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STABLE SETS FOR $(P_6, K_{2,3})$-FREE GRAPHS

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Abstract

The Maximum Stable Set (MS) problem is a well known NP-hard problem. However different graph classes for which MS can be efficiently solved have been detected and the augmenting graph technique seems to be a fruitful tool to this aim. In this paper we apply a recent characterization of minimal augmenting graphs [22] to prove that MS can be solved for $(P_6,K_{2,3})$-free graphs in polynomial time, extending some known results.

Keywords: graph algorithms, stable sets, P_6-free graphs.

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1. Introduction

A *stable set* in a graph G is a set of pairwise nonadjacent vertices of G. The Maximum Stable Set (MS) problem is that of determining a stable set of maximum cardinality of a graph G. The MS problem is NP-hard, even under strong restrictions [13]. The following specific graphs are mentioned later. A P_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j < k$. A C_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j \leq k-1$ (index arithmetic modulo k). A $K_{p,q}$, for $p, q \geq 1$, is a complete bipartite graph with sides of cardinality p and q respectively. A $K_{1,3}$ is also called a *claw*. Given two graphs G_1, G_2, let $G_1 + G_2$ denote the graph obtained as a disjoint union of G_1 and G_2.

Let us say that a graph G is *F-free* if no induced subgraph of G is isomorphic to a given graph F. If G is F_1-free and F_2-free for given graphs F_1 and F_2, then let us say that G is *(F_1, F_2)-free*.

Let us say that a graph is of *type T* if it is a subdivided claw or a path, i.e. if it is a tree with at most one vertex of degree 3 and the other vertices of degree less than 3. Then a graph of type T which is different from a path contains three paths, each one from the vertex of degree 3 to respectively the three vertices of degree 1: then it can be denoted as $T_{i,j,k}$, where i, j, k stand for the length of such three paths (e.g. a $T_{1,1,1}$ is a claw).

Alekseev [1, 4] proved that MS remains NP-hard in the class of F-free graphs whenever F is a graph of which at least one component is not of type T.

Notice that if MS is polynomial for F-free graphs, for a given graph F, then MS is polynomial for $P_1 \cup F$ graphs, where $P_1 \cup F$ is the graph formed by the disjoint union of an isolated vertex and F: in fact, for any graph $G = (V, E)$, the MS problem can be solved by solving the same problem on each its subgraph $G[V \setminus N(v)]$, for $v \in V$.

Let us consider the computational complexity of MS for F-free graphs, for every 5-vertex graph F.

Assume that F is connected. If F is not of type T, then MS remains NP-hard for F-free graphs by Alekseev's result. If F is of type T, then F is either a fork (a $fork$ has vertices a, b, c, d, e and edges ab, bc, cd, ce) or a P_5. If F is a fork, then MS is polynomial for F-free graphs [2, 3], also in its weighted version [21]: notice that then MS is polynomial for F'-free graphs, for every induced subgraph F' of a fork. If F is a P_5, then the computational complexity of MS is unknown for F-free graphs.

Assume that F is disconnected. If at least one component of F is not of type T, then MS remains NP-hard for F-free graphs by Alekseev's result. Then assume that every component of F is of type T. If F has an isolated vertex, then the remaining four vertices of F either form an induced subgraph of a fork, or form a P_2+P_2, or form a $4P_1$ (i.e., a stable set of four vertices): then by the above remarks and since MS is polynomial for $P_2 + P_2$-free graphs [11] and clearly for $5P_1$-free graphs, MS is polynomial for F-free graphs. If F has no isolated vertices, i.e., F is a $P_2 + P_3$, then MS is polynomial for F-free graphs [23].

Summarizing, if F is a 5-vertex graph, then the computational complexity of MS is unknown for F-free graphs only in case $F = P_5$. Also the computational complexity of MS is unknown for F-free graphs, where F is a connected graph of type T with more than 5 vertices, in particular for P_t-free graphs for $t \geq 6$.

In this paper we prove that MS can be solved for (P_6,$K_{2,3}$)-free in polynomial time. That extends the following analogous results concerning:

(i) (P_5,$K_{2,3}$)-free graphs, see [15] where the result holds even for (P_5,$K_{m,m}$)-free graphs (see [27] for the weighted case) and

(ii) (P_6,C_4)-free graphs, see [7, 26] (see [7] for the weighted case). Let us recall that, since a $K_{2,3}$ contains a C_4, MS remains NP-hard for $K_{2,3}$-free graphs [29].

Two topics are linked to this paper: the first is the study of P_6-free graphs (with particular reference to MS for subclasses of these graphs); the second is the augmenting graph technique (see e.g. [17] for a survey on this topic), which is a fruitful approach to detect graph classes for which MS can be solved in polynomial time, and which we apply in this paper with particular reference to a recent characterization of minimal augmenting graphs [22].

