

## ITERATED NEIGHBORHOOD GRAPHS

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### Abstract

The *neighborhood graph*  $N(G)$  of a simple undirected graph  $G = (V, E)$  is the graph  $(V, E_N)$  where  $E_N = \{\{a, b\} \mid a \neq b, \{x, a\} \in E \text{ and } \{x, b\} \in E \text{ for some } x \in V\}$ . It is well-known that the neighborhood graph  $N(G)$  is connected if and only if the graph  $G$  is connected and non-bipartite.

We present some results concerning the *k-iterated neighborhood graph*  $N^k(G) := N(N(\dots N(G)))$  of  $G$ . In particular we investigate conditions for  $G$  and  $k$  such that  $N^k(G)$  becomes a complete graph.

**Keywords:** neighborhood graph, 2-step graph, neighborhood completeness number.

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### 1. INTRODUCTION AND DEFINITIONS

All graphs considered here are undirected and finite without loops and multiple edges.

**Definition.** The *neighborhood graph*  $N(G)$  of a graph  $G = (V, E)$  is the graph  $(V, E_N)$  where  $E_N = \{\{a, b\} \mid a \neq b, \{x, a\} \in E \text{ and } \{x, b\} \in E \text{ for some } x \in V\}$ .

Several aspects of neighborhood graphs were investigated in the last thirty years (cf. [1–3, 5, 6, 9–14, 16]). Some of these papers use the notation *2-step graph* or *competition graph* instead of neighborhood graph. As the latter name indicates, the neighborhood graph  $N(G)$  of an undirected graph  $G$  is closely related to the competition graph  $C(D)$  of a digraph  $D$ . Surveys of competition graphs can be found in Kim [7], Lundgren [8] and Roberts [15].

With  $d_G(x, y)$  and  $d(x : G)$  we denote the distance of  $x, y \in V$  in  $G$  and the degree of  $x \in V$  in  $G$ , respectively. Further we use the neighborhood sets  $N_G(x) = \{z \in V \mid \{x, z\} \in E\}$  and  $N_G(x, y) = N_G(x) \cap N_G(y)$ . Definitions not explicitly given here can be found in [4].

First, we summarize some simple results on neighborhood graphs from the literature mentioned above.

**Proposition 1.** *Let  $G = (V, E)$  be a connected graph and  $N(G) = (V, E_N)$  its neighborhood graph. Then the following hold:*

- (a)  $N(G)$  has at most two connected components.
- (b)  $N(G)$  is connected if and only if  $G$  is non-bipartite.
- (c) If  $G$  is 2-connected and non-bipartite, then  $N(G)$  is also 2-connected and non-bipartite.
- (d) For each  $n \geq 5$  and  $p \geq 2$  with  $2p \leq n$  there is a  $p$ -connected, non-bipartite graph  $G$  with  $n$  vertices, such that the neighborhood graph  $N(G)$  has connectivity 2.
- (e) For the path  $P_n$  with  $n$  vertices:  $N(P_n) \cong P_{\lfloor \frac{n}{2} \rfloor} \cup P_{\lfloor \frac{n}{2} \rfloor}$ .
- (f) For the cycle  $C_n$  with  $n$  vertices:  $N(C_{2k+1}) \cong C_{2k+1}$ ,  $N(C_{2k}) \cong C_k \cup C_k$  (for  $k \geq 3$ ) and  $N(C_4) \cong P_2 \cup P_2$ .
- (g) For the complete graph  $K_n$  with  $n$  vertices:  $N(K_n) \cong K_n$ ,  $n \neq 2$  (note that  $G = C_{2n+1}$  and  $G = K_n$ ,  $n \neq 2$ , are the only connected graphs with  $N(G) \cong G$  (cf. Brigham and Dutton [3])).
- (h) For the complete bipartite graph  $K_{m,n}$  with  $m + n$  vertices:  $N(K_{m,n}) \cong K_m \cup K_n$ .
- (i) For the wheel  $W_n$  with  $n + 1$  vertices:  $N(W_n) \cong K_{n+1}$ .

Properties (e)–(i) lead to the question what happens if the construction of the neighborhood graph is iterated:

**Definition.** For a positive integer  $k \in \mathbb{N}^+$ , the  $k$ -iterated neighborhood graph  $N^k(G)$  of a graph  $G$  is the neighborhood graph of  $N^{k-1}(G)$ , where  $N^0(G) := G$ .

In this paper we consider the following problems:

**Problem 1.** What is the structure of  $N^k(G)$ , for large  $k$ ?

**Problem 2.** Under which conditions  $N^k(G) \cong K_n$ , for sufficiently large  $k$ ?

**Problem 3.** If  $G$  fulfils the conditions mentioned in Problem 2, what is the minimum  $k$  such that  $N^k(G) \cong K_n$ ?

The answers of Problems 1 and 2 follow from the results of Exoo and Harary [5]; we discuss these problems in the (short) Section 2. Section 3 contains the main results of this paper. There we determine the minimum  $k$  mentioned in Problem 3 for a certain class of graphs and give upper bounds for  $k$  being better than those from [5].

## 2. THE STRUCTURE OF $N^k(G)$ FOR LARGE $k$

Summarizing the results of Lemma 1–3 of [5] we obtain immediately the following theorem solving Problem 2. Here we present another (short) proof using arguments which prepare several ideas used in Section 3.

**Theorem 2.** *Let  $G = (V, E)$  be a graph with  $n > 1$  vertices. Then there exists  $k \in \mathbb{N}$  with  $N^k(G) \cong K_n$  if and only if  $G$  is connected, non-bipartite and  $G \not\cong C_{2p+1}$  (for  $p > 1$ ).*

**Proof.** Let  $n = |V| > 1$ . If  $G$  is an odd cycle  $C_{2p+1}$ ,  $p > 1$ , or bipartite or not connected then, by Proposition 1 (b) and (f),  $N^k(G) \not\cong K_n$  for all  $k \in \mathbb{N}$ . Therefore the three conditions (connected, non-bipartite and  $G \not\cong C_{2p+1}$ ,  $p > 1$ ) are necessary for the existence of  $k \in \mathbb{N}$  with  $N^k(G) \cong K_n$ .

Now let  $G$  fulfil these conditions and  $v \in V$  be a vertex with the degree  $d(v : G) = p \geq 3$ . Then the neighborhood  $N_G(v)$  induces a  $p$ -clique  $K_p$  in the neighborhood graph  $N^1(G)$ .

We prove that for  $k, p \in \mathbb{N}^+$  with  $3 \leq p < n$  the existence of a  $p$ -clique  $K_p$  in  $N^k(G)$  implies the existence of a  $(p+1)$ -clique  $K_{p+1}$  in  $N^{k+2}(G)$ .

