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THE VERTEX MONOPHONIC NUMBER OF A GRAPH

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Abstract

For a connected graph G of order $p \geq 2$ and a vertex x of G, a set $S \subseteq V(G)$ is an *x-monophonic set* of G if each vertex $v \in V(G)$ lies on an $x - y$ monophonic path for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G, denoted by $m_x(G)$. An x-monophonic set of cardinality $m_x(G)$ is called a *m_x-set* of G. We determine bounds for it and characterize graphs which realize these bounds. A connected graph of order p with vertex monophonic numbers either $p-1$ or $p-2$ for every vertex is characterized. It is shown that for positive integers a, b and $n \geq 2$ with $2 \leq a \leq b$, there exists a connected graph G with $rad_m G = a, diam_m G = b$ and $m_x(G) = n$ for some vertex x in G. Also, it is shown that for each triple m, n and p of integers with $1 \leq n \leq p - m - 1$ and $m \geq 3$, there is a connected graph G of order p, monophonic diameter m and $m_x(G) = n$ for some vertex x of G.

Keywords: monophonic path, monophonic number, vertex monophonic number.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x, y)$ is called an *$x-y$ geodesic*. The *neighbourhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighbourhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$. A vertex v is a *simplicial vertex* if the subgraph induced by its neighbours is complete. A *nonseparable graph* is connected, nontrivial, and has no cut vertices. A *block* of a graph is a maximal nonseparable subgraph. A *connected block graph* is a connected graph in which each of its blocks is complete. A *caterpillar* is a tree for which the removal of all the end vertices gives a path. The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [1, 7] and further studied in [2, 3].

The concept of vertex geodomination number was introduced in [8] and further studied in [9]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-geodominating set* of G if each vertex v of G lies on an $x - y$ geodesic in G for some element y in S. The minimum cardinality of an *x-geodominating set* of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a *g_x-set*.

For vertices x and y in a connected graph G, the *detour distance* $D(x, y)$ is the length of a longest $x - y$ path in G. The *closed interval* $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of G, while for $S \subseteq V$, $I_D[S] = \bigcup_{x,y \in S} I_D[x, y]$. A set S of vertices is a *detour set* if $I_D[S] = V$, and the minimum cardinality of a detour set is the *detour number* $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set*. The detour number of a graph was introduced in [4] and further studied in [5]. The concept of vertex detour number was introduced in [10]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-detour set* if each vertex v of G lies on an $x - y$ detour in G for some element y in S. The minimum cardinality of an x-detour set of G is defined as the *x-detour number* of G and is denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a *d_x-set* of G.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. The *closed interval* $I_m[x, y]$ consists of all vertices lying on some $x - y$ monophonic path of G. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from

u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max \{d_m(v,u) : u \in V(G)\}$. The *monophonic radius*, $rad_m G$ of G is $rad_m G = \min \{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m G$ of G is $diam_m G = \max \{e_m(v) : v \in V(G)\}$. The monophonic distance was introduced and studied in [11]. The following theorems will be used in the sequel.

Theorem 1 [6]**.** *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) *v is a cut vertex of G.*

(ii) *There exist vertices u and w distinct from v such that v is on every $u-w$ path.*

(iii) *There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u-w$ path.*

Theorem 2 [6]**.** *Every nontrivial connected graph has at least two vertices which are not cut vertices.*

Theorem 3 [6]**.** *Let G be a connected graph with at least three vertices. The following statements are equivalent:*

(i) *G is a block.*

(ii) *Every two vertices of G lie on a common cycle.*

Theorem 4 [9]**.** *Let G be a connected graph of order $p \geq 3$ with exactly one cut vertex. Then the following are equivalent:*

(i) $g(G) = p - 1$.

(ii) $G = K_1 + \cup m_j K_j$, *where* $\Sigma m_j \geq 2$.

(iii) $g_x(G) = p - 1$ *or* $p - 2$ *for any vertex x in G.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. VERTEX MONOPHONIC NUMBER

Definition. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-monophonic set* if each vertex v of G lies on an $x-y$ monophonic path in G for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G and is denoted by $m_x(G)$ or simply m_x. An x-monophonic set of cardinality $m_x(G)$ is called a *m_x-set* of G.

We observe that for any vertex x in G, x does not belong to any m_x-set of G.