Concerning the first topic: the class of P_6-free graphs is a natural extension of that of P_5-free graphs. The first characterization of such graphs was maybe given in [6]. Then further results were introduced also recently, see e.g. [10, 12, 16, 18, 19, 20]. In particular structural properties of P_6-free graphs were directly applied to define polynomial time algorithms to solve the MS problem (also for its weighted version) for subclasses of these graphs, such as (P_6,triangle)-free [9], (P_6,$K_{1,p}$)-free [24], (P_6, C_4)-free [7, 26] and (P_6,diamond)-free graphs [28]. Let us observe that results on MS for subclasses of P_6-free graphs may keep their own interest even if the complexity of MS for P_5-free graphs should be determined. In fact: if MS should (be shown to) remain NP-hard for P_5-free graphs, then MS would remain NP-hard for P_6-free graphs too; if MS should (be shown to) be polynomial for P_5-free graphs, then according to the aforementioned Alekseev's result the class of P_6-free graphs would be one of the three minimal classes (the other ones are that of $T_{1,1,3}$-free graphs and that of $T_{1,2,2}$-free graphs), defined by forbidding a single connected subgraph, for which the computational complexity of MS would be unknown.

Concerning the second topic: the augmenting graph technique to solve the MS problem derives directly from the well-known augmenting technique to solve the Maximum Matching problem, and the first application to MS of such a technique was maybe introduced in [25, 30] for claw-free graphs. Then further results were introduced also recently, see e.g. [5, 14, 22]. Let us observe that in [5] the authors prove that while applying the augmenting graph technique one can treat banner-free graphs (a *banner* has vertices a, b, c, d, e and edges ab, bc, be, cd, de) as C_4-free graphs; in particular the mentioned results of [5, 14, 22] deal with subclasses of banner-free graphs; in this manuscript we consider a subclass of $K_{2,3}$-free graphs (i.e., that is an extension of the application of the augmenting graph technique in a different direction).

2. Preliminaries

For any missing notation or references, let us refer to [8]. Let $G = (V, E)$ be a finite undirected graph and let $|V| = n$, $|E| = m$. For every $u \in V$, let $N(u) = \{v \in V : uv \in E\}$ be the set of *neighbors* of u. Let $N[v] = N(v) \cup \{v\}$. Let U, W be two subsets of V. Let $N(U) = \{v \in V \setminus U :$ there exists $u \in U$ such that $uv \in E\}$. Let $N[U] = N(U) \cup U$. Let $N_W(U) = N(U) \cap W$; if $U = \{u\}$, then let us simply write $N_W(u)$. Let us say that $v \in V$ *dominates* U if v is adjacent to each vertex of U.

Let $G[U]$ denote the subgraph of G induced by $U \subseteq V$. A *component of* G is the vertex-set of a maximal connected subgraph of G. The *distance* $d(v, w)$ between $v, w \in V$ is the number of edges in a shortest path from v to w.

Let S be a stable set of G. A bipartite graph $H = (H_1, H_2, F)$ is called an *augmenting graph* for S if $H_2 \subseteq S$, $H_1 \subseteq V \setminus S$, $N(H_1) \cap (S \setminus H_2) = \emptyset$, and $|H_1| > |H_2|$. The following theorem is well known and not difficult to prove (see e.g. [17]).

Theorem 1. *Let S be a stable set S of a graph G. Then S is not maximum if and only if there exists an augmenting graph for S.*

Replacement of the vertices of H_2 in S by the vertices of H_1 is called the *H-augmentation* of S (in particular, $|H_1| - |H_2|$ is the increment). Then the following algorithm correctly solves the MS problem for any graph G and points out that the difficulty of the problem can be directly linked to that of detecting augmenting graphs for stable sets.

Algorithm Alpha

Input: a graph $G = (V, E)$.
Output: a maximum stable set S of G.

Step 1. Compute any stable set S of G.
Step 2. Check if there exists a (minimal) augmenting graph for S, say H.
Step 3. If the answer is *no*, then return S. STOP.
Step 4. If the answer is *yes*, then apply H-augmentation to S. Go to Step 2.

A *stable system of representatives* (shortly ssr) *of* $U \subseteq V$ is a stable set $T \subseteq V \setminus U$, with $|T| = |U|$, such that $G[T \cup U]$ has a matching of $|T| = |U|$ elements, i.e., one can write $U = \{u_1, \ldots, u_m\}$ and $T = \{t_1, \ldots, t_m\}$ so that $(u_i, t_i) \in E$ for $i = 1, \ldots, m$.

A *minimal augmenting graph* for S is an augmenting graph for S that is not the induced supergraph of any other augmenting graph for S. Notice that every minimal augmenting graph is connected. Let us report the following result from [22].

Lemma 2 [22]**.** *Let $G = (V, E)$ be a graph, S be a maximal stable set of G, and $v \in V \setminus S$. If v belongs to a minimal augmenting graph (H_1, H_2, F) for S, then $H_1 \setminus \{v\}$ admits an* ssr *in H_2.*

Theorem 2 of [6] implies that every connected P_6-free graph $G = (V, E)$ admits a vertex v such that $d(v, u) \leq 3$ for every $u \in V$. Theorem 2 of [20] implies that every connected P_6-free bipartite graph admits two such special vertices, belonging respectively to the two sides of the bipartite graph. The following observation points out that, in a connected P_6-free bipartite graph G, a sufficient condition for a vertex to enjoy the above property is to have maximum degree in G among the vertices of its side.

