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CHARACTERIZING CARTESIAN FIXERS AND MULTIPLIERS

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Abstract

Let $G \,\square\, H$ denote the Cartesian product of the graphs G and H. In 2004, Hartnell and Rall [On dominating the Cartesian product of a graph and K_2, Discuss. Math. Graph Theory 24(3) (2004), 389–402] characterized prism fixers, i.e., graphs G for which $\gamma(G \,\square\, K_2) = \gamma(G)$, and noted that $\gamma(G \square K_n) \geq \min\{|V(G)|, \gamma(G)+n-2\}$. We call a graph G a consistent fixer if $\gamma(G \square K_n) = \gamma(G)+n-2$ for each n such that $2 \leq n < |V(G)| - \gamma(G)+2$, and characterize this class of graphs.

Also in 2004, Burger, Mynhardt and Weakley [On the domination number of prisms of graphs, Dicuss. Math. Graph Theory **24**(2) (2004), 303–318] characterized prism doublers, i.e., graphs G for which $\gamma(G \,\square\, K_2) = 2\gamma(G)$. In general $\gamma(G \,\square\, K_n) \leq n\gamma(G)$ for any $n \geq 2$. We call a graph attaining equality in this bound a Cartesian n-multiplier and also characterize this class of graphs.

Keywords: Cartesian product, prism fixer, Cartesian fixer, prism doubler, Cartesian multiplier, domination number.

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1. Introduction

We generally follow the notation and terminology of [5]. For two graphs G and H, the Cartesian product $G \square H$ is the graph with vertex set $V(G) \times V(H)$ and vertex (v_i, u_j) adjacent to (v_k, u_l) if and only if (a) $v_i v_k \in E(G)$ and $u_j = u_l$, or (b) $v_i = v_k$ and $u_j u_l \in E(H)$. The graph $G \square K_2$ is called the *prism* of G.

As usual $\gamma(G)$ denotes the domination number of G. A set $D \subseteq V(G)$ is called a γ-*set* if it is a dominating set with $|D| = \gamma(G)$. The domination number $\gamma(G \square K_2)$ of the prism of G lies between $\gamma(G)$ and $2\gamma(G)$. The edgeless graph $G = \overline{K_m}$ attains equality in the lower bound, whereas $\gamma(K_m \square K_2) = 2\gamma(K_m)$.

In 2004, Hartnell and Rall [4] characterized graphs G, called *prism fixers*, for which $\gamma(G \square K_2) = \gamma(G)$. A γ-set D of G is called a *symmetric γ-set* if D can be partitioned into two nonempty subsets D_1 and D_2 such that $V(G) - N[D_1] = D_2$ and $V(G) - N[D_2] = D_1$. We write $D = D_1 \cup D_2$ for convenience. A symmetric γ-set $D = D_1 \cup D_2$ is called *primitive* if $|D_i| = 1$ for at least one i.

Theorem 1 [4]. *A connected graph G is a prism fixer if and only if G has a symmetric γ-set.*

Hartnell and Rall generalized the lower bound for $\gamma(G \square K_2)$ to $\gamma(G \square K_n)$ by utilizing one of their results in [3]. They confirmed that the lower bound is sharp by providing a family of graphs attaining equality.

Corollary 2 [4]. *For any graph G and $n \geq 2$, $\gamma(G \square K_n) \geq \min\{|V(G)|, \gamma(G) + n - 2\}$.*

Note that $\gamma(G \square K_n) = |V(G)|$ for the edgeless graph $G = \overline{K_m}$. Also, if $n \geq |V(G)| - \gamma(G) + 2$, then $\min\{|V(G)|, \gamma(G) + n - 2\} = |V(G)|$. A minimum domination strategy is to take all vertices in a single copy of G as a dominating set, hence $\gamma(G \square K_n) = |V(G)|$.

For $2 \leq n < |V(G)| - \gamma(G) + 2$, Corollary 2 gives a nontrivial lower bound, and a graph G is called a *Cartesian n-fixer* if $\gamma(G \square K_n) = \gamma(G) + n - 2$. We henceforth simply refer to a Cartesian n-fixer as an *n-fixer*. Furthermore, if G is an n-fixer for each n such that $2 \leq n < |V(G)| - \gamma(G) + 2$, then G is called a *consistent fixer*. We characterize these graphs in Section 2. In Section 3 we discuss graphs that are n-fixers for only some values of n in the range $2 \leq n < |V(G)| - \gamma(G) + 2$. In 2004, Burger, Mynhardt and Weakley [1] characterized *prism doublers*, i.e., graphs G for which $\gamma(G \square K_2) = 2\gamma(G)$. In general $\gamma(G \square K_n) \leq n\gamma(G)$ for any $n \geq 2$, and a graph attaining equality in this upper bound is called a *Cartesian n-multiplier*. Once again, we refer to such a graph simply as an *n-multiplier*. In Section 4 we follow a similar argument to that in [1] to characterize n-multipliers.