Example 5. For the graph G given in Figure 1, the minimum vertex monophonic sets and the vertex monophonic numbers are given in Table 1.1.

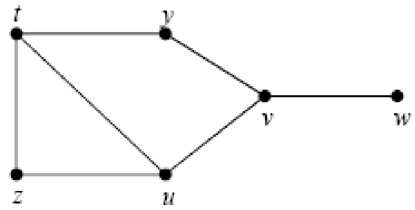

Figure 1

Vertex	Minimum vertex monophonic sets	Vertex monophonic number
t	{*z*,*w*}	2
y	{*z*,*w*}	2
z	{*w*}	1
u	{*z*,*w*,*y*}	3
v	{*z*,*w*}	2
w	{*z*}	1

Table 1.1

Theorem 6. *Let x be a vertex of a connected graph G.*

(i) *Every simplicial vertex of G other than the vertex x (whether x is simplicial vertex or not) belongs to every m_x-set.*

(ii) *No cut vertex of G belongs to any m_x-set.*

Proof. (i) Let x be a vertex of G. Then x does not belong to any m_x-set of G. Let $u \neq x$ be a simplicial vertex and S_x a m_x-set of G. Suppose that $u \notin S_x$. Then u is an internal vertex of an $x - y$ monophonic path, say P, for some $y \in S_x$. Let v and w be the neighbors of u on P. Then v and w are not adjacent and so u is not a simplicial vertex, which is a contradiction.

(ii) Let y be a cut vertex of G. Then by Theorem 1, there exists a partition of the set of vertices $V - \{y\}$ into subsets U and W such that for any vertex $u \in U$ and $w \in W$, the vertex y is on every $u - w$ path. Hence, if $x \in U$, then for any vertex w in W, y lies on every $x - w$ path so that y is an internal vertex of an $x - w$ monophonic path. Let S_x be any m_x-set of G. Suppose that $S_x \cap W = \emptyset$.

Then for any $w_1 \in W$, there exists an element z in S_x such that w_1 lies in some $x - z$ monophonic path $P : x = z_0, z_1, \ldots, w_1, \ldots, z_n = z$ in G. Now, the $x - w_1$ subpath of P and $w_1 - z$ subpath of P both contain y so that P is not a path in G, which is a contradiction. Hence $S_x \cap W \neq \emptyset$. Let $w_2 \in S_x \cap W$. Then y is an internal vertex of an $x - w_2$ monophonic path. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on an $x - y$ monophonic path also lies on an $x - w_2$ monophonic path. Hence it follows that S is an x-monophonic set of G, which is a contradiction since S_x is a minimum x-monophonic set of G. Thus y does not belong to any m_x-set. Similarly, if $x \in W$, y does not belong to any m_x-set. If $x = y$, then obviously y does not belong to any m_x-set. ■

Note 7. In Theorem 6, even if x is a simplicial vertex of G, x does not belong any m_x-set.

Corollary 8. *Let T be a tree with t end-vertices. Then $m_x(T) = t - 1$ or t according as x is an end-vertex or not. In fact, if W is the set of all end-vertices of T, then $W - \{x\}$ is the unique m_x-set of T.*

Proof. Let W be the set of all end-vertices of T. It follows from Note 7 and Theorem 6 that $W - \{x\}$ is the unique m_x-set of T for any end-vertex x in T and W is the unique m_x-set of T for any cut vertex x in T. Thus $W - \{x\}$ is the unique m_x-set of T for any vertex x in T. ■

Theorem 9. *For any vertex x in a graph G, $1 \leq m_x(G) \leq p - 1$.*

Proof. It is clear from the definition of a m_x-set that $m_x(G) \geq 1$. Also, since the vertex x does not belong to any m_x-set, it follows that $m_x(G) \leq p - 1$. ■

Remark 10. The bounds for $m_x(G)$ in Theorem 9 are sharp. The cycle $C_n (n \geq 4)$ has $m_x(C_n) = 1$ for every vertex x in C_n. Also, the non-trivial path P_n has $m_x(P_n) = 1$ for any end vertex x in P_n. The complete graph K_p has $m_x(K_p) = p - 1$ for every vertex x in K_p.

Now we proceed to characterize graphs G of order p for which the upper bound in Theorem 9 is attained.