Observation 3. *Let $H = (H_1, H_2, E)$ be a connected bipartite P_6-free graph. Let $v \in H_1$ be a vertex such that v has maximum degree in H among the vertices of H_1. Then $d(v, h) \leq 3$ for every $h \in H_1 \cup H_2$.*

Proof. By contradiction assume that there exists $h \in H_1 \cup H_2$ such that $d(v, h) = 4$. Since G is connected bipartite, $h \in H_1$. Let v, a, u, b, h be the vertices inducing a shortest path from v to h. By the maximum degree of v (and since u is adjacent to b), there exists a vertex $a' \in H_2$ such that a' is adjacent to v and nonadjacent to u. Notice that a' is also nonadjacent to h, since $d(v, h) = 4$. Then a', v, a, u, b, h induce a P_6 (contradiction). ■

Let G be a connected P_6-free graph. Let S be a maximal but not maximum stable set of G, and let $H = (H_1, H_2, F)$ be a minimal augmenting graph for S. Let us say that a vertex $v \in H_1$ such that v has maximum degree in H among the vertices of H_1 is a *nail* of H. By Observation 3 and the aforementioned observation that H is connected, if v is a nail of H, then $d(v, h) \leq 3$ for every $h \in H_1 \cup H_2$.

3. Stable Sets for (P_6,$K_{2,3}$)-free Graphs

Throughout this section let $G = (V, E)$ be a connected (P_6,$K_{2,3}$)-free graph, and S be a maximal stable set of G. To solve MS for G we apply Algorithm Alpha. Then let us prove that Step 2 of Algorithm Alpha, referring to minimal augmenting graphs, can be efficiently executed. To this end, since every minimal augmenting graph for S contains at least one nail, let us proceed as follows.

Let us show that if a vertex v of G is a nail of a minimal augmenting graph $H = (H_1, H_2, F)$ for S, then H can be efficiently detected. Then let us fix a vertex $v \in V \setminus S$ and assume that v is a nail of a minimal augmenting graph $H = (H_1, H_2, F)$ for S (then H is connected). Let us write $A = N_S(v)$, $B = N(A) \setminus N[v]$, and $C = (S \setminus A) \cap N(B)$. Then by the definition of a nail and by Observation 3 one can assume that:

(1) H is a subgraph of $G[A \cup B \cup C \cup \{v\}]$, i.e., $H_1 \subseteq B \cup \{v\}$ and $H_2 \subseteq A \cup C$.

(2) No vertex of B has in $A \cup C$ more neighbors than v in A: if this does not happen, then one can delete all the vertices of B which have in $A \cup C$ more neighbors than v in A (since v is a nail of H).

Furthermore, since G is $K_{2,3}$-free, the following fact holds:

(3) Each vertex of B has degree 1 or 2 in A.

Let $A = \{a_1, \ldots, a_h\}$ and $C^* = C \cap H_2 = \{c_1, \ldots, c_k\}$.

To show that H can be efficiently detected, let us distinguish between the case in which $C^* = \emptyset$ and the case in which $C^* \neq \emptyset$.

3.1. The case in which $C^* = \emptyset$

In this case, the difficulty is to check if A admits an ssr in B.

Lemma 4. *Let $\bar{b}_i \in B \cap N(a_i)$ for $i = 1, 2, 3$ be pairwise nonadjacent. Assume that $\bar{b}_1$ and $\bar{b}_2$ are nonadjacent to any vertex of $\{a_4, \ldots, a_h\}$. Then one can check if $\{a_4, \ldots, a_h\}$ admits an ssr in $B \setminus N[\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}]$ in $O(n + m)$ time.*

Proof. First let us prove a claim.

Claim 5. *Let $\bar{p}, \bar{q} \in B \setminus N[\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}]$. Let $\bar{p} \in N(a_p)$ and $\bar{q} \in N(a_q)$ for any $p, q \in \{4, \ldots, h\}$. If $\bar{p}$ is nonadjacent to a_q, then $\bar{p}$ is nonadjacent to $\bar{q}$.*

Proof. By contradiction assume that $\bar{p}$ is adjacent to $\bar{q}$. By (3), to avoid a P_6 formed by either $\bar{b}_1, a_1, v, a_q, \bar{q}, \bar{p}$ or $\bar{b}_2, a_2, v, a_q, \bar{q}, \bar{p}$ one may without loss of generality that $\bar{p}$ is adjacent to a_1, and $\bar{q}$ is adjacent to a_2. By (3): $\bar{q}$ is nonadjacent to a_p, and both $\bar{p}$ and $\bar{q}$ are nonadjacent to a_3. Then, since by (3) $\bar{b}_3$ can not be adjacent to both a_p and a_q, either $\bar{b}_3, a_3, v, a_p, \bar{p}, \bar{q}$ or $\bar{b}_3, a_3, v, a_q, \bar{q}, \bar{p}$ induce a P_6 (contradiction). □

Let us write $A^* = \{a_4, \ldots, a_h\}$ and $B^* = B \setminus N[\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}]$.

