y] = \emptyset\) for all \(x, y \in S\). Each of the remaining vertices of \(S\) \(k\)-dominates \((k+1)^2\) vertices of \(G\). Further, \(|V(G)| - (k+1)(k+2)\) is a multiple of \((k+1)^2\) and hence it follows that \(S\) is an efficient \(k\)-dominating set of \(G\). Hence, by Lemma 3.1, we have \(\gamma_{kf}(G) = \gamma_k(G) = |S| = \left\lceil \frac{n}{k+1} \right\rceil\).

**Theorem 3.9.** For the graph \(G = P_3 \square P_n\), we have \(\gamma_{2f}(G) = \gamma_2(G) = \left\lceil \frac{n}{3} \right\rceil\).

**Proof.** If \(n \equiv 1 \pmod{3}\), then the result follows from Theorem 3.8. Suppose \(n \equiv 0 \pmod{3}\) or \(2 \pmod{3}\). Let \(n = 3q\), \(q \geq 1\) or \(n = 3q + 2\), \(q \geq 0\). Let \(P_3 = (u_0, u_1, u_2)\) and \(P_n = (v_0, v_1, \ldots, v_{n-1})\) so that \(V(G) = \{(u_i, v_j) : 0 \leq i \leq 2\}, 0 \leq j \leq n-1\)\). Now \(D = \{(u_i, v_j) : j \equiv 1 \pmod{3}\}\) is a \(\gamma_2\)-set of \(G\) with \(|D| = \left\lceil \frac{n}{3} \right\rceil\) and hence \(\gamma_2(G) = \left\lceil \frac{n}{3} \right\rceil\). Further \(f = \chi_D\) is a 2-pack of \(G\) with \(|f| = \left\lceil \frac{n}{3} \right\rceil\). Let \(S_1 = \{(u_0, v_j) : j \equiv 0 \pmod{6}\}\), \(S_2 = \{(u_2, v_j) : j \equiv 3 \pmod{6}\}\), \(S = S_1 \cup S_2\). Then \(g = \chi_S\) is a 2-pack of \(G\) with \(|g| = \left\lceil \frac{n}{3} \right\rceil\). Hence \(\gamma_{2f}(G) = \left\lceil \frac{n}{3} \right\rceil\).

**Observation 3.10.** The graph \(G = P_3 \square P_3\) does not have an efficient 2-dominating set. In fact the set \(S = \{(u_0, v_0), (u_2, v_3)\}\) efficiently 2-dominates 14 vertices of \(G\) and the vertex \((u_0, v_4)\) is not 2-dominated by \(S\). Further if \(S\) is any 2-dominating set of \(G\) with \(|S| = \gamma_2(G) = 2\), then at least one vertex of \(G\) is 2-dominated by both vertices of \(S\). This shows that the converse of Lemma 3.1 is not true.

**Theorem 3.11.** For the linear benzenoid chain \(G = B(h)\), we have

\[
\gamma_{kf}(G) = \gamma_k(G) = \left\lceil \frac{h+1}{2} \right\rceil + 1 \quad \text{if} \quad k = 2 \quad \text{and} \quad h \equiv 0 \pmod{2},
\]

\[
\left\lceil \frac{h}{2} \right\rceil \quad \text{if} \quad k \geq 3 \quad \text{and} \quad h \equiv \left\lceil \frac{h}{2} \right\rceil \pmod{k}.
\]

**Proof.** Since \(G = B(h)\) is a subgraph of \(P_2 \square P_{2h+1}\), we take \(V(G) = \{(u_i, v_j) : i = 0, 1, 0 \leq j \leq 2h\}\), where \(P_2 = (u_0, u_1)\) and \(P_{2h+1} = (v_0, v_1, \ldots, v_{2h})\). Clearly, \(|V(G)| = 4h + 2\). Any vertex \(u \in V(G)\) \(k\)-dominates at most 4\(k\) vertices of \(G\) and hence \(\gamma_k(G) \geq \left\lceil \frac{4h+2}{4k} \right\rceil\).