Theorem 11. *For any graph G, $m_x(G) = p - 1$ if and only if deg $x = p - 1$.*

Proof. Let $m_x(G) = p - 1$. Suppose that deg $x < p - 1$. Then there is a vertex u in G which is not adjacent to x. Since G is connected, there is a monophonic path from x to u, say P, with length greater than or equal to 2. It is clear that $(V(G) - V(P)) \cup \{u\}$ is an x-monophonic set of G and hence $m_x(G) \leq p - 2$, which is a contradiction.

Conversely, if deg $x = p - 1$, then all other vertices of G are adjacent to x and hence all these vertices form the m_x-set. Thus $m_x(G) = p - 1$. ■

Corollary 12. *A graph G is complete if and only if $m_x(G) = p - 1$ for every vertex x in G.*

Now we proceed to characterize graphs for which the lower bound in Theorem 9 is attained. For this, we introduce the following definition.

Definition. Let x be any vertex in G. A vertex y in G is said to be an *x-monophonic superior vertex* if for any vertex z with $d_m(x, y) < d_m(x, z)$, z lies on an $x - y$ monophonic path.

Example 13. For any vertex x in the cycle C_n $(n \geq 4)$, $V(C_n) - N[x]$ is the set of all x-monophonic superior vertices.

Theorem 14. *For a vertex x in a graph G, $m_x(G) = 1$ if and only if there exists an x-monophonic superior vertex y in G such that every vertex of G is on an $x - y$ monophonic path.*

Proof. Let $m_x(G) = 1$ and let $S_x = \{y\}$ be a m_x-set of G. If y is not an x-monophonic superior vertex, then there is a vertex z in G with $d_m(x, y) < d_m(x, z)$ and z does not lie on any $x - y$ monophonic path. Thus S_x is not a m_x-set of G, which is a contradiction. The converse is clear from the definition. ■

The *n-dimensional cube* or *hypercube* Q_n is the simple graph whose vertices are the n-tuples with entries in $\{0, 1\}$ and whose edges are the pairs of n-tuples that differ in exactly one position.

Example 15. For $n \geq 2$, $m_x(Q_n) = 1$ for every vertex x in Q_n. Let $x = (a_1, a_2, \ldots, a_n)$ be any vertex in Q_n, where $a_i \in \{0, 1\}$. Let $y = (a'_1, a'_2, \ldots, a'_n)$ be another vertex of Q_n such that a'_i is the complement of a_i. Let u be any vertex in Q_n. For convenience, let $u = (a_1, a'_2, a_3, \ldots, a_n)$. Then u lies on the $x - y$ geodesic $x = (a_1, a_2, \ldots, a_n)$, $(a_1, a'_2, a_3, \ldots, a_n)$, $(a'_1, a'_2, a_3, \ldots, a_n)$, $(a'_1, a'_2, a'_3, \ldots, a_n), \ldots, (a'_1, a'_2, \ldots, a'_{n-1}, a_n)$, $(a'_1, a'_2, \ldots,$
$a'_n) = y$ and so u lies on an $x - y$ monophonic path.
Hence $m_x(Q_n) = 1$ for every vertex x in Q_n.

Theorem 16. (i) *For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 5)$, $m_x(W_n) = n - 1$ or 1 according as x is K_1 or x is in C_{n-1}.*

(ii) *Let $K_{m,n}$ $(m, n \geq 2)$ be a complete bipartite graph with bipartition (V_1, V_2) Then $m_x(K_{m,n})$ is $m - 1$ or $n - 1$ according as x is in V_1 or x is in V_2.*

Proof. (i) Let x be the vertex of K_1. Then by Theorem 11, $m_x(W_n) = n - 1$.

Let $C_{n-1} : u_1, u_2, u_3, \ldots, u_{n-1}, u_1$ be the cycle of W_n. Let x be any vertex in C_{n-1}, say $x = u_1$. It is clear that u_i $(i = 3, 4, \ldots, n - 2)$ is an x-monophonic superior vertex and every vertex of G lies on an $x - u_i$ monophonic path. Then by Theorem 14, $m_x(W_n) = 1$

(ii) Let $x \in V_1$. Then it is clear that $V_1 - \{x\}$ is a minimum x-monophonic set of G and so $m_x(K_{m,n}) = m-1$. Similarly, for any vertex $x \in V_2$, $m_x(K_{m,n}) = n-1$. ■

Now we characterize graphs G of order p having vertex monophonic number $m_x(G)$ equaling either $p-1$ or $p-2$ for every vertex x in G. First, we prove the following theorem.