By Proposition 1(b),  $N^k(G)$  is connected. Since  $p < n$ , there is a vertex  $u$  in the  $p$ -clique  $K_p$  having a neighbor  $u' \in V(G) \setminus V(K_p)$  in  $N^k(G)$ . Consequently, in  $N^{k+1}(G)$  — in addition to  $K_p$  — the set  $(V(K_p) \setminus \{u\}) \cup \{u'\}$  induces a second  $p$ -clique. Therefore, in  $N^{k+2}(G)$  also the vertices  $u$  and  $u'$  are adjacent (in  $N^{k+1}(G)$  they have common neighbors in  $V(K_p) \setminus \{u\}$  and  $V(K_p) \cup \{u'\}$  induces a  $(p+1)$ -clique (cf. Figure 1)). ■

Proposition 1 and Theorem 2 imply the following corollary, which solves Problem 1 (the result is established in [5] and also mentioned in [3]).

**Corollary 3.** *For an arbitrary graph  $G = (V, E)$  and sufficiently large  $k \in \mathbb{N}$ ,  $N^k(G)$  consists of odd cycles and (possibly trivial) complete graphs.*

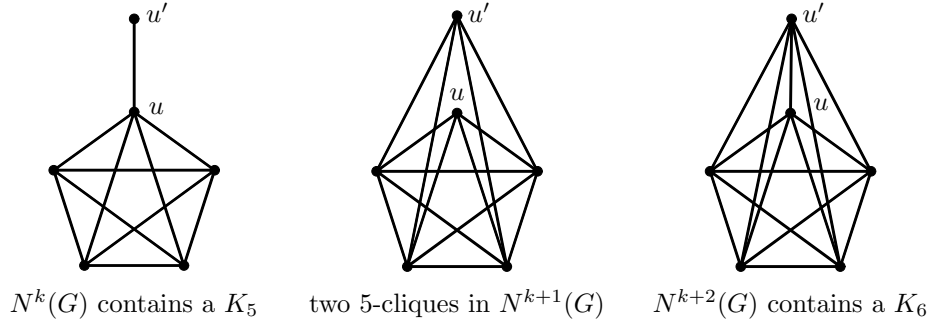


Figure 1. An example with  $p = 5$ .

### 3. THE NEIGHBORHOOD COMPLETENESS NUMBER

Now we turn to Problem 3. To determine the minimum  $k$  such that  $N^k(G)$  is complete could be interesting in connection with graph algorithms; this motivates the definition:

**Definition.** For  $G = (V, E)$  connected, non-bipartite and  $G \not\cong C_{2p+1}$  (for  $p > 1$ ), we define the *neighborhood completeness number* of  $G$  by

$$cn(G) := \min\{k \in \mathbb{N} \mid N^k(G) \cong K_n\}.$$

The only result concerning the neighborhood completeness number can be found in [5]. Let  $G$  be a connected graph with  $n$  vertices which is neither bipartite nor an odd cycle. If  $C$  is a cycle of length  $2k+1$  in  $G$ ,  $d$  is the maximum least distance from a vertex not on  $C$  to a vertex on  $C$  and  $r := \log_2 d$ , then  $N^{r+2k+1}(G) = K_n$ . Hence

$$(EH) \quad cn(G) \leq r + 2k + 1.$$

The sharpness of this bound will be discussed at the end of Subsection 3.2. Before, in Subsection 3.1, we determine the neighborhood completeness number for a special class of graphs. This result is used in the following to improve the bound (EH) for  $cn(G)$  for arbitrary non-bipartite graphs  $G$ .

#### 3.1. A special class of graphs: $l$ -cliques with a tail

**Definition.** For  $l \geq 3$  and  $s \geq 1$ , let  $K_l^s$  be the graph  $(V, E)$  defined by

$$V = \{1, 2, \dots, l, l+1, \dots, l+s\},$$

$$E = \{\{i, j\} \mid 1 \leq i < j \leq l\} \cup \{\{l, l+1\}, \{l+1, l+2\}, \dots, \{l+s-1, l+s\}\}.$$

Hence,  $K_l^s$  consists of a complete graph  $K_l$  with  $l$  vertices and a "tail" of length  $s$  (cf. Figure 2). We start with a lemma describing several structural properties of  $N^k(K_l^s)$ , for  $l \geq 3$ .

We denote by  $\langle v_1, v_2, \dots, v_t \rangle = \langle v_1, v_2, \dots, v_t \rangle_{N^k(G)}$  the subgraph of  $N^k(G)$  induced by the vertices  $v_1, v_2, \dots, v_t \in V(N^k(G))$ .

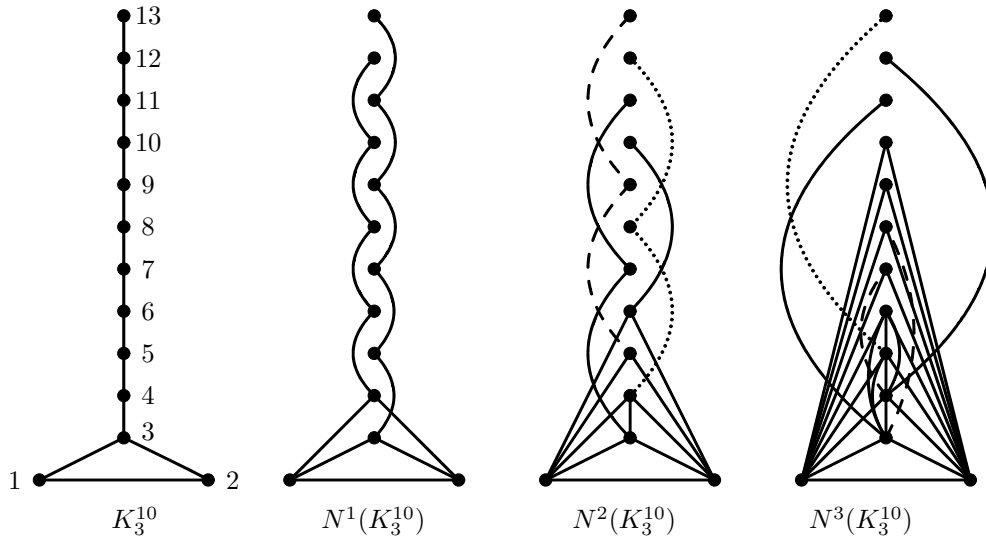


Figure 2. An example to Lemma 4.

