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# EDGE MAXIMAL $C_{2k+1}$-EDGE DISJOINT FREE GRAPHS

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**Abstract**

For two positive integers $r$ and $s$, $\mathcal{G}(n;r,s)$ denotes to the class of graphs on $n$ vertices containing no $r$ of $s$-edge disjoint cycles and $f(n;r,s) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;r,s)\}$. In this paper, for integers $r \geq 2$ and $k \geq 1$, we determine $f(n;r,2k+1)$ and characterize the edge maximal members in $\mathcal{G}(n;r,2k+1)$.

**Keywords:** extremal graphs, edge disjoint, cycles.

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## 1. Introduction

The graphs considered in this paper are finite, undirected and have no loops or multiple edges. Most of the notations that follow can be found in [5]. For a given graph $G$, we denote the vertex set of a graph $G$ by $V(G)$ and the edge set by $E(G)$. The cardinalities of these sets are denoted by $\nu(G)$ and $\mathcal{E}(G)$, respectively. The cycle on $n$ vertices is denoted by $C_n$.

Let $G_1$ and $G_2$ be graphs. The union of $G_1$ and $G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Two graphs $G_1$ and $G_2$ are vertex disjoint if and only if $V(G_1) \cap V(G_2) = \emptyset$; $G_1$ and $G_2$ are edge disjoint if $E(G_1) \cap E(G_2) = \emptyset$. If $G_1$ and $G_2$ are vertex disjoint, we denote their union by $G_1 + G_2$. The intersection $G_1 \cap G_2$ of graphs $G_1$ and $G_2$ is defined similarly, but in this case we need to assume that $V(G_1) \cap V(G_2) \neq \emptyset$. The join $G \vee H$ of two vertex disjoint graphs $G$ and $H$ is the graph obtained from $G + H$ by joining each vertex of $G$ to each vertex of $H$. For two vertex disjoint subgraphs $H_1$ and $H_2$ of $G$, we let $E_G(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}_G(H_1, H_2) = |E_G(H_1, H_2)|$.

In this paper we consider the Turán-type extremal problem with the odd edge disjoint cycles being the forbidden subgraph. Since a bipartite graph contains no odd cycles, the non-bipartite graphs have been considered by some authors. First, we recall some notations and terminologies. For a positive integer $n$ and a set of graphs $\mathcal{F}$, let $\mathcal{G}(n; \mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on $n$ vertices, and

$$f(n; \mathcal{F}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; \mathcal{F})\}.$$

For simplicity, in the case when $\mathcal{F}$ consists only of one member $C_s$, where $s$ is an odd integer, we write $\mathcal{G}(n; s) = \mathcal{G}(n; \mathcal{F})$ and $f(n; s) = f(n; \mathcal{F})$.

An important problem in extremal graph theory is that of determining the values of the function $f(n; \mathcal{F})$. Further, characterize the extremal graphs $\mathcal{G}(n; \mathcal{F})$ where $f(n; \mathcal{F})$ is attained. For a given $r$, the edge maximal graphs of $\mathcal{G}(n; r)$ have been studied by a number of authors [1, 2, 3, 7, 8, 9, 10, 12]. In 1998, Jia [11] proved the following result:

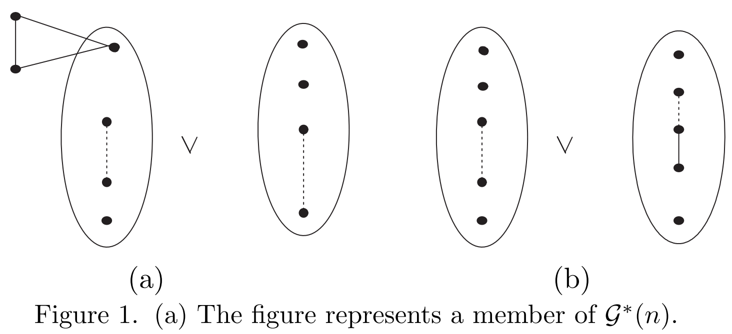

Figure 1. (a) The figure represents a member of $\mathcal{G}^*(n)$.
(b) The figure represents a member of $\Omega(n, 2)$.