Theorem 17. *Let G be a graph with k cut vertices. Then every vertex of G is either a cut vertex or a simplicial vertex if and only if $m_x(G) = p-k$ or $p-k-1$ for any vertex x in G.*

Proof. Let G be a graph with every vertex of G is either a cut vertex or a simplicial vertex. Since x does not belong to any m_x-set of G, it follows from Theorem 6 that $m_x(G) = p-k$ or $p-k-1$ according as x is a cut vertex or a simplicial vertex.

Conversely, suppose that $m_x(G) = p-k$ or $p-k-1$ for any vertex x in G. Suppose that there is a vertex x in G which is neither a cut vertex nor a simplicial vertex. Since x is not a simplicial vertex, the subgraph induced by $N(x)$ is not complete and hence there exist u and v in $N(x)$ such that $d(u,v) = 2$. Also, since x is not a cut vertex of G, $G - \{x\}$ is connected and hence there exists a $u-v$ geodesic say $P : u, u_1, \ldots, u_n, v$ in $G - \{x\}$. Then $P \cup \{v, x, u\}$ is a shortest cycle, say C, containing both the vertices u and v with length at least 4 in G. Let R be the set of all cut vertices of G. We consider two cases.

Case 1 u or v is not a cut vertex of G. Assume that u is not a cut vertex of G. Clearly, x lies on a $u-v$ monophonic path and hence $V(G) - (R \cup \{u, x\})$ is a u-monophonic set of G. Therefore $m_u(G) \le p-k-2$, which is a contradiction to the assumption.

Case 2. u and v are cut vertices of G. By Theorem 1, there exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for vertices $u_1 \in U$ and $w_1 \in W$, the vertex v is on every $u_1 - w_1$ path. Assume that $x \in U$. Let y be a vertex in W with maximum monophonic distance from v in W. By choice of y, y is not a cut vertex of G. Since the order of the cycle C is at least 4, $V(G) - (R \cup \{x, y\})$ is a y-monophonic set of G and so $m_y(G) \le p-k-2$, which is a contradiction to the assumption. Hence every vertex of G is either a cut vertex or a simplicial vertex. ■

Corollary 18. *Let G be a connected block graph with number of cut vertices k. Then $m_x(G) = p-k$ or $p-k-1$ for any vertex x in G.*

Proof. Let G be a connected block graph. Then every vertex of G is either a cut vertex or a simplicial vertex and hence by Theorem 17, $m_x(G) = p-k$ or $p-k-1$ for any vertex x in G. ■

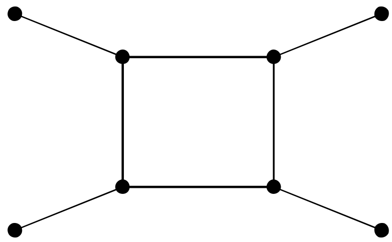

Figure 2

Note 19. The converse of Corollary 18 is not true. For the graph G given in Figure 2, $k = 4$ and $m_x(G) = p - k$ or $p - k - 1$ for any vertex x in G. However, it is not a connected block graph.

Theorem 20. *Let G be a connected graph. Then $G = K_1 + \bigcup m_jK_j$ if and only if $m_x(G) = p - 1$ or $p - 2$ for any vertex x in G.*

Proof. Let $G = K_1 + \cup m_jK_j$. Then G has at most one cut vertex. If G has no cut vertex, then $G = K_p$ and so by Corollary 12, $m_x(G) = p - 1$ for every vertex x in G. Suppose that G has exactly one cut vertex. Then all the remaining vertices are simplicial and hence by Theorem 17, $m_x(G) = p - 1$ or $p - 2$ for any vertex x in G.

Conversely, suppose that $m_x(G) = p - 1$ or $p - 2$ for any vertex x in G. If $p = 2$, then $G = K_2 = K_1 + K_1$. If $p \geq 3$, then by Theorem 2, there exists a vertex x, which is not a cut vertex of G. If G has two or more cut vertices, then by Theorem 6, $m_x(G) \leq p - 3$, which is a contradiction. Thus, the number of cut vertices k of G is at most one.