**Lemma 4.** *Let  $k, l, s \in \mathbb{N}$  with  $l \geq 3$  and  $s \geq 1$ . Then the following hold for  $N^k(K_l^s)$ :*

- (a) *If  $2^k - 1 \leq s$ , then there are exactly  $2^k$   $l$ -cliques containing the  $(l - 1)$ -clique  $\langle 1, 2, \dots, l - 1 \rangle$ , namely  $\langle 1, 2, \dots, l - 1, l \rangle, \langle 1, 2, \dots, l - 1, l + 1 \rangle, \dots, \langle 1, 2, \dots, l - 1, l + 2^k - 1 \rangle$ .*
- (b) *If  $2^k \leq s$ , then all the edges between  $\{1, 2, \dots, l + 2^k - 1\}$  and  $\{l + 2^k, l + 2^k + 1, \dots, l + s\}$  have the form  $\{x, x + 2^k\}$ .  
These edges exist for all  $x \in \{l, l + 1, \dots, l + \min\{2^k - 1, s - 2^k\}\}$ .*
- (c) *If  $2^k - 1 \leq s$ , then  $\langle l + 2^k - 1, l + 2^k, \dots, l + s \rangle$  is the union of the vertex disjoint paths  $(y, y + 2^k, y + 2 \cdot 2^k, y + 3 \cdot 2^k, \dots)$ , where  $y \in \{l + 2^k - 1, l + 2^k, \dots, l + \min\{2^{k+1} - 2, s - 2^k\}\}$ .  
(Therefore, these paths contain only edges of the form  $\{x, x + 2^k\}$ , where  $x \in \{l + 2^k - 1, l + 2^k, \dots, l + s - 2^k\}$ .)*
- (d) *If  $k \geq 1$  and  $2^{k-1} - 1 \leq s$ , then  $\langle 1, 2, \dots, l + 2^{k-1} - 1 \rangle$  is a maximal clique.*

Before proving Lemma 4, as an example we consider  $K_3^{10}$  (cf. Figure 2).

Note that the dashed edges  $\{3, 8\}$  and  $\{4, 7\}$  in  $N^3(K_3^{10})$  (and corresponding edges in  $N^k(K_3^{10})$  ( $k > 3$ )) will be of no account in our investigations. In reference to the Lemma, these edges connect a vertex of the maximum clique of  $N^k(K_3^{10})$  (cf. (d)) with a vertex from the set  $\{2^{k-1} + l, 2^{k-1} + l + 1, \dots, 2^k + l - 1\}$ , which

is contained in one of the triangles (i.e.  $l$ -cliques with  $l = 3$ , cf. (a)), but not in the maximum clique.

Obviously, in  $N^{k+1}(K_3^{10})$  these edges “disappear” since they are included in the maximum clique of  $N^{k+1}(K_3^{10})$ .

Now we verify Lemma 4 by induction on  $k$ :

**Proof.** Let  $n := l + s$ .

$k = 0$ .

- (a) Because  $N^0(K_l^s) = K_l^s$  there is exactly  $2^0 = 1$   $l$ -clique, namely  $\langle 1, 2, \dots, l \rangle$ .
- (b) The only edge between  $\{1, 2, \dots, l\}$  and  $\{l+1, l+2, \dots, n\}$  is  $\{l, l+1\}$ .
- (c)  $\langle l, l+1, \dots, n \rangle$  is the path  $(l, l+1, \dots, n)$ .
- (d) Not applicable.

$k = 1$ .

- (a) There are  $2^1 = 2$   $l$ -cliques:  $\langle 1, 2, \dots, l-1, l \rangle$  and  $\langle 1, 2, \dots, l-1, l+1 \rangle$ .
- (b) The edges between  $\{1, 2, \dots, l+1\}$  and  $\{l+2, l+3, \dots, n\}$  are  $\{l, l+2\}$  and  $\{l+1, l+3\}$ .
- (c)  $\langle l+1, l+2, \dots, n \rangle$  is the (disjoint) union of the paths  $(l+1, l+3, l+5, \dots)$  and  $(l+2, l+4, l+6, \dots)$ .
- (d)  $\langle 1, 2, \dots, l \rangle$  is a maximum — and, therefore, also maximal — clique.

$k \geq 2$ .

**Induction hypotheses:** (a)–(d) are true for all  $k' \leq k-1$ .

For technical reasons and a better comprehension of the following, we formulate the induction hypotheses for  $k' = k-1$  in detail.

In  $N^{k-1}(K_l^s)$  it holds:

- (a') If  $2^{k-1} + l - 1 \leq n$ , then there are exactly  $2^{k-1}$   $l$ -cliques over the  $(l-1)$ -clique  $\langle 1, 2, \dots, l-1 \rangle$ , namely  $\langle 1, 2, \dots, l-1, l \rangle, \langle 1, 2, \dots, l-1, l+1 \rangle, \dots, \langle 1, 2, \dots, l-1, 2^{k-1} + l - 1 \rangle$ .
- (b') Between  $\{1, 2, \dots, 2^{k-1} + l - 1\}$  and  $\{2^{k-1} + l, 2^{k-1} + l + 1, \dots, n\}$  there are only edges of the form  $\{x, x + 2^{k-1}\}$ .  
These edges exist for all  $x \in \{l, l+1, \dots, \min\{2^{k-1} + l - 1, n - 2^{k-1}\}\}$ .
- (c')  $\langle 2^{k-1} + l - 1, 2^{k-1} + l, \dots, n \rangle_{N^{k-1}(K_l^s)}$  is the union of the vertex disjoint paths  $(y, y + 2^{k-1}, y + 2 \cdot 2^{k-1}, y + 3 \cdot 2^{k-1}, \dots)$ , where  $y \in \{2^{k-1} + l - 1, 2^{k-1} + l, \dots, \min\{2^k + l - 2, n - 2^{k-1}\}\}$ .  
(Therefore, these paths contain only edges of the form  $\{x, x + 2^{k-1}\}$ , where  $x \in \{2^{k-1} + l - 1, 2^{k-1} + l, \dots, n - 2^{k-1}\}$ .)
- (d') If  $2^{k-2} + l - 1 \leq n$ , then  $\langle 1, 2, \dots, 2^{k-2} + l - 1 \rangle_{N^{k-1}(K_l^s)}$  is a maximal clique.

**Induction steps.**

At first, we mention the following.

- (o) In  $N^k(K_l^s)$ , there exist the edges  $\{x, x + 2^k\}$  for each  $x \in \{1, 2, \dots, n - 2^k\}$ .

Verification of (o).