For $i = 4, \ldots, h$ let $D_i = \{b \in B^* : N_{A^*}(b) = \{a_i\}\}$. Notice that by Claim 5 all the vertices in D_i, for $i = 4, \ldots, h$, have no neighbors in $B^* \setminus D_i$. Then, since one has to check if A^* admits an ssr in B^*, one can proceed as follows. For every $D_i \neq \emptyset$: delete all the vertices of D_i except from one. Denote as B^*_{one} what remains of B^*.

For $i, j = 4, \ldots, h$ let $D_{i,j} = \{b \in B^*_{one} : N_{A^*}(b) = \{a_i, a_j\}\}$. Notice that the vertices in $D_{i,j}$ are mutually adjacent (since G is $K_{2,3}$-free), and that by Claim 1 all the vertices in $D_{i,j}$, for $i, j = 4, \ldots, h$, have no neighbors in $B^*_{one} \setminus D_{ij}$. Then, since one has to check if A^* admits an ssr in B^*_{one}, one can proceed as follows. For every $D_{i,j} \neq \emptyset$: delete all the vertices of $D_{i,j}$ except from one. Denote as B^*_{two} what remains of B^*_{one}.

Now by (3) and by Claim 5 B^*_{two} is a stable set. Then to check if A^* admits an ssr in B^*_{two} it is enough to check if the bipartite graph $G[A^* \cup B^*_{two}]$ admits a matching of $h - 3$ elements. Since G is P_6-free, that can be done in linear time as shown in [12]. Then the lemma follows. ■

Lemma 6. *Assume that $C^* = \emptyset$. Then H can be detected in $O(n^3 m)$ time.*

Proof. Since $C^* = \emptyset$, by Lemma 2 one has to check if A admits an ssr in B. If $|A| \leq 3$, then the assertion can be easily proved. Then assume that $|A| \geq 4$. If A admits an ssr in B, then there exists a vertex $b \in N(a_1)$ belonging to such an ssr. For every $\bar{b}_1 \in N(a_1)$ one can check if $\bar{b}_1$ belongs to such an ssr, as follows.

First assume that $\bar{b}_1$ has degree 1 in A. Then for every $\bar{b}_2 \in N(a_2) \setminus N[\bar{b}_1]$ do:

1. if $\bar{b}_2$ has degree 1 in A, then for every $\bar{b}_3 \in N(a_3) \setminus N[\{\bar{b}_1, \bar{b}_2\}]$ check if $\{a_4, \ldots, a_m\}$ admits an ssr in $B \setminus N[\{\bar{b}_1, \bar{b}_2, \bar{b}_3\}]$, according to Lemma 4;

2. if $\bar{b}_2$ has degree 2 in A, then: if $\bar{b}_2$ is adjacent to a_1, then one can proceed similarly to the previous case; if $\bar{b}_2$ is adjacent to a_i, with $i \neq 1, 2$, then one can assume without loss of generality that $i = 3$ and proceed similarly to the previous case.

Then assume $\bar{b}_1$ has degree 2 in A. Then $\bar{b}_1$ is adjacent to some a_i with $i \neq 1$. Then one can assume without loss of generality that $i = 2$ and proceed similarly to the case in which $\bar{b}_1$ has degree 1 in A. ∎

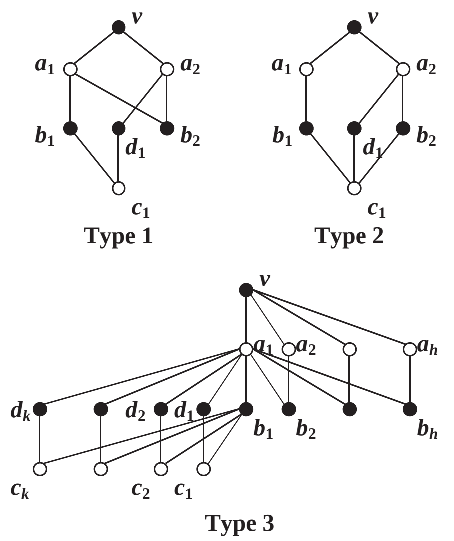

Figure 1

3.2. The case in which $C^* \neq \emptyset$

In this case, let us show that H can be just of three types each of which can be efficiently detected. By Lemma 2, let $\{\{b_1, \ldots, b_h\}, \{d_1, \ldots, d_k\}\}$ be a partition of $H_1 \setminus \{v\}$, such that $\{b_1, \ldots, b_h\}$ is an ssr of A, and $\{d_1, \ldots, d_k\}$ is an ssr of C^* (in $H_1 \setminus \{v\}$). Referring to Figure 1, let us say that H is of:

- Type 1 if: $A = \{a_1, a_2\}$, $C^* = \{c_1\}$; b_1 is nonadjacent to a_2; b_2 is adjacent to a_1; d_1 is adjacent to a_2 and nonadjacent to a_1; c_1 is adjacent to b_1 and nonadjacent to b_2;
- Type 2 if: $A = \{a_1, a_2\}$, $C^* = \{c_1\}$; b_1 is nonadjacent to a_2; b_2 is nonadjacent to a_1; d_1 is adjacent to a_2 and nonadjacent to a_1; c_1 is adjacent to b_1, b_2;
- Type 3 if: a_1 is adjacent to b_i for every $i \geq 2$; a_1 is adjacent to d_j for every $j \geq 2$; a_i is nonadjacent to b_t for every $i \geq 2$ and $t \neq i$; a_i is nonadjacent to d_j for every $i \geq 2$ and $j \geq 1$; b_1 is adjacent to c_i for every $i \geq 1$; c_j is nonadjacent to b_t for every $j \geq 2$ and $t \geq 1$; c_j is nonadjacent to d_t for every $j \geq 2$ and $t \neq i$.