Case 1. $k = 0$. Then the graph G is a block. If $p = 3$, then $G = K_3 = K_1 + K_2$. For $p \geq 4$, we claim that G is complete. If G is not complete, then there exist two vertices x and y in G such that $d(x, y) \geq 2$. By Theorem 3, x and y lie on a common cycle and hence x and y lie on a smallest cycle $C : x, x_1, \ldots, y, \ldots, x_n, x$ of length at least 4. Then $V(G) - \{x, x_1, x_n\}$ is an x-monophonic set of G and so $m_x(G) \leq p - 3$, which is a contradiction to the assumption. Hence G is the complete graph K_p and so $G = K_1 + K_{p-1}$.

Case 2. $k = 1$. Let x be the cut vertex of G. If $p = 3$, then $G = P_3 = K_1 + m_jK_1$, where $\Sigma m_j = 2$. If $p \geq 4$, we claim that $G = K_1 + \cup m_jK_j$, where $\Sigma m_j \geq 2$. It is enough to prove that every block of G is complete. Suppose that there exists a block B, which is not complete. Let u and v be two vertices in B such that $d(u, v) \geq 2$. Then by Theorem 3, both u and v lie on a common cycle so that u and v lie on a smallest cycle of length at least 4. Then as in *Case* 1, $m_u(G) \leq p - 3$, which is a contradiction. Thus every block of G is complete so that $G = K_1 + \cup m_jK_j$, where K_1 is the vertex x and $\Sigma m_j \geq 2$. ∎

Theorem 21. *Let G be a connected graph of order $p \geq 3$ with exactly one cut vertex. Then $G = K_1 + \cup m_j K_j$, where $\Sigma m_j \geq 2$ if and only if $m_x(G) = p - 1$ or $p - 2$ for any vertex x in G.*

Proof. The proof is contained in Theorem 20. ∎

Theorem 22. *Let G be a connected graph of order $p \geq 3$ with exactly one cut vertex. Then the following are equivalent:*

(i) $g(G) = p - 1$.

(ii) $G = K_1 + \cup m_j K_j$, *where* $\Sigma m_j \geq 2$.

(iii) $g_x(G) = p - 1$ *or* $p - 2$ *for any vertex* x *in* G.

(iv) $m_x(G) = p - 1$ *or* $p - 2$ *for any vertex* x *in* G.

Proof. This follows from Theorems 4 and 21. ∎

Now, Corollary 12 and Theorem 20 lead to the natural question whether there exists a graph G for which $m_x(G) = p-2$ for every vertex x in G. This is answered in the next theorem.

Theorem 23. *There is no graph G of order p with $m_x(G) = p - 2$ for every vertex x in G.*

Proof. Suppose that there exists a graph G with $m_x(G) = p - 2$ for every vertex x in G. Let x be any vertex of G. Let S_x be a m_x-set of G so that $m_x(G) = |S_x| = p - 2$. Since $x \notin S_x$ and $m_x(G) = p - 2$, there exists exactly one vertex $y \neq x$ such that $y \notin S_x$. Hence y lies on the monophonic path x, y, w for some $w \in S_x$ and so y lies on the $x - w$ geodesic in G of length 2. We consider two cases.

Case 1. y is not a cut vertex of G. Then $G - \{y\}$ is connected and so there is an $x - w$ geodesic, say P, in $G - \{y\}$. Thus $C : P \cup (w, y, x)$ is a smallest cycle of length greater than or equal to 4. Hence $V(G) - \{x, y, w\}$ is a y-monophonic set of G and hence $m_y(G) \leq p - 3$, which is a contradiction to the assumption.