For  $x \geq l$ , in  $N^k(K_l^s)$  the existence of  $\{x, x + 2^k\}$  follows from the existence of the edges  $\{x, x + 2^{k-1}\}$ ,  $\{x + 2^{k-1}, (x + 2^{k-1}) + 2^{k-1} = x + 2^k\}$  in  $N^{k-1}(K_l^s)$  (cf. the induction hypotheses (b'), (c')), since, obviously,  $x$  and  $x + 2^k$  are common neighbors of  $x + 2^{k-1}$  in  $N^{k-1}(K_l^s)$ .

For  $x \in \{1, 2, \dots, l-1\}$ , additionally to (b') and (c') also (a') is needed to ensure  $\{x, x + 2^{k-1}\}, \{x + 2^{k-1}, x + 2^k\} \in E(N^{k-1}(K_l^s))$ .

Now we show (a)–(d).

(a) Let  $2^k + l - 1 \leq n$ . Since the  $2^{k-1}$   $l$ -cliques  $\langle 1, 2, \dots, l-1, l \rangle$ ,  $\langle 1, 2, \dots, l-1, l+1 \rangle$ ,  $\dots$ ,  $\langle 1, 2, \dots, l-1, 2^{k-1} + l - 1 \rangle$  from  $N^{k-1}(K_l^s)$  (cf. (a')) are complete subgraphs, they exist also in  $N^k(K_l^s)$ . Because of (a') and (o) in  $N^{k-1}(K_l^s)$  each vertex  $x \in \{l, l+1, \dots, 2^{k-1} + l - 1\}$  has at least the neighbors  $1, 2, \dots, l-1$  and  $x + 2^{k-1}$ . Hence, in  $N^k(K_l^s)$  there are the  $l$ -cliques  $\langle 1, 2, \dots, l-1, 2^{k-1} + l \rangle$ ,  $\langle 1, 2, \dots, l-1, 2^{k-1} + l + 1 \rangle$ ,  $\dots$ ,  $\langle 1, 2, \dots, l-1, 2^k + l - 1 \rangle$ . In  $N^k(K_l^s)$ , there are no other  $l$ -cliques over the  $(l-1)$ -clique  $\langle 1, 2, \dots, l-1 \rangle$ , since (a'), (b') imply that, in  $N^{k-1}(K_l^s)$ , all neighbors  $x$  of the vertices  $1, 2, \dots, l-1$  are contained in  $\{1, 2, \dots, 2^{k-1} + l - 1\}$  and, moreover, every vertex  $x \in \{1, 2, \dots, 2^{k-1} + l - 1\}$  in the set  $\{2^{k-1} + l, 2^{k-1} + l + 1, \dots, n\}$  has only the neighbor  $y = x + 2^{k-1}$ . Therefore, owing to  $y = x + 2^{k-1} \leq 2^{k-1} + 2^{k-1} + l - 1 = 2^k + l - 1$ , in  $N^k(K_l^s)$ , the  $l$ -cliques  $\langle 1, 2, \dots, l-1, l \rangle$ ,  $\langle 1, 2, \dots, l-1, l+1 \rangle$ ,  $\dots$ ,  $\langle 1, 2, \dots, l-1, 2^k + l - 1 \rangle$  include all these neighbors  $y$ , which are the only possible candidates for building  $l$ -cliques containing the vertices  $1, 2, \dots, l-1$ . This completes the proof of (a).

(b) Without loss of generality, let  $2^k + l \leq n$ , otherwise there is nothing to show. Because of (o) it suffices to show that the edges of the form  $\{x, x + 2^k\}$ , where  $x \in \{l, l+1, \dots, \min\{2^k + l - 1, n - 2^k\}\}$ , are the only edges between the sets  $\{1, 2, \dots, 2^k + l - 1\}$  and  $\{2^k + l, 2^k + l + 1, \dots, n\}$ .

In  $N^{k-1}(K_l^s)$ , between  $z \in \{1, 2, \dots, 2^{k-1} + l - 1\}$  and  $\{2^{k-1} + l, 2^{k-1} + l + 1, \dots, n\}$  there are only edges of the form  $\{z, z + 2^{k-1}\}$  (cf. (b')). This implies, for the end vertices of such edges,  $z \in \{l, l+1, \dots, 2^{k-1} + l - 1\}$  and  $z + 2^{k-1} \in \{2^{k-1} + l, 2^{k-1} + l + 1, \dots, 2^k + l - 1\}$ .

Now let  $x + 2^k \in \{2^k + l, 2^k + l + 1, \dots, n\}$  with  $x \in \{l, l+1, \dots, 2^k + l - 1\}$  and assume  $y \in \{1, 2, \dots, 2^k + l - 1\} \setminus \{x\}$  is another neighbor of  $x + 2^k$  in  $N^k(K_l^s)$ . Then, in  $N^{k-1}(K_l^s)$ , there are vertices  $z$  and  $z'$  such that  $z$  is a common neighbor of  $x$  and  $x + 2^k$ , as well as  $z'$  is a common neighbor of  $y$  and  $x + 2^k$ . Clearly,  $x + 2^k > 2^{k-1} + l - 1$  and, consequently, owing to (b') and (c') this implies  $z = x + 2^k - 2^{k-1}$  or  $z = x + 2^k + 2^{k-1}$ . Since  $z$  is also a neighbor of  $x$  in  $N^{k-1}(K_l^s)$ , the only possibility is  $z = x + 2^k - 2^{k-1} = x + 2^{k-1} \in \{2^{k-1} + l, 2^{k-1} + l + 1, \dots, n\}$ .

Analogously, we obtain  $z' = x + 2^{k-1}$ . Consequently,  $z = z' = x + 2^{k-1}$  has the three pairwise distinct neighbors  $x, y, x + 2^k$  in  $N^{k-1}(K_l^s)$ , in contradiction to  $z \geq 2^{k-1} + l$  and (b') and (c'), what excludes other neighbors than  $z - 2^{k-1}$ ,

$z + 2^{k-1}$ . Thus (b) holds.

(c) Due to (o), the existence (and, obviously, the disjointness) of the paths  $(y, y + 2^k, y + 2 \cdot 2^k, y + 3 \cdot 2^k, \dots)$  is clear, for all  $y \in \{2^k + l - 1, 2^k + l, \dots, \min\{2^{k+1} + l - 2, n - 2^k\}\}$ .

Assume, there are  $x, x' \in \{2^k + l - 1, 2^k + l, \dots, n\}$  with  $x < x', x' \neq x + 2^k$ , and  $\{x, x'\} \in E(N^k(K_l^s))$ . Then, in  $N^{k-1}(K_l^s)$ , there must be a common neighbor  $z$  of  $x$  and  $x'$ .