**Theorem 1** (Jia). *Let $G \in \mathcal{G}(n; 5)$, $n \geq 10$. Then $\mathcal{E}(G) \leq \lfloor (n-2)^2/4 \rfloor + 3$. Furthermore, equality holds if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$. Figure 1(a) displays a member of $\mathcal{G}^*(n)$.*

Jia, also conjectured that $f(n; 2k+1) \leq \lfloor (n-2)^2/4 \rfloor + 3$ for all $n \geq 4k+2$. In 2007, Bataineh, confirmed positively the conjecture. In fact, he proved the following result:

**Theorem 2** (Bataineh). *Let $k \geq 3$ be a positive integer and $G \in \mathcal{G}(n; 2k+1)$. Then for large $n$, $\mathcal{E}(G) \leq \lfloor (n-2)^2/4 \rfloor + 3$.*

*Furthermore, equality holds if and only if $G \in \mathcal{G}^*(n)$ where $\mathcal{G}^*(n)$ is as above.*

Let $\mathcal{G}(n; r, s)$ denote to the class of graphs on $n$ vertices containing no $r$ of $s$-edge disjoint cycles and

$$f(n; r, s) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n; r, s)\}.$$

Note that

$$\mathcal{G}(n; 2, s) \subseteq \mathcal{G}(n; 3, s) \subseteq \cdots \subseteq \mathcal{G}(n; r, s).$$

Let $\Omega(n, r)$ denote to the class of graphs obtained by adding $r-1$ edges to the complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Figure 1(b) displays a member of $\Omega(n, 2)$.

The Turán-type extremal problem with two odd edge disjoint cycles being the forbidden subgraph, was studied by Bataineh and Jaradat [2]. In fact, they only established partial results by proving the following:

**Theorem 3** (Bataineh and Jaradat). *Let $k = 1, 2$ and $G \in \mathcal{G}(n; 2, 2k+1)$. Then for large $n$,*

$$\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + 1.$$

*Furthermore, equality holds if and only if $G \in \Omega(n, 2)$.*

In this paper, we continue the work initiated in [2] by generalizing and extending the above theorem. In fact, we determine $f(n; r, 2k+1)$ and characterize the edge maximal members in $\mathcal{G}(n; r, 2k+1)$. Now, we state a number of results, which play an important role in proving our result.

**Lemma 4** (Bondy and Murty). *Let $G$ be a graph on $n$ vertices. If $\mathcal{E}(G) > n^2/4$, then $G$ contains a cycle of length $r$ for each $3 \leq r \leq \lfloor (n+3)/2 \rfloor$.*

**Theorem 5** (Brandt). *Let $G$ be a non-bipartite graph with $n$ vertices and more than $\lfloor (n-1)^2/4 \rfloor + 1$ edges. Then $G$ contains all cycles of length between 3 and the length of the longest cycle.*

In the rest of this paper, $N_G(u)$ stands for the set of neighbors of $u$ in the graph $G$. Moreover, $G[X]$ denotes to the subgraph induced by $X$ in $G$.

## 2. Edge-Maximal $C_{2k+1}$-edge Disjoint Free Graphs

In this section, we determine $f(n;r,2k+1)$ and characterize the edge maximal members in $\mathcal{G}(n;r,2k+1)$. Observe that $\Omega(n,r) \subseteq \mathcal{G}(n;r,2k+1)$ and every graph in $\Omega(n,r)$ contains $\lfloor n^2/4 \rfloor + r - 1$ edges. Thus, we have established that

$$f(n;r,2k+1) \geq \lfloor n^2/4 \rfloor + r - 1. \tag{1}$$

In the following work, we establish that equality holds. Further we characterize the edge maximal members in $\mathcal{G}(n;r,2k+1)$.

**Theorem 6.** *Let $k \geq 1, r \geq 2$ be two positive integers and $G \in \mathcal{G}(n;r,2k+1)$. For large $n$,*

$$\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + r - 1.$$

*Furthermore, equality holds if and only if $G \in \Omega(n,r)$.*

***Proof.*** We prove the theorem using induction on $r$.
**Step 1.** We show the result for $r = 2$. Note that by Theorem 3, it is enough to prove the result for $k \geq 3$. Let $G \in \mathcal{G}(n,2,2k+1)$. If $G$ does not have a cycle of length $2k+1$, then by Lemma 4, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$. So, we need to consider the case when $G$ has cycles of length $2k+1$. Assume $C = x_1x_2 \ldots x_{2k+1}x_1$ be a cycle of length $2k+1$ in $G$. Consider $H = G - \{e_1 = x_1x_2, e_2 = x_2x_3, \ldots, e_{2k+1} = x_{2k+1}x_1\}$. Observe that $H$ cannot have $2k+1$-cycle as otherwise $G$ would have two edge disjoint $2k+1$-cycles. We now consider two cases according to $H$:

*Case* 1. $H$ is not a bipartite graph. If $k \geq 2$, then by Theorems 1 and 2

$$\mathcal{E}(H) \leq \lfloor (n-2)^2/4 \rfloor + 3.$$

But, $\mathcal{E}(G) = \mathcal{E}(H) + 2k + 1 \leq \lfloor \frac{(n-2)^2}{4} \rfloor + 2k + 4 \leq \lfloor \frac{n^2}{4} \rfloor - n + 2k + 5$. Thus, for $n \geq 2k+5$, we have $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + 1$. If $k = 1$, then by Theorems 5 $\mathcal{E}(H) \leq \lfloor (n-1)^2/4 \rfloor + 1$. And so, by using the same argument as in the above, we get that for $n \geq 7$,

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

*Case* 2. $H$ is a bipartite graph. Let $X$ and $Y$ be the partition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is when $|X| = \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lceil \frac{n}{2} \rceil$. Thus, $\mathcal{E}(H) \leq \lfloor \frac{n^2}{4} \rfloor$. Restore the edges of the cycle $C$. We now consider the following subcases:

(2.1). One of $X$ and $Y$ contains two edges of the cycle, say $e_i$ and $e_j$ belong to $X$. Let $y_1, y_2, \ldots, y_{k-1}$ be a set of vertices in $X - \{x_i, x_{i+1}, x_j, x_{j+1}\}$. We split this subcase into two subcases:

(2.1.1). $i$ and $j$ are not consecutive. Then $|N_Y(x_i) \cap N_Y(x_{i+1}) \cap N_Y(x_j) \cap N_Y(x_{j+1}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \cdots \cap N_Y(y_{k-1})| \leq k+2$, as otherwise $G$ contains two edge disjoint $2k+1$-cycles. Thus,

$$\mathcal{E}_G(\{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k+2)|Y| + k + 2.$$

So,

$$\begin{aligned}\mathcal{E}(G) &= \mathcal{E}_G(X - \{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) \\ &\quad + \mathcal{E}_G(\{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \mathcal{E}(G[X]) + \mathcal{E}(G[Y]) \\ &\leq (|X| - k - 3)|Y| + (k+2)|Y| + k + 2 + 2k + 1 \\ &\leq |X||Y| - |Y| + 3k + 3 \leq (|X| - 1)|Y| + 3k + 3.\end{aligned}$$

Observe that $|X| + |Y| = n$. The maximum of the above equation is when $|Y| = \lceil \frac{n-1}{2} \rceil$ and $|X| - 1 = \lfloor \frac{n-1}{2} \rfloor$. Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 3k + 3.$$

Hence, for $n \geq 6k + 7$, we have $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + 1$.

(2.1.2). $i$ and $j$ are consecutive, say $j = i + 1$. Then by following, word by word, the same arguments as in (2.1.1) and by taking into the account that $|N_Y(x_i) \cap N_Y(x_{i+1}) \cap N_Y(x_{j+2}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \cdots \cap N_Y(y_{k-1})| \leq k+1$ and so $\mathcal{E}(\{x_i, x_{i+1}, x_{i+2}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k+1)|Y| + k + 1$, we get the same inequality.

(2.2). $\mathcal{E}(G[X]) = 1$ and $\mathcal{E}(G[Y]) = 0$ or $\mathcal{E}(G[X]) = 0$ and $\mathcal{E}(G[Y]) = 1$, say $e_1 \in E(G[X])$. Then $G - e_1$ is a bipartite graph and so as in the above $\mathcal{E}(G - e_1) \leq \lfloor \frac{n^2}{4} \rfloor$. Thus, $\mathcal{E}(G) = \mathcal{E}(G - e_1) + 1 \leq \lfloor \frac{n^2}{4} \rfloor + 1$.

One can observe from the above arguments that for $r = 2$ the only time we have equality is in case when $G$ is obtained by adding an edge to the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. This gives rise to the class $\Omega(n, 2)$.