For $A, B \subseteq V(G)$, we abbreviate "A dominates B" to "$A \succ B$"; if $B = V(G)$ we write $A \succ G$ and if $B = \{b\}$ we write $A \succ b$. Further, $N(v) = \{u \in V(G) :$

$uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ denote the *open* and *closed neighbourhoods*, respectively, of a vertex v of G. The *closed neighbourhood* of $S \subseteq V(G)$ is the set $N[S] = \bigcup_{s \in S} N[s]$, the *open neighbourhood* of S is $N(S) = \bigcup_{s \in S} N(s)$, while $N\{S\}$ denotes the set $N(S) - S$.

Consider two graphs G and H, with vertex sets labelled $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ respectively. Vertices (v_i, u_j) of the Cartesian product $G \,\square\, H$ are labelled $v_{i,j}$ for convenience. The subgraph induced by all vertices that differ from a given vertex $v_{i,j}$ only in the first [second] coordinate, is known as the (Cartesian) *G-layer* [*H-layer*] through $v_{i,j}$.

We often consider projections $p_G : V(G \,\square\, H) \to V(G)$ and $p_H : V(G \,\square\, H) \to V(H)$. A general vertex $v_{i,j}$ of $G \,\square\, H$ has as first coordinate the vertex $p_G(v_{i,j}) = v_i \in V(G)$ and second coordinate $p_H(v_{i,j}) = u_j \in V(H)$. The *preimage* $p_G^{-1}(v_i)$ of a vertex v_i in G is the set of vertices in $G \,\square\, H$ that have v_i as first coordinate, that is, the vertex set of the H-layer through $v_{i,j}$ for any j. The *preimage* of $A \subseteq V(G)$ is the set $p_G^{-1}(A) = \bigcup_{v \in A} p_G^{-1}(v)$. The projection p_G and preimage p_G^{-1} are abbreviated to p and p^{-1} respectively.

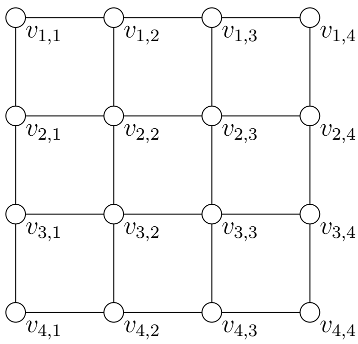

Figure 1. The Cartesian product $P_4 \,\square\, P_4$.

As an example, consider the graph $P_4 \,\square\, P_4$ in Figure 1. For this graph we have $p(\{v_{1,3}, v_{3,2}\}) = \{v_1, v_3\}$, while $p^{-1}(\{v_1, v_3\}) = \{v_{i,j} : i = 1, 3,\ j = 1, 2, 3, 4\}$. Lastly, a dominating set W of $G \,\square\, H$ can be partitioned into sets $W_1, W_2, \ldots, W_n$, where W_i is a subset of vertices in the i^{th} G-layer. We write $W = W_1 \cup W_2 \cup \cdots \cup W_n$ when this partition is clear from the context.

2. Consistent Fixers

Hartnell and Rall [4] provided examples of graphs that show that the lower bound in Corollary 2 is sharp. Let G_k be the graph with vertex set $V(G_k) = \{v\} \cup \{x_i, y_i, z_i : i = 1, 2, \ldots, k\}$, and edge set $\{vx_i, x_iy_i, y_iz_i, z_iv : i = 1, 2, \ldots, k\}$.

(The 4-cycles $G_k[\{v, x_i, y_i, z_i\}]$ share a common vertex v, $i = 1, 2, \ldots, k$.) Then $\gamma(G_k) = k + 1$ and $D = \{(y_i, u_1) : i = 1, 2, \ldots, k\} \cup \{(v, u_j) : j = 2, 3, \ldots, n\}$ is a dominating set of $G_k \,\square\, K_n$ of cardinality $k + n - 1 = \gamma(G_k) + n - 2$. The graph G_3 is illustrated in Figure 2. If $k > \frac{n-2}{2}$, then $|V(G_k)| = 3k + 1 > k + n - 1$ and hence $\gamma(G_k \,\square\, K_n) = \gamma(G_k) + n - 2$.