If  $z \leq 2^{k-1} + l - 1$ , then (because of (b')) the only edge in  $N^{k-1}(K_l^s)$  between  $z$  and vertices in  $\{2^{k-1} + l, 2^{k-1} + l + 1, \dots, n\}$  is the edge  $\{z, z + 2^{k-1}\}$ . This implies the contradiction  $x = z + 2^{k-1} = x'$ .

If  $z > 2^{k-1} + l - 1$ , then (because of (b') and (c'))  $x < x'$  induces  $x = z - 2^{k-1}$  and  $x' = z + 2^{k-1}$  and, therefore,  $x' = x + 2 \cdot 2^{k-1} = x + 2^k$  incompatible with the assumption.

(d) Let  $2^{k-1} + l - 1 \leq n$ . In  $N^{k-1}(K_l^s)$  the vertices  $2, 3, \dots, 2^{k-1} + l - 1$  are common neighbors of 1 (because of (a')). Hence,  $\langle 2, 3, 4, \dots, 2^{k-1} + l - 1 \rangle_{N^k(K_l^s)}$  is a clique. Analogously, we obtain that  $\langle 1, 3, 4, 5, \dots, 2^{k-1} + l - 1 \rangle_{N^k(K_l^s)}$  is a clique. Because, in  $N^{k-1}(K_l^s)$ , the vertex 3 is a common neighbor of the vertices 1 and 2, it follows  $\{1, 2\} \in E(N^k(K_l^s))$ , and  $\langle 1, 2, \dots, 2^{k-1} + l - 1 \rangle_{N^k(K_l^s)}$  is a clique.

Assume, the clique  $\langle 1, 2, \dots, 2^{k-1} + l - 1 \rangle_{N^k(K_l^s)}$  is not maximal.

In  $N^k(K_l^s)$ , let  $z \geq 2^{k-1} + l$  be the smallest vertex being adjacent to all vertices  $x \in \{1, 2, \dots, 2^{k-1} + l - 1\}$ .

In  $N^{k-1}(K_l^s)$ , it follows that  $z$  has to have a common neighbor with every vertex  $x \in \{1, 2, \dots, 2^{k-1} + l - 1\}$ . The induction hypotheses (b') and (c') imply that there are at most two neighbors of  $z$  in  $N^{k-1}(K_l^s)$ , namely  $z - 2^{k-1}$  and  $z + 2^{k-1}$ .

In  $N^{k-1}(K_l^s)$ , because of (b') and  $z + 2^{k-1} > (2^{k-1} + l - 1) + 2^{k-1}$ , the vertex  $z + 2^{k-1}$  has no neighbor in the set  $\{1, 2, \dots, 2^{k-1} + l - 1\}$ . Therefore,  $z - 2^{k-1}$  is adjacent to all vertices  $x \in \{1, 2, \dots, 2^{k-1} + l - 1\}$ . Since  $z - 2^{k-1}$  cannot be adjacent to itself, this implies  $z - 2^{k-1} \geq 2^{k-1} + l$ . Hence,  $z - 2^{k-1} > 2^{k-2} + l - 1$  and  $\langle 1, 2, \dots, 2^{k-2} + l - 1, z - 2^{k-1} \rangle_{N^{k-1}(K_l^s)}$  is a clique in  $N^{k-1}(K_l^s)$ . This contradicts the maximality of the clique  $\langle 1, 2, \dots, 2^{k-2} + l - 1 \rangle_{N^{k-1}(K_l^s)}$  (cf. (d')).

Therefore, the clique  $\langle 1, 2, \dots, 2^{k-1} + l - 1 \rangle_{N^k(K_l^s)}$  is maximal and the proof of (d) is complete. ■

**Theorem 5.** For  $l \geq 3$  and  $s \geq 1$ ,  $cn(K_l^s) = \lceil 1 + \log_2(s + 1) \rceil$ .

**Proof.** Let  $n = l + s$ . For  $2^{k-1} + l - 1 \leq n$ , from part (d) of Lemma 4 it follows that  $\langle 1, 2, \dots, 2^{k-1} + l - 1 \rangle_{N^k(K_l^s)}$  is a maximal clique in  $N^k(K_l^s)$ .



This implies that  $N^k(K_l^s)$  is complete if and only if  $2^{k-1} + l - 1 \geq n$ , which is equivalent to  $k - 1 \geq \log_2(n - l + 1) = \log_2(s + 1)$ , i.e.  $k \geq 1 + \log_2(s + 1)$ . Therefore,  $cn(K_l^s) = \lceil 1 + \log_2(s + 1) \rceil$ . ■

**3.2. The general case**

In this section, let  $G = (V, E)$  be connected, non-bipartite and not an odd cycle. For the first definition we suppose that  $G$  contains an  $l$ -clique ( $l \geq 3$ ).

**Definition.** Let  $K_l$  be an  $l$ -clique ( $l \geq 3$ ) in  $G = (V, E)$  and  $\mathcal{W} = \{w_1, \dots, w_q\}$  a system of paths in  $G$  such that  $V \setminus V(K_l) \subseteq V(\mathcal{W}) := \bigcup_{i=1}^q V(w_i)$  and every path  $w_i \in \mathcal{W}$  has exactly one end vertex  $v_i$  in common with  $K_l$ , for  $i \in \{1, \dots, q\}$ . The subgraph  $G_{K_l, \mathcal{W}} = K_l \cup w_1 \cup \dots \cup w_q = (V, E')$  with  $V = V(K_l) \cup V(w_1) \cup \dots \cup V(w_q)$  and  $E' = E(K_l) \cup E(w_1) \cup \dots \cup E(w_q) \subseteq E$  will be referred to as a  $K_l$ -path-covering of  $G$ . The paths  $w_1, \dots, w_q$  are called *tails*.

Note that the tails are not necessarily disjoint. Moreover, they cover all vertices of  $G - K_l$  (and, additionally, the end vertices  $v_1, \dots, v_q \in (\bigcup_{i=1}^q V(w_i)) \cap V(K_l)$ ) but not necessarily all edges of  $G - K_l$  (cf. Figure 3).