Case 1. \(k = 2\) and \(h \equiv 0 \pmod{2}\). In this case we have \(\gamma_2(G) \geq \left\lceil \frac{4h+2}{8} \right\rceil = \frac{h}{2} + 1\). Now let \(S_1 = \{(u_0, v_j) : j \equiv 0 \pmod{8}\}\), \(S_2 = \{(u_1, v_j) : j \equiv 4 \pmod{8}\}\) and \(S = S_1 \cup S_2\). Clearly, for any \(x, y \in S\), \(d(x, y) \geq 5\) and hence \(N_2[x] \cap N_2[y] = \emptyset\). Also \(|S| = \left\lceil \frac{2h+1}{4} \right\rceil = \frac{h}{2} + 1\). Now \((u_0, v_0)\) and exactly one of the vertices \((u_0, v_{2h})\) or \((u_1, v_{2h})\) is in \(S\) and each of these two vertices \(2\)-dominates exactly 5 vertices of \(G\). Each of the remaining vertices of \(S\) \(2\)-dominates 8 vertices of \(G\). Further \(|V(G)| - 10 = 4h - 8 = 8\left\lceil \frac{h}{2} \right\rceil - 1\), which is a multiple of 8 and hence it follows that \(S\) is an efficient 2-dominating set of \(G\). Hence \(\gamma_{2f}(G) = \gamma_2(G) = |S| = \frac{h}{2} + 1\).
Case 2. \( k \geq 3 \) and \( h \equiv \left\lfloor \frac{k}{2} \right\rfloor \pmod{k} \). Let \( h = kq + \left\lfloor \frac{k}{2} \right\rfloor, q \geq 1 \). In this case we have \( \gamma_k(G) \geq \left\lceil \frac{4h}{k} + 2 \right\rceil = \left\lceil \frac{h}{k} \right\rceil \). Now let \( S_1 = \{(u_0, v_j) : j \equiv (k-1) \pmod{4k}\} \), \( S_2 = \{(u_1, v_j) : j \equiv (3k-1) \pmod{4k}\} \) and \( S = S_1 \cup S_2 \). Clearly, \( d(x, y) = (2k + 1)r, r \geq 1 \) for all \( x, y \in S \), hence \( N_k[x] \cap N_k[y] = \emptyset \). Also \( |S| = \left\lceil \frac{2h-(k-1)}{2k} \right\rceil = \left\lceil \frac{h}{k} \right\rceil \).

Now, when \( k \) is odd, exactly one of the vertices \( (u_0, v_{2h}) \) or \( (u_1, v_{2h}) \) is in \( S \) and it \( k \)-dominates \( 2k+1 \) vertices. When \( k \) is even, exactly one of the vertices \( (u_0, v_{2h-1}) \) or \( (u_1, v_{2h-1}) \) are in \( S \) and it \( k \)-dominates \( 2k+3 \) vertices. The vertex \( (u_0, v_{k-1}) \) \( k \)-dominates \( 4k-1 \) vertices. In both cases the number of vertices of \( G \) which are not \( k \)-dominated by these two vertices is a multiple of \( 4k \) and each of the remaining vertices of \( S \) \( k \)-dominates \( 4k \) vertices of \( G \). Hence it follows that \( S \) is an efficient \( k \)-dominating set of \( G \) so that \( \gamma_kf(G) = \gamma_k(G) = |S| = \left\lceil \frac{h}{k} \right\rceil \).

Conclusion. In this paper we have determined the fractional \( k \)-domination number of several families of graphs. We have also obtained several families of graphs for which \( \gamma_kf(G) = \gamma_k(G) \). The study of the fractional version of distance \( k \)-irredundance and distance \( k \)-independence remains open. Slater has mentioned several efficiency parameters such as irredundance and influence in Chapter 1 of [10]. One can investigate these parameters for fractional distance domination. The following are some interesting problems for further investigation.

1. Characterize the class of graphs \( G \) for which \( \gamma_kf(G) = \frac{n}{k+1} \).
2. Characterize the class of graphs \( G \) with \( \gamma_kf(G) = \gamma_k(G) \).
3. Determine \( \gamma_kf(P_r \square P_s) \) for \( r, s \geq 4 \).

Acknowledgement

We are thankful to the National Board for Higher Mathematics, Mumbai, for its support through the project 48/5/2008/R&D-II/561, awarded to the first author. The second author is thankful to the UGC, New Delhi for the award of FIP teacher fellowship during the XI\textsuperscript{th} plan period. We are also thankful to the referees for their helpful suggestions.

References


Received 22 December 2010
Revised 12 August 2011
Accepted 16 August 2011