**Step 2.** Assume that the result is true for $r - 1$. We now show the result is true for $r \geq 3$. To accomplish that we use similar arguments to those in Step 1. Let $G \in \mathcal{G}(n; r, 2k+1)$. If $G$ contains no $r - 1$ edge disjoint cycles of length $2k + 1$, then by the inductive step $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + r - 2$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + r - 1$. So, we need to consider the case when $G$ has $r - 1$ edge disjoint cycles of length $2k + 1$. Assume that $\{C^i = x_{i1}, x_{i2}, \ldots, x_{i2k+1}, x_{i1}\}_{i=1}^{r-1}$ be the set of cycles of length $2k + 1$. Consider $H = G - \cup_{i=1}^{r-1} E(C^i)$. Observe that $H$ cannot have $2k + 1$-cycles as otherwise $G$ would have $r$ of edges disjoint $2k + 1$-cycles. As in Step 1, we consider two cases:

*Case* I. $H$ is not a bipartite graph. If $k \geq 2$, then by Theorems 1 and 2 $\mathcal{E}(H) \leq \lfloor (n-2)^2/4 \rfloor + 3$. Thus, $\mathcal{E}(G) = \mathcal{E}(H) + (r-1)(2k+1) \leq \lfloor \frac{n^2}{4} \rfloor + (r-1) - n + 4 + 2k(r-1)$. Hence, for $n > 4 + 2k(r-1)$, we have $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + r - 1$. If $k = 1$, then by Theorems 5 $\mathcal{E}(H) \leq \lfloor (n-1)^2/4 \rfloor + 1$.

By using the same argument as in the above, we get that for $n \geq 4(r-1)+1$,

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

*Case* II. $H$ is a bipartite graph. Let $X$ and $Y$ be the partition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is when $|X| = \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lceil \frac{n}{2} \rceil$. Thus, $\mathcal{E}(H) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. Now, we consider the following two subcases:

(II.I). There is $1 \leq m \leq r-1$ such that $C^m$ contains at least two edges, say $e_i = x_{mi}x_{m(i+1)}$ and $e_j = x_{mj}x_{m(j+1)}$, joining vertices of one of $X$ and $Y$ , say $X$. Let $y_1, y_2, \ldots, y_{k-1}$ be a set of vertices in $X - \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}\}$. To this end we have two subcases:

(II.I.I). $i$ and $j$ are not consecutive. Then $|N_Y(x_{mi}) \cap N_Y(x_{m(i+1)}) \cap N_Y(x_{mj}) \cap N_Y(x_{m(j+1)}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \cdots \cap N_Y(y_{k-1})| \leq k+2$, as otherwise $H \cup \{e_i, e_j\}$ contains two edges disjoint $2k+1$-cycles and so $G$ contains $r$ edge disjoint cycles of length $2k+1$. Thus, as in (2.1.1) of Step 1,

$$\mathcal{E}_H(\{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k+2)|Y| + k + 2.$$

And so,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(H) + |\cup_{i=1}^{r-1} E(C^i)| \\
&= \mathcal{E}_H(X - \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) \\
&+ \mathcal{E}_H(\{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y) + |\cup_{i=1}^{r-1} E(C^i)| \\
&\leq (|X| - k - 3)|Y| + (k+2)|Y| + k + 2 + (r-1)(2k+1) \\
&= (|X| - 1)|Y| + k + 2 + (r-1)(2k+1).
\end{aligned}$$

Moreover, the maximum of the above inequality is when $|Y| = \lceil \frac{n-1}{2} \rceil$ and $|X| - 1 = \lfloor \frac{n-1}{2} \rfloor$. Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + k + 2 + (r-1)(2k+1).$$

Hence, for $n \geq 6k(r-1) + 7$, we have $\mathcal{E}(G) < \lfloor \frac{n^2}{4} \rfloor + (r-1)$.

(II.I.II). $i$ and $j$ are consecutive, say $j = i + 1$. Then by following word by word the same arguments as in (2.1.2) of Step 1 and (II.I.I) of Step 2, we get the same inequality

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + (r-1).$$

(II.II). Each $1 \leq m \leq r-1$, $C^m$ has exactly one edge belonging to one of $X$ and $Y$. Let $e$ be the edge of $C^1$ that belongs to one of $X$ and $Y$. Then $G - e \in \mathcal{G}(n; r-1, 2k+1)$ and so by inductive step, $\mathcal{E}(G) = \mathcal{E}(G-e) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 2 + 1 = \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1$.

We now characterize the extremal graphs. Throughout the proof, we notice that the only time we have equality is in case when $G$ obtained by adding $r-1$ edges to the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. This gives rise to the class $\Omega(n, r)$. This completes the proof of the theorem. ■

From Theorem 6 and the inequality (1), we get that for $k \geq 1$, $r \geq 2$ and large $n$, $f(n; r, 2k+1) = \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1$.

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