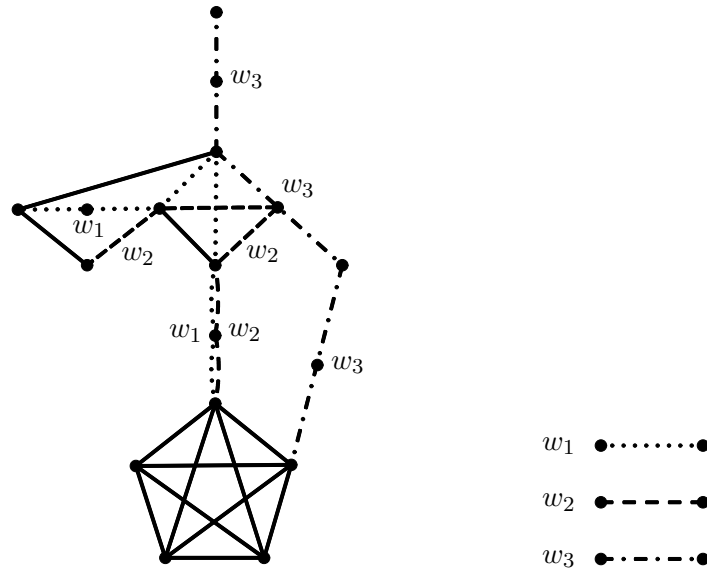


Figure 3. A  $K_5$ -path-covering  $G_{K_5, \mathcal{W}} = K_5 \cup w_1 \cup w_2 \cup w_3$ .

$K_l$ -path-coverings are suitable auxiliaries to give an upper bound for the neighborhood completeness number of arbitrary graphs. In the case of connected graphs containing an  $l$ -clique ( $l \geq 3$ ), this upper bound is the same as in the previous subsection.

Obviously, if the connected graph  $G$  contains an  $l$ -clique  $K_l$  ( $l \geq 3$ ), then there is also a  $K_l$ -path-covering  $G_{K_l, \mathcal{W}}$  in  $G$  and vice versa.

**Theorem 6.** *Let  $G_{K_l, \mathcal{W}} = K_l \cup w_1 \cup \dots \cup w_q$  be a  $K_l$ -path-covering of a graph  $G = (V, E)$ . If  $s$  is the maximum length of the tails  $w_1, \dots, w_q$ , then  $cn(G) \leq \lceil 1 + \log_2(s + 1) \rceil$ .*

**Proof.** It suffices to show that  $cn(G_{K_l, \mathcal{W}}) \leq \lceil 1 + \log_2(s + 1) \rceil$ .

So let  $u, v \in V$  be arbitrary vertices of  $G_{K_l, \mathcal{W}}$  and  $t := \lceil 1 + \log_2(s + 1) \rceil$ . Without loss of generality, let  $w_x$  and  $w_y$  be tails such that  $u \in V(K_l) \cup V(w_x)$  and  $v \in V(K_l) \cup V(w_y)$ , respectively. (Note that also the special cases  $u \in V(K_l) \setminus V(w_x)$  or  $v \in V(K_l) \setminus V(w_y)$  or  $w_x = w_y$  or  $w_x \neq w_y$  and  $V(w_x) \cap V(w_y) \neq \emptyset$  are possible.)

Since  $K_l \cup w_x \cong K_l^{r_x}$ , where  $r_x \leq s$  denotes the length of the path  $w_x$ , by Theorem 5 it follows that  $N^t(K_l \cup w_x)$  is complete. Consequently, due to Lemma 4(a), in  $N^{t-1}(K_l \cup w_x)$  the vertex  $u$  has at least  $l - 1$  neighbors in the vertex set  $V(K_l)$ . Clearly, the same holds for the vertex  $v$  in  $N^{t-1}(K_l \cup w_y)$ . Because of  $l \geq 3$ , in  $N^{t-1}(K_l \cup w_x \cup w_y)$  the vertices  $u$  and  $v$  have at least  $l - 2 \geq 1$  common neighbors (in  $V(K_l)$ ). Therefore, they are adjacent in  $N^t(G_{K_l, \mathcal{W}})$ . So  $N^t(G_{K_l, \mathcal{W}})$  is complete. ■

To obtain a class of graphs where the bound of Theorem 6 is sharp, we consider graphs  $\widehat{G}$  having a  $K_l$ -path-covering with a longest tail  $w_i$ , such that only the end vertex  $v_i \in V(K_l)$  of  $w_i$  has neighbors in  $V(\widehat{G}) \setminus V(w_i)$ ; more precisely:

**Corollary 7.** *Let  $\widehat{G}_{K_l, \mathcal{W}} = K_l \cup w_1 \cup \dots \cup w_q$  be a  $K_l$ -path-covering of a graph  $\widehat{G} = (V, E)$ . If the length of the tail  $w_1$  is equal to the maximum tail length  $s$  of  $w_1, \dots, w_q$  and all vertices of  $V(w_1) \setminus V(K_l)$  except the end vertex, which has the degree one, have the degree two in  $\widehat{G}$ , then  $cn(\widehat{G}) = \lceil 1 + \log_2(s + 1) \rceil$ .*

**Proof.** If  $w_1 = (u_1, u_2, \dots, u_{s+1})$  and  $V(K_l) \cap V(w_1) = \{u_1\}$ , then  $\widehat{G} = U \cup w_1$ , where  $U = \langle V(\widehat{G}) \setminus \{u_2, u_3, \dots, u_{s+1}\} \rangle_{\widehat{G}}$ . With  $l := |V(\widehat{G})| - s$ , the graph  $\widehat{G}$  is isomorphic to an edge-deleted subgraph of  $K_l^s$ , i.e. to a subgraph containing all  $l + s$  vertices of  $K_l^s$ . Because of  $cn(K_l^s) = \lceil 1 + \log_2(s + 1) \rceil$ ,  $cn(\widehat{G}) \geq cn(K_l^s)$  and Theorem 6 we obtain the assertion. ■

For graphs  $G$  containing an  $l$ -clique  $K_l$  ( $l \geq 3$ ), Theorem 6 gives an upper bound for the neighborhood completeness number  $cn(G)$ . Now we consider graphs without such cliques. So let  $G$  be a triangle-free graph. The basic idea is the following:

Since  $G$  is non-bipartite and is not isomorphic to an odd cycle, there must be a vertex  $v \in V(G)$  having a degree  $d := d(v : G) \geq 3$ . The neighborhood  $N_G(v)$  of  $v$  in  $G$  induces a  $d$ -clique  $K_d$  in the neighborhood graph  $N(G)$ . Let  $N(G)_{K_d, \mathcal{W}} = K_d \cup w_1 \cup \dots \cup w_q$  be a  $K_d$ -path-covering of  $N(G)$  and  $\hat{s}$  be the maximum tail length of  $N(G)_{K_d, \mathcal{W}}$ .