Lemma 7. *Assume that $C^* \neq \emptyset$. Then H is of* Type 1*, or* 2*, or* 3.

Proof. Since $C^* \neq \emptyset$ and H is a minimal augmenting graph for S, there is a vertex in $\{b_1, \ldots, b_h\}$ adjacent to a vertex in $\{c_1, \ldots, c_k\}$, otherwise $\{v\} \cup \{a_1, \ldots, a_h\} \cup \{b_1, \ldots, b_h\}$ is an augmenting graph for S.

Assume without loss of generality that b_1 is adjacent to c_1. Then by (2) with respect to b_1, one has $|A| \geq 2$.

Claim 8. *Exactly one of the following cases holds:*
(i) *d_1 dominates $A \setminus \{a_1\}$, or*
(ii) *d_1 is adjacent to a_1.*

Proof. By (2) with respect to d_1, statements (i) and (ii) can not hold at the same time. Then let us assume that d_1 is nonadjacent to a_1, and prove that d_1 dominates $A \setminus \{a_1\}$. By contradiction assume that there exists a vertex in $A \setminus \{a_1\}$ nonadjacent to d_1, say a_2 without loss of generality. To avoid that $d_1, c_1, b_1, a_1, v, a_2$ induce a P_6, b_1 is adjacent to a_2. Then by (2) with respect to b_1, one has $A \setminus \{a_1, a_2\} \neq \emptyset$. Furthermore by (3), b_1 is nonadjacent to any vertex in $A \setminus \{a_1, a_2\}$. Then to avoid that d_1, c, b_1, a_1, v, a_3 induce a P_6, d_1 is adjacent to a_3. Let us consider b_2. Notice that b_2 is nonadjacent to a_1 (otherwise a_1, a_2, v, b_1, b_2 induce a $K_{2,3}$) and to c_1 (otherwise $a_1, v, a_2, b_2, c_1, d_1$ induce a P_6). Furthermore b_2 is nonadjacent to a_3: in fact otherwise to avoid that $a_1, b_1, a_2, b_2, a_3, b_3$ induce a P_6, one has that either b_3 is adjacent to a_2 (but then b_2, b_3, v, a_2, a_3 induce a $K_{2,3}$) or b_3 is adjacent to a_1 (but then $a_2, b_1, a_1, b_3, a_3, d_1$ induce a P_6). Then $b_2, a_2, b_1, c_1, d_1, a_3$ induce a P_6 (contradiction). □

According to Claim 8 let us consider the following cases.

Case 1. d_1 dominates $A \setminus \{a_1\}$ (and is nonadjacent to a_1). Then by (3), $|A \setminus \{a_1\}| \leq 2$.

Case 1.1. $|A \setminus \{a_1\}| = 1$. Then d_1 is adjacent to a_2. By (2) with respect to b_1, b_1 is nonadjacent to a_2. To avoid that $b_2, a_2, d_1, c_1, b_1, a_1$ induce a P_6, b_2 is adjacent either to a_1 or to c_1 (not to both by (2)).

Assume that b_2 is adjacent to a_1 (and is nonadjacent to c_1). Let us show that $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce a minimal augmenting graph of Type 1. To this end, let us show that no extension of this graph is possible, i.e., that $C^* = \{c_1\}$ (and thus $D = \{d_1\}$). By contradiction assume that $C^* \setminus \{c_1\} \neq \emptyset$. Then every vertex of $C^* \setminus \{c_1\}$ is nonadjacent to any vertex of $\{b_1, b_2, d_1\}$, by (2). But then $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce an augmenting graph, i.e., this possible extension would not be a minimal augmenting graph. Then H is of Type 1.

Assume that b_2 is adjacent to c_1 (and is nonadjacent to a_1). Let us show that $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce a minimal augmenting graph of Type 2. To this end, let us show that no extension of this graph is possible, i.e., that $C^* = \{c_1\}$ (and thus $D = \{d_1\}$). By contradiction assume that $C^* \setminus \{c_1\} \neq \emptyset$. Then every vertex of $C^* \setminus \{c_1\}$ is nonadjacent to any vertex of $\{b_1, b_2, d_1\}$, by (2). But then $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce an augmenting graph, i.e., this possible extension would not be a minimal augmenting graph. Then H is of Type 2.

Case 1.2. $|A \setminus \{a_1\}| = 2$. Then d_1 is adjacent to a_2 and a_3. Then to avoid a $K_{2,3}$: b_2 is nonadjacent to a_3, and b_3 is nonadjacent to a_2. Furthermore, by (2) let us assume without loss of generality that b_1 is nonadjacent to a_3.