Case 2. y is a cut vertex of G. If deg $y = p - 1$, then by Theorem 11, $m_y(G) = p - 1$, which is a contradiction. If deg $y \leq p - 2$, then there exists a vertex u in G such that $d(u, y) \geq 2$. It is clear that $V(G) - I_m[u, y]$ is an u-monophonic set in G and so $m_u(G) \leq p - 3$, which is a contradiction to the assumption. Thus there is no graph G with $m_x(G) = p - 2$ for every vertex x in G. ∎

Theorem 24. *For every non-trivial tree T with monophonic diameter d_m, $m_x(T) = p - d_m$ or $p - d_m + 1$ for any vertex x in T if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let P be a monophonic path of length d_m. Let k be the number of end vertices of T and l be the number of internal vertices of T other than the internal vertices of P. Then $d_m - 1 + l + k = p$. By Corollary 8, $m_x(T) = k$ or $k - 1$ for any vertex x in G and so $m_x(T) = p - d_m - l + 1$ or $p - d_m - l$ for any vertex x in T. Hence $m_x(T) = p - d_m + 1$ or $p - d_m$ for any vertex x in T if and only if $l = 0$, if and only if all the internal vertices of T lie on the monophonic diametral path P, if and only if T is a caterpillar. ∎

For any connected graph G, $rad_mG \leq diam_mG$. It is shown in [11] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the vertex monophonic number can be prescribed.

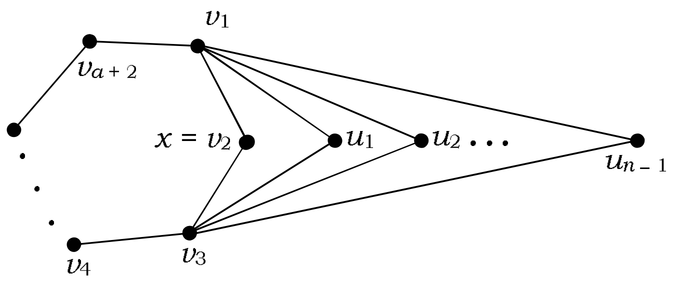

Figure 3

Theorem 25. *For positive integers a, b and $n \geq 2$ with $2 \leq a \leq b$, there exists a connected graph G with $rad_mG = a, diam_mG = b$ and $m_x(G) = n$ for some vertex x in G.*

Proof. We prove this theorem by considering four cases.

Case 1. $a = b$. Let $C_{a+2} : v_1, v_2, \ldots, v_{a+2}, v_1$ be a cycle of order $a + 2$. Let G be the graph obtained from C_{a+2} by adding $n - 1$ new vertices $u_1, u_2, \ldots, u_{n-1}$ and joining each vertex u_i $(1 \leq i \leq n - 1)$ to both v_1 and v_3. The graph G is shown in Figure 3. It is easily verified that the monophonic eccentricity of each vertex of G is a and so $rad_mG = diam_mG = a$. Also, for the vertex $x = v_2$, it is clear that $S = \{v_{a+2}, u_1, u_2, \ldots, u_{n-1}\}$ is a minimum x-monophonic set of G and so $m_x(G) = n$.

Case 2. $b = a + 1$. Let $C_{a+2} : v_1, v_2, \ldots, v_{a+2}, v_1$ be a cycle of order $a + 2$. Let G be the graph obtained from C_{a+2} by adding n new vertices $u_1, u_2, \ldots, u_n$ and joining each vertex u_i $(1 \leq i \leq n - 2)$ to both v_1 and v_3; joining the vertices u_{n-1}, u_n to v_{a+2}; and joining the vertices u_{n-1} and u_n. The graph G is shown in Figure 4. It is easily verified that $e_m(v_i) = a$ for $i = 1, 3, 4, \ldots, a + 2$ and $e_m(v_2) = a + 1$; $e_m(u_i) = a + 1$ for $i = 1, 2, 3, \ldots, n - 2$.