Then, owing to Theorem 6,

$$(*) \quad cn(G) = cn(N(G)) + 1 \leq \lceil 1 + \log_2(\hat{s} + 1) \rceil + 1.$$

Following this idea, in Theorem 8 we give a bound for  $cn(G)$  which uses only parameters of the graph  $G$ , not of its neighborhood graph  $N(G)$ . First, for a cycle  $C$  in  $G$  let  $l(C)$  be the length of  $C$  and  $s_{max}(C) := \max\{d_G(C, v) \mid v \in V\}$ , where  $d_G(C, v) := \min\{d_G(x, v) \mid x \in V(C)\}$ , i.e.  $s_{max}(C)$  is the maximum distance of any vertex in  $G$  from the cycle  $C$ .

**Theorem 8.** *Let  $G = (V, E)$  be triangle-free, connected, non-bipartite and not an odd cycle. Moreover, let  $s' := \min \left\{ \frac{l(C)-1}{2} + \left\lceil \frac{s_{max}(C)}{2} \right\rceil \mid C \text{ is an odd cycle in } G \right\}$ . Then,  $cn(G) \leq \lceil 2 + \log_2(s' + 1) \rceil$ .*

**Proof.** Because of Theorem 6 and (\*), it suffices to show that there is a  $K_d$ -path-covering ( $d \geq 3$ ) of  $N(G)$  with the maximum tail length  $\hat{s} \leq s'$ .

Let  $\tilde{C}$  be an odd cycle in  $G$  such that  $s' = \frac{l(\tilde{C})-1}{2} + \left\lceil \frac{s_{max}(\tilde{C})}{2} \right\rceil$ , where  $s'$  is defined as above.

Moreover, let  $\mathcal{W}_{\tilde{C}} = \{\tilde{w}_1, \dots, \tilde{w}_p\}$  be a system of paths of length at most  $s_{max}(\tilde{C})$  in  $G$  such that  $V \setminus V(\tilde{C}) \subseteq V(\mathcal{W}_{\tilde{C}}) := \bigcup_{i=1}^p V(\tilde{w}_i)$  and every path  $\tilde{w}_i \in \mathcal{W}_{\tilde{C}}$  has exactly one end vertex  $v_i$  in common with  $\tilde{C}$ , for  $i \in \{1, \dots, p\}$ .

In the following, we investigate the subgraph  $U := \tilde{C} \cup \tilde{w}_1 \cup \dots \cup \tilde{w}_p$  of  $G$ . Obviously, it suffices to prove the existence of a  $K_d$ -path-covering ( $d \geq 3$ ) of  $N(U)$  with a maximum tail length  $\hat{s} \leq s'$ .

For this end, let  $v \in V(\tilde{C}) \cap V(\tilde{w}_1)$  and  $d := d(v : U) \geq 3$  be the degree of  $v$  in  $U$ .

Furthermore, let  $K_d = \langle N_U(v) \rangle_{N(U)}$  be the  $d$ -clique induced in the neighborhood graph  $N(U)$  by the neighborhood  $N_U(v)$  of  $v$  in  $U$ .

At first we verify that the distance of each vertex  $u \in V$  from  $K_d$  in  $N(U)$  is at most  $s'$ , i.e.

$$(**) \quad \hat{s} = \max\{d_{N(U)}(K_d, u) \mid u \in V\} \leq s',$$

where  $d_{N(U)}(K_d, u) := \min\{d_{N(U)}(x, u) \mid x \in V(K_d)\}$ .

Let  $v' \in V$  be a vertex with  $d_{N(U)}(K_d, v') = \hat{s}$ . If  $v' \in N_U(v)$ , then  $d_{N(U)}(K_d, v') = 0$  and there is nothing to prove.

If  $v' \in V(\tilde{C}) \setminus N_U(v)$ , then in  $\langle V(\tilde{C}) \rangle_U$  there is path of even length  $t \leq l(\tilde{C}) - 1$  from one vertex in  $N_U(v) \cap V(\tilde{C})$  to the vertex  $v'$ ; therefore  $\hat{s} \leq \frac{t}{2} \leq \frac{l(\tilde{C})-1}{2} \leq s'$ .

Now let  $v' \in V(\mathcal{W}_{\tilde{C}}) \setminus (V(\tilde{C}) \cup N_U(v))$ ; in detail, let  $v' \in V(\tilde{w}_j) \setminus (V(\tilde{C}) \cup N_U(v))$ , where  $j \in \{1, 2, \dots, p\}$ .

Then it is easy to see that in  $U$  there is a path of (even) length at most  $(l(\tilde{C}) - 1) + l(\tilde{w}_j) \leq (l(\tilde{C}) - 1) + s_{max}(\tilde{C})$  from  $v'$  to one of the vertices in

$V(\tilde{C}) \cap N_U(v)$ . Therefore, in  $N(U)$  there is a path of length at most  $\frac{l(\tilde{C})-1}{2} + \lceil \frac{s_{max}(\tilde{C})}{2} \rceil = s'$  from  $K_d$  to  $v'$  and (\*\*) is true.

Because of (\*\*) in  $N(U)$  there exists a system  $\mathcal{W} = \{w_1, \dots, w_q\}$  of paths of maximum length  $\hat{s} \leq s'$  such that  $N(U)_{K_d, \mathcal{W}} = K_d \cup w_1 \cup \dots \cup w_q$  is a  $K_d$ -path-covering of  $N(U)$  which has a maximum tail length  $\hat{s} \leq s'$ ; this completes the proof. ■

We conjecture that the bound given in Theorem 8 is sharp for many graphs  $C_q^s$  consisting of a cycle  $C$  of odd length  $l(C) = q$  and a tail  $w$  of length  $l(w) = s$ . The computation of  $cn(C_q^s)$  for a set of pairs  $(q, s)$  lead to

**Conjecture 9.** *If  $q \geq 3$  is odd and  $s \geq 1$ , then  $cn(C_q^s) = \lceil 1 + \log_2(s + q - 2) \rceil$ .*

For  $q = 3$ , Theorem 5 proves the conjecture, because of  $K_3^s = C_3^s$  and  $n - 2 = s + 1$ . In the case  $q > 3$  for  $C_q^s$  due to  $l(C) = q$  odd and  $s_{max}(C) = s$  it follows  $s' = \frac{l(C)-1}{2} + \lceil \frac{s_{max}(C)}{2} \rceil = \frac{q-1}{2} + \lceil \frac{s}{2} \rceil$ . For  $s$  even (i.e.  $n = q + s$  odd) we obtain  $s' = \frac{q+s-1}{2} = \frac{n-1}{2}$  and for  $s$  odd (i.e.  $n$  even)  $s' = \frac{q+s}{2} = \frac{n}{2}$ .