To avoid that $b_3, a_3, v, a_1, b_1, c_1$ induce a P_6, b_3 is adjacent either to c_1 or to a_1. If b_3 is adjacent to a_1, then b_2 is adjacent to a_1. Otherwise $a_1, b_3, a_3, d_1, a_2, b_2$ induce a P_6, then b_1 is nonadjacent to a_2. Otherwise a_1, a_2, v, b_1, b_2 induce a $K_{2,3}$ but then $b_1, a_1, b_2, a_2, d_1, a_3$ induce a P_6. If b_3 is adjacent to c_1, then b_2 is adjacent to c_1. Otherwise $b_2, a_2, v, a_3, b_3, c_1$ induce a P_6, then b_1 is nonadjacent to a_2. Otherwise a_2, c_1, b_1, b_2, d_1 induce a $K_{2,3}$ but then $a_2, v, a_3, b_3, c_1, b_1$ induce a P_6.

Case 2. d_1 is adjacent to a_1 (and does not dominate $A \setminus \{a_1\}$). By (2) with respect to b_1, b_1 is nonadjacent to at least one vertex of $A \setminus \{a_1\}$, say a_h. To avoid that $b_h, a_h, v, a_1, b_1, c_1$ induce a P_6, b_h is adjacent either to c_1 or to a_1 (not to both, otherwise a_1, c_1, d_1, b_1, b_h induce a $K_{2,3}$).

Case 2.1. b_h is adjacent to c_1 (and is nonadjacent to a_1). Then to avoid that $a_i, v, a_h, b_h, c_1, b_1$ induce a P_6, for all $i = 2, \ldots, h-1$, a_i is adjacent either to b_1 or to b_h. By (3) this implies that $|A| \leq 4$.

Assume that $|A| = 2$, i.e., $h = 2$. Then b_1 and d_1 are nonadjacent to a_2, by (2). Let us show that $v, a_1, a_2, b_1, b_2, c_1, d_1$ induce a minimal augmenting graph of Type 2, up to symmetry. By symmetry the proof is similar to that given in Case 1.1. Then H is of Type 2.

Assume that $|A| = 3$, i.e., $h = 3$. To avoid that $a_2, v, a_3, b_3, c_1, b_1$ induce a P_6, a_2 is adjacent either to b_1 or to b_3. If a_2 is adjacent to b_1 and nonadjacent to b_3, then: to avoid that $b_2, a_2, b_1, c_1, b_3, a_3$ induce a P_6, b_2 is adjacent either to a_3 or to c_1; if b_2 is adjacent to a_3 (and then is nonadjacent to a_1 by (3)), then $a_1, b_1, a_2, b_2, a_3, b_3$ induce a P_6; if b_2 is adjacent to c_1, then $a_1, v, a_2, b_2, c_1, b_3$

induce a P_6. If a_2 is adjacent to b_3 and nonadjacent to b_1, then by symmetry one obtains a contradiction as well. If a_2 is adjacent to both b_1 and b_3, then: d_1 is nonadjacent to a_2 (otherwise a_1, a_2, d_1, b_1, v induce a $K_{2,3}$), b_2 is nonadjacent to a_1 (otherwise a_1, a_2, b_2, b_1, v induce a $K_{2,3}$), b_2 is nonadjacent to c_1 (otherwise a_2, c_1, b_1, b_2, b_3 induce a $K_{2,3}$); then $b_2, a_2, v, a_1, d_1, c_1$ induce a P_6.

Assume that $|A| = 4$, i.e., $h = 4$. Then one can apply an argument similar to that of the previous paragraph, with b_4 instead of b_3, to obtain a contradiction.

Case 2.2. b_h is adjacent to a_1 (and is nonadjacent to c_1). By (3), b_h is nonadjacent to any vertex of $\{a_2, \ldots, a_{h-1}\}$.

Claim 9. *c_1 is nonadjacent to any vertex of $\{b_2, \ldots, b_{h-1}\}$ (and then of $\{b_2, \ldots, b_h\}$).*

Proof. By contradiction, assume that c_1 is adjacent to a vertex of $\{b_2, \ldots, b_{h-1}\}$, say b_i, for some $i \in \{2, \ldots, h-1\}$; by (2) b_i can not be adjacent to both a_1 and a_h; then either $c_1, b_i, a_i, v, a_1, b_h$ (if b_i is nonadjacent to a_1) or $c_1, b_i, a_i, v, a_h, b_h$ (if b_i is nonadjacent to a_h) induce a P_6 (contradiction). □

Claim 10. *a_1 is adjacent to every vertex of $\{b_2, \ldots, b_{h-1}\}$ (and then of $\{b_2, \ldots, b_h\}$).*

Proof. By contradiction assume that a_1 is nonadjacent to a vertex of $\{b_2, \ldots, b_{h-1}\}$, say b_i for some $i \in \{2, \ldots, h-1\}$: then a_i is adjacent to b_1, otherwise $c_1, b_1, a_1, v, a_i, b_i$ induce a P_6. It follows, by (3) with respect to b_1, that at most one vertex of $\{b_2, \ldots, b_{h-1}\}$ is nonadjacent to a_1, namely b_i. Without loss of generality let us say that $b_i = b_2$: then a_2 is adjacent to b_1. Then by (2) with respect to b_1, one has $|A| \geq 3$. Notice that for all $t = 3, \ldots, h$, a_t is adjacent to b_2, otherwise $a_t, b_t, a_1, b_1, a_2, b_2$ induce a P_6. Then by (3) with respect to b_2, one has $|A| = 3$. Then $b_2, a_3, b_3, a_1, b_1, c_1$ induce a P_6 (contradiction). □