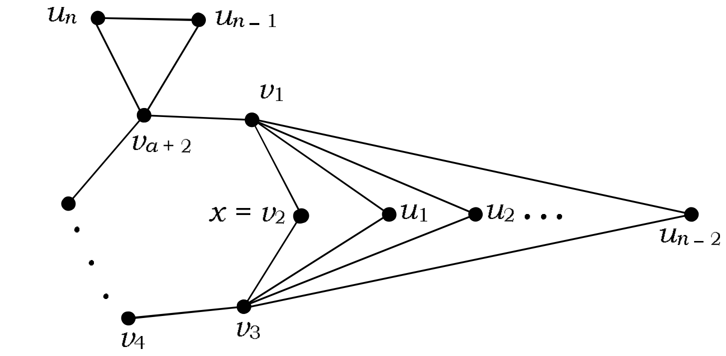

Figure 4

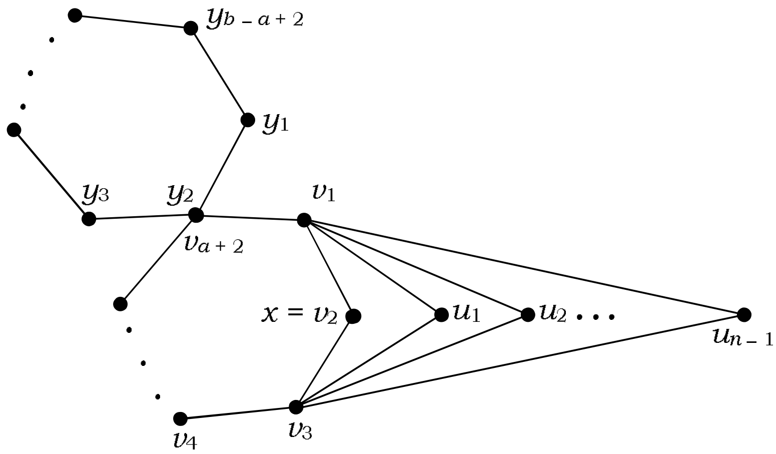

Figure 5

Hence $rad_m G = a$ and $diam_m G = a + 1 = b$. Also, for the vertex $x = v_2$, it is clear that $S = \{u_1, u_2, \ldots, u_n\}$ is a minimum x-monophonic set of G and so $m_x(G) = n$.

Case 3. $a + 2 \leq b \leq 2a$. Let $C_{a+2} : v_1, v_2, \ldots, v_{a+2}, v_1$ be a cycle of order $a+2$ and let $C_{b-a+2} : y_1, y_2, \ldots, y_{b-a+2}, y_1$ be a cycle of order $b-a+2$. Let G be the graph obtained by first identifying the vertex v_{a+2} of C_{a+2} and the vertex y_2 of C_{b-a+2}, and then adding $n-1$ new vertices $u_1, u_2, \ldots, u_{n-1}$ and joining each vertex u_i $(1 \leq i \leq n-1)$ to both v_1 and v_3. The graph G is shown in Figure 5. It is easily verified that $a \leq e_m(z) \leq b$ for any vertex z in G. Also, since $e_m(v_1) = a$ and $e_m(v_2) = b$, we have $rad_m G = a$ and $diam_m G = b$. Also, for the vertex $x = v_2$, it is clear that $S = \{u_1, u_2, \ldots, u_n\}$ is a minimum x-monophonic set of G and so $m_x(G) = n$.

Case 4. $b > 2a$. Let $P_{2a-1} : v_1, v_2, \ldots, v_{2a-1}$ be a path of order $2a-1$. Let G be the graph obtained from the wheel $W_n = K_1 + C_{b+2}$ and the complete