Therefore,

$$\begin{aligned} \lceil 2 + \log_2(s' + 1) \rceil &= \begin{cases} \lceil 2 + \log_2(\frac{n+1}{2}) \rceil & \text{if } n \text{ is odd,} \\ \lceil 2 + \log_2(\frac{n+2}{2}) \rceil & \text{if } n \text{ is even,} \end{cases} \\ &= \begin{cases} \lceil 1 + \log_2(n + 1) \rceil & \text{if } n \text{ is odd,} \\ \lceil 1 + \log_2(n + 2) \rceil & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Provided that Conjecture 9 is true, for all odd  $q > 3$  and all  $s \geq 1$  the bound in Theorem 8 is sharp for  $C_q^s$  if and only if

$$\lceil \log_2(n - 2) \rceil = \begin{cases} \lceil \log_2(n + 1) \rceil & \text{if } n \text{ is odd,} \\ \lceil \log_2(n + 2) \rceil & \text{if } n \text{ is even,} \end{cases}$$

where  $n = q + s$ .

By computer, we verified Conjecture 9 (and, therefore, the sharpness of the bound in Theorem 8) for  $C_q^s$  if  $q \in \{5, 7, 9, 21\}$  and  $s \in \{1, 2, \dots, 35 - q\}$ .

To give one of the examples in detail, consider  $C_7^4$ . By computer, we obtained  $cn(C_7^4) = 5$  and from  $q = 7, s = 4, n = 11$  it follows  $\lceil 1 + \log_2(n - 2) \rceil = \lceil 1 + \log_2(11 - 2) \rceil = 5$  as well as  $\lceil 1 + \log_2(n + 1) \rceil = \lceil 1 + \log_2(11 + 1) \rceil = 5$ .

We close this subsection with the remark that, for infinitely many graphs, our results are better than the bound (EH) of Exoo and Harary [5] given at the beginning of Section 3. As a first example, consider  $K_3^{10}$  (cf. Figure 2). Then Theorem 5 yields  $cn(K_3^{10}) = 5$ , but from (EH) we would obtain  $cn(K_3^{10}) \leq \lceil \log_2 10 + 3 \rceil = 7$ . As a second example, for  $C_{21}^4$  Theorem 8 provides the bound  $cn(C_{21}^4) \leq \lceil 2 + \log_2 13 \rceil = 6$ , and from (EH) it follows  $cn(C_{21}^4) \leq \lceil \log_2 4 + 21 \rceil = 23$ .

In general, with increasing length of the (odd) cycle considered in the graph, the bound (EH) becomes more blurred.

### 3.3. Neighborhood completeness number and diameter

We can observe that the diameter  $diam(G)$  (the maximum distance between two vertices in the graph  $G$ ) is closely related to the neighborhood completeness number  $cn(G)$ . But at least in the class of graphs consisting of a clique  $K_l$  ( $l \geq 3$ ) and some vertex disjoint tails, the length  $s$  ( $s \geq 1$ ) of a longest tail is a more elegant measure to determine  $cn(G)$ . For illustration, consider the graph  $K_l^{s,s}$  consisting of an  $l$ -clique  $K_l$  with two (vertex disjoint) tails of length  $s$ . Because of  $diam(K_l^s) = s + 1$  and  $diam(K_l^{s,s}) = 2s + 1$  Corollary 7 implies

**Remark 10.**  $cn(K_l^s) = \lceil 1 + \log_2(diam(K_l^s)) \rceil$  and  $cn(K_l^{s,s}) = \lceil \log_2(diam(K_l^{s,s}) + 1) \rceil$ .

Hence, using the diameter, we obtain two different formulas for the neighborhood completeness numbers  $cn(K_l^s)$  and  $cn(K_l^{s,s})$ . By contrast, using the length  $s$  of a longest tail as a parameter, we obtain one and the same formula for both types of graphs: Corollary 7 leads to  $cn(K_l^s) = \lceil 1 + \log_2(s + 1) \rceil = cn(K_l^{s,s})$ , since the length of a longest tail is the same (namely  $s$ ) in both  $K_l^s$  and  $K_l^{s,s}$ .

A recent result of Schweitzer [17] immediately implies

**Theorem 11** [17]. *If  $G$  is connected, non-bipartite and not an odd cycle, then  $\log_2(diam(G)) \leq cn(G) \leq \lceil 2 + \log_2(diam(G)) \rceil$ .*

Note that  $2 + \log_2(diam(G))$  is not an upper bound for  $cn(G)$ : taking the above example  $C_7^4$  we obtain  $diam(C_7^4) = 7$  and  $cn(C_7^4) = 5 > 2 + \log_2(7)$ .

For special classes of graphs the upper bound in Theorem 11 follows from our results. Additionally to  $K_l^s$  and  $K_l^{s,s}$  (cf. Remark 10) we mention the following two classes:

(A) Consider the graphs  $\widehat{G}$  being investigated in Corollary 7, which have a  $K_l$ -path-covering with a longest tail  $w_1$  of length  $s = l(w_1)$ , such that only the end vertex  $v_1 \in V(K_l)$  of  $w_1$  has neighbors in  $V(\widehat{G}) \setminus V(w_1)$ . The diameter of such a graph is at least  $s + 1$ , consequently  $cn(\widehat{G}) = \lceil 1 + \log_2(s + 1) \rceil < \lceil 2 + \log_2(diam(\widehat{G})) \rceil$ .

(B) Similarly, using Theorem 8 we obtain a corresponding result for certain triangle-free, connected, non-bipartite graphs being no odd cycles.

Let  $G$  be a unicyclic graph consisting of a cycle  $C$  of odd length  $q > 3$  and several trees (one with at least two vertices), where each of the trees has exactly one end vertex in common with  $C$ .

Moreover, let  $\mathcal{W}_C = \{w_1, \dots, w_p\}$  be a system of paths of length at most  $s := s_{max}(C)$  in  $G$  such that  $V \setminus V(C) \subseteq V(\mathcal{W}_C) := \bigcup_{i=1}^p V(w_i)$  and every path  $w_i \in \mathcal{W}_C$  has exactly one end vertex  $v_i$  in common with  $C$ , for  $i \in \{1, \dots, p\}$ . Since at least one of the trees in  $G$  is nontrivial,  $s \geq 2$  is valid.

Then  $\text{diam}(G) \geq \frac{q-1}{2} + s > \frac{q-1}{2} + \lceil \frac{s}{2} \rceil = \frac{l(C)-1}{2} + \lceil \frac{s_{\max}(C)}{2} \rceil = s'$ . Theorem 8 implies  $cn(G) \leq \lceil 2 + \log_2(s' + 1) \rceil \leq \lceil 2 + \log_2(\text{diam}(G)) \rceil$ .

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