Let us write $B_1 = \{b_2, \ldots, b_h\}$. By Claim 10, a_1 dominates B_1. Then by (3) every vertex $b_i \in B_1$ is adjacent in A only to vertices a_1, a_i. Then b_1 and d_1 are nonadjacent to any vertex of $\{a_2, \ldots, a_h\}$, otherwise a $K_{2,3}$ arises. Let us show that the possible extensions of this graph lead to the conclusion that H is of Type 3.

Then let us assume that $C^* \setminus \{c_1\} \neq \emptyset$. Since $C^* \setminus \{c_1\} \neq \emptyset$ and H is a minimal augmenting graph for S, there is a vertex in $C^* \setminus \{c_1\}$ adjacent to a vertex in $B_1 \cup \{b_1, d_1\}$, otherwise $\{v\} \cup \{a_1, \ldots, a_h\} \cup \{b_1, \ldots, b_h\} \cup \{c_1, d_1\}$ is an augmenting graph for S.

Claim 11. *Every vertex of $C^* \setminus \{c_1\}$ is nonadjacent to any vertex of B_1.*

Proof. By contradiction assume without loss of generality by symmetry that c_k is adjacent to b_h. Then c_k is adjacent to each vertex of $B_1 \setminus \{b_h\}$, otherwise a P_6 arises (namely, $c_k, b_h, a_h, v, a_i, b_i$ for every $b_i \in B_1 \setminus \{b_h\}$). Then $|B_1| \leq 2$ otherwise a $K_{2,3}$ arises involving a_1 and c_k. If $|B_1| = 1$, then one has a contradiction to (2) with respect to b_h. If $|B_1| = 2$, then: b_1 is nonadjacent to c_k, otherwise $c_1, b_1, c_k, b_h, a_h, v$ induce a P_6; then d_k is nonadjacent to a_1, otherwise a_1, c_k, b_2, b_h, d_k induce a $K_{2,3}$; then d_k is adjacent to c_1, otherwise $d_k, c_k, b_h, a_1, b_1, c_1$ induce a P_6; then $v, a_1, b_1, c_1, d_k, c_k$ induce a P_6 (contradiction). □

By the above and by Claim 11, at least one vertex of $C^* \setminus \{c_1\}$ is adjacent to b_1 or to d_1: without loss of generality by symmetry, let us say to b_1. Let $C_1^* = \{c \in C^* \setminus \{c_1\} : c \text{ is adjacent to } b_1\}$. Then $C_1^* \neq \emptyset$.

Claim 12. *For every pair (c_j, d_j) with $c_j \in C_1^*$ one has that: d_j is adjacent to a_1, d_j is nonadjacent to any vertex of $A \setminus \{a_1\}$, d_j is nonadjacent to c_1, c_j is nonadjacent to d_1.*

Proof. First let us show that d_j is adjacent to a_1. By contradiction assume that d_j is nonadjacent to a_1. To avoid that $d_j, c_j, b_1, a_1, v, a_i$ for $i = 2, \ldots, h$ induce a P_6, d_j dominates $A \setminus \{a_1\}$. Then by (2) d_j is nonadjacent to c_1. Then $c_1, b_1, c_j, d_j, a_i, b_i$, for $i > 1$, induce a P_6 (contradiction). Then d_j is adjacent to a_1. Since G is $K_{2,3}$-free one obtains: d_j is nonadjacent to any vertex of $A \setminus \{a_1\}$; d_j is nonadjacent to c_1; c_j is nonadjacent to d_1. □

Finally let us prove that $C_1^* = C^* \setminus \{c_1\}$, i.e., that $(C^* \setminus \{c_1\}) \setminus C_1^* = \emptyset$. By contradiction assume that $(C^* \setminus \{c_1\}) \setminus C_1^* \neq \emptyset$. Since H is a minimal augmenting graph, there exists a vertex $c_q \in (C^* \setminus \{c_1\}) \setminus C_1^*$ adjacent to some vertex d_p such that $c_p \in C_1^* \cup \{c_1\}$ (also by Claim 10). In particular c_q is adjacent to d_1, otherwise $c_p \in C^* \setminus \{c_1\}$ and then $c_q, d_p, c_p, b_1, c_1, d_1$ induce a P_6 (also by Claim 11). Then d_q is adjacent to a_1: in fact otherwise to avoid that $d_q, c_q, d_1, a_1, v, a_2$ induce a P_6, d_q is adjacent to a_2; then to avoid that $b_2, a_2, d_q, c_q, d_1, c_1$ induce a P_6, d_q is adjacent to c_1; then $c_q, d_q, c_1, b_1, a_1, v$ induce a P_6. Furthermore d_q is nonadjacent to c_1, otherwise a_1, c_1, d_q, d_1, b_1 induce a $K_{2,3}$. Now, recalling that $C_1^* \neq \emptyset$, let us consider a vertex $c_j \in C_1^*$. Then d_q is adjacent to c_j, otherwise $d_q, c_q, d_1, c_1, b_1, c_j$ induce a P_6 (also by Claim 12). Then a_1, c_j, d_q, d_j, b_1 induce a $K_{2,3}$ (contradiction).