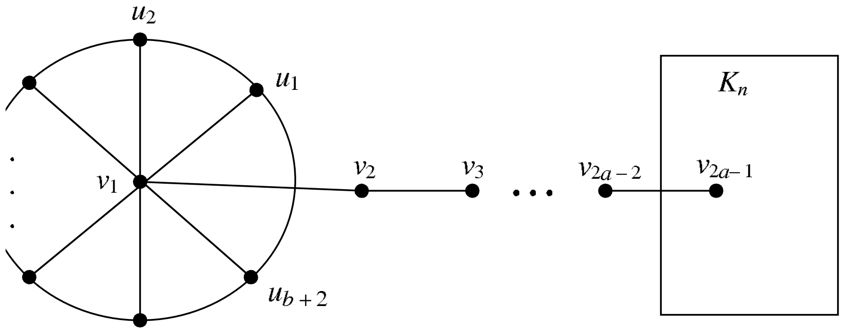

Figure 6

graph K_n by identifying the vertex v_1 of P_{2a-1} with the central vertex of W_n, and the vertex v_{2a-1} of P_{2a-1} with a vertex of K_n. The graph G is shown in Figure 6. Since $b > 2a$, we have $e_m(x) = b$ for any vertex $x \in V(C_{b+2})$. Also, $e_m(x) = 2a$ for any vertex $x \in V(K_n) - \{v_{2a-1}\}$; $a \leq e_m(x) \leq 2a-1$ for any vertex $x \in V(P_{2a-1})$; and $e_m(x) = a$ for the central vertex x of P_{2a-1}. Thus $rad_m G = a$ and $diam_m G = b$. Let $S = V(K_n) - \{v_{2a-1}\}$ be the set of all simplicial vertices of G. Then by Theorem 6(i), every m_x-set of G contains S for the vertex $x = u_2$. It is clear that S is not an x-monophonic set of G and so $m_x(G) > |S| = n - 1$. Then $S' = S \cup \{u_{b+2}\}$ is an x-monophonic set of G and so $m_x(G) = n$. ■

In the following, we construct a graph of prescribed order, monophonic diameter and vertex monophonic number under suitable conditions.

Theorem 26. *For each triple m, n and p of integers with $1 \leq n \leq p - m - 1$ and $m \geq 3$, there is a connected graph G of order p, monophonic diameter m and $m_x(G) = n$ for some vertex x of G.*

Proof. *Case* 1. $n = 1$. Let G be a graph obtained from the cycle C_{m+2} : $u_1, u_2, \ldots, u_{m+2}, u_1$ of order $m+2$ by adding $p-m-2$ new vertices $w_1, w_2, \ldots, w_{p-m-2}$ and joining each vertex w_i $(1 \leq i \leq p-m-2)$ to both u_1 and u_3. The graph G has order p and monophonic diameter m and is shown in Figure 7. It is clear that $\{u_{m+1}\}$ is an x-monophonic set of G for the vertex $x = u_1$ and so $m_x(G) = 1$.

Case 2. $2 \leq n \leq p - m - 1$. Let G be a graph obtained from the cycle $C_{m+1} : u_1, u_2, \ldots, u_{m+1}, u_1$ of order $m + 1$ by
(i) adding $n - 1$ new vertices $v_1, v_2, \ldots, v_{n-1}$ and joining each vertex v_i $(1 \leq i \leq n-1)$ to u_1; and
(ii) adding $p - m - n$ new vertices $w_1, w_2, \ldots, w_{p-m-n}$ and joining each vertex w_i $(1 \leq i \leq p - m - n)$ to both u_1 and u_3. The graph G has order p and monophonic diameter m and is shown in Figure 8. Let $S = \{v_1, v_2, \ldots, v_{n-1}\}$ be the set of all simplicial vertices of G.

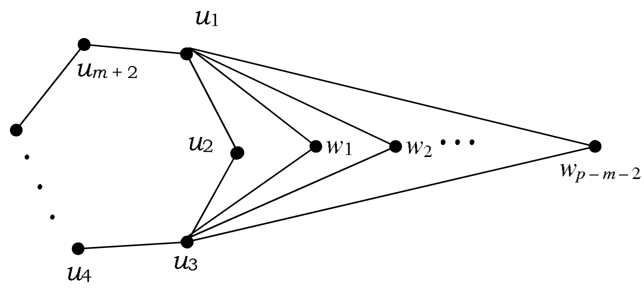

Figure 7

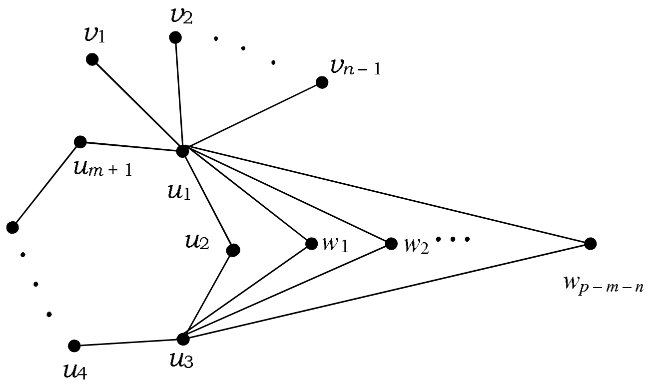

Figure 8

Then by Theorem 6(i), every x-monophonic set of G contains S for the vertex $x = u_1$. It is clear that S is not an x-monophonic set of G and so $m_x(G) > n-1$. Then $S' = S \cup \{u_m\}$ is an x-monophonic set of G and so $m_x(G) = n$. ■

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