Then $C_1^* = C^* \setminus \{c_1\}$. Then by the above claims, H is of Type 3. This completes the proof of the lemma. ■

Lemma 13. *Assume that $C^* \neq \emptyset$. Then H can be detected in $O(n^3 m)$ time.*

Proof. By Lemma 7, H is of Type 1, or 2, or 3. Let us observe that one can easily determine the sets A, B, and C.

If H is of Type 1, see Figure 1, then let us proceed as follows. Clearly it is necessary that $|A| = 2$. Then for each vertex $b \in B \setminus N(a_2)$ (where b represents b_1) such that b has exactly one neighbor in C, say c_1, one has to check if there exists a stable set of $B \setminus N(b)$, say x, y (where x and y represent b_2 and d_1 respectively) with x adjacent to a_1, a_2 and nonadjacent to c_1, and with y adjacent to a_2, c_1 and nonadjacent to a_1 (then one should proceed similarly by interchanging a_1 with a_2, for a symmetry check).

If H is of Type 2, see Figure 1, then one can proceed in a similar way. Then assume that H is of Type 3. Then let us proceed as follows. Let us describe the procedure in case $|C^*| \geq 2$. The case in which $|C^*| = 1$ can be similarly treated. Let us say that a vertex of $H_1 \setminus \{v\}$ is *critical* for H if it has more than two neighbors in H. Then H contains one critical vertex, namely vertex b_1.

Let us say that a vertex $b \in B$ is *green* if it is a candidate to be critical for H, i.e., if $|N(b) \cap A| = 1$ and $|N(b) \cap C| \geq 2$. Thus there exists at least one green vertex which is critical for H. Let $b \in B$ be a green vertex. Let $N(b) \cap A = \{a_1\}$ (without loss of generality), and $N(b) \cap C = \{\tilde{c}_1, \ldots, \tilde{c}_m\}$. For every vertex $s \in A \cup C$ with $s \neq a_1$ let $M(s) = \{b' \in B : N(b') \cap (A \cup C) = \{s, a_1\}\}$.

Let $\tilde{d_j} \in M(\tilde{c}_j)$ for some $j \in \{1, \ldots, m\}$. Then every vertex $\tilde{d_r} \in M(\tilde{c}_r) \setminus (N(b) \cup N(\tilde{d_j}))$ in nonadjacent to any vertex $\tilde{d_t} \in M(\tilde{c}_t) \setminus (N(b) \cup N(\tilde{d_j}))$ for every $r, t \neq j$, otherwise $\tilde{d_r}, \tilde{d_t}, \tilde{c}_t, b, \tilde{c}_j, \tilde{d_j}$ induce a P_6.

Let $\tilde{b_i} \in M(a_i)$ for some $i \in \{2, \ldots, h\}$. Then every vertex $\tilde{b}_r \in M(a_r) \setminus N(\tilde{b_i})$ is nonadjacent to any vertex $\tilde{b}_t \in M(a_t) \setminus N(\tilde{b_i})$ for every $r, t \neq i$, otherwise $\tilde{b}_r, \tilde{b}_t, a_t, v, a_i, \tilde{b}_i$ induce a P_6.

Furthermore, if $|A| \geq 3$, then every vertex $\tilde{d_j} \in M(\tilde{c}_j)$ for $j = 1, \ldots, m$ is nonadjacent to any vertex $\tilde{b_i} \in M(a_i)$ for $i = 2, \ldots, h$.

Otherwise $\tilde{c}_j, \tilde{d_j}, \tilde{b}_i, a_i, v, a_{i+i}$ (or a_{i-i}) induce a P_6. Then by the above a green vertex b is critical for H if and only if there exists a pair of nonadjacent vertices, namely $\tilde{b_2}$ and $\tilde{d_1}$, with $\tilde{b_2} \in M(a_2)$ and $\tilde{d_1} \in M(\tilde{c}_1)$, such that $[M(a_i) \setminus (N(b) \cup N(\tilde{b_2}) \cup N(\tilde{d_1})) \neq \emptyset$, for all $i = 3, \ldots, h]$ AND $[M(\tilde{c_j}) \setminus (N(b) \cup N(\tilde{b_2}) \cup N(\tilde{d_1})) \neq \emptyset$, for all $j = 2, \ldots, k]$. Since that can be checked in $O(n^2 m)$ for every green vertex b, the lemma follows. ■

3.3. Summarizing

Then to solve MS for $(P_6, K_{2,3})$-free graphs one can apply Algorithm Alpha, referring to minimal augmenting graphs, whose Step 2 can be handled by Lemmas 6, 7 and 13.

Theorem 14. *The MS problem can be solved for $(P_6, K_{2,3})$-free graphs in $O(n^4 m)$ time.*

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