For the graph G_3 in Figure 2, let $D_1 = \{y_1, y_2, y_3\}$ and $D_2 = \{v\}$, and note that $D = D_1 \cup D_2$ is a primitive symmetric γ-set of G_3. In general, any graph G that has a primitive symmetric γ-set satisfies $\gamma(G \,\square\, K_n) = \gamma(G) + n - 2$ for any $2 \le n < |V(G)| - \gamma(G) + 2$:

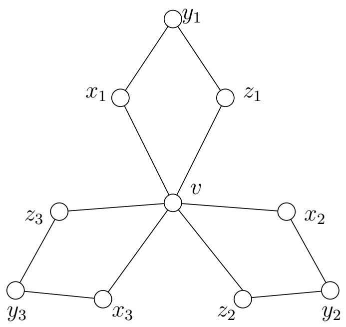

Figure 2. The graph G_3.

Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $D = D_1 \cup D_2$ be a primitive symmetric γ-set of G with $D_2 = \{x\}$. Figure 3 illustrates the dominating set $W = \{(v, u_1) : v \in D_1\} \cup \{(x, u_i) : i = 2, 3, \ldots, n\}$ of $G \,\square\, K_n$ of cardinality $\gamma(G) + n - 2$. In the first G-layer, the set $Y = V(G) - D$ is dominated by $\{(v, u_1) : v \in D_1\}$, and in the i^{th} G-layer Y is dominated by (x, u_i), $i \ge 2$.

The question now arises whether graphs with primitive symmetric γ-sets are the only n-fixers. Our characterization will show that this is not the case.

We first state some useful properties of a graph having a symmetric γ-set.

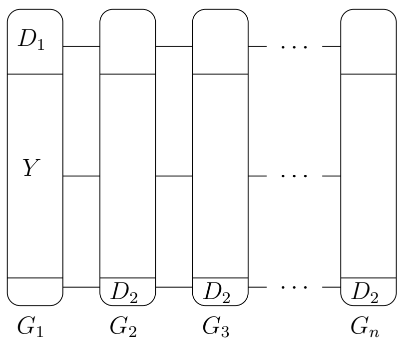

Figure 3. A domination strategy for $G \,\square\, K_n$ if G has a primitive symmetric γ-set.

Observation 3 [4]**.**

(i) *Let G be a connected graph with symmetric γ-set $D = D_1 \cup D_2$ and let $Y = V(G) - D$. Then*

(a) $N[D_i] = D_i \cup Y$, $i = 1, 2$,

(b) *D is an independent set,*

(c) *the sets $\{N(x)\}_{x \in D_i}$ are disjoint, and these sets form a partition of Y,*

(d) *each vertex in D is adjacent to at least two vertices in Y.*

(ii) *Let G be a graph with at least one symmetric γ-set, but no primitive symmetric γ-set, and let $Y = V(G) - D$. Then $\gamma(G[Y]) > 1$.*

(iii) *If G is a 2-fixer and $W = W_1 \cup W_2$ is a γ-set of $G \,\square\, K_2$, then $p(W_1) \cup p(W_2)$ is a symmetric γ-set of G.*

Suppose G is a 2-fixer with no primitive symmetric γ-set and $\gamma(G \square K_3) = \gamma(G)+1$. Then a minimum domination strategy for the Cartesian product $G \square K_3$ will never be to take a γ-set of $G \,\square\, K_2$ and select one vertex in the third G-layer, as we show next.

Lemma 4. *Let G be a connected 3-fixer with symmetric γ-set $D = D_1 \cup D_2$, but no primitive symmetric γ-set. Then no γ-set $W = W_1 \cup W_2 \cup W_3$ of $G \,\square\, K_3$ has $p(W_1) = D_1$, $p(W_2) = D_2$ and $|W_3| = 1$.*

Proof. Let $D = D_1 \cup D_2$ be a symmetric γ-set of G with $|D_1|, |D_2| \geq 2$ and let $Y = V(G) - D$. Suppose $W = W_1 \cup W_2 \cup W_3$ is a γ-set of $G \,\square\, K_3$, with $p(W_1) = D_1$, $p(W_2) = D_2$ and $W_3 = \{(x, u_3)\}$. Then $x \succ Y$. If $x \notin D$, then $x \in Y$ and so $\gamma(G[Y]) = 1$, contradicting Observation 3(ii). So assume $x \in D$, say $x \in D_2$, and let $z \in D_2 - \{x\}$. Then z is adjacent to some vertex in Y, hence x and z have a common neighbour in Y, contradicting Observation 3(i)(c). ■

We now provide a characterization of consistent fixers. We only consider connected graphs and also require G to have at least three vertices; since $\gamma(G) \leq \frac{1}{2}|V(G)|$ for any connected graph G, this requirement ensures that a value $n \geq 3$ is included in the range $2 \leq n < |V(G)| - \gamma(G) + 2$.

Theorem 5. *Let G be a connected graph of order at least 3. Then G is a consistent fixer if and only if*

(i) *G has a primitive symmetric γ-set, or*

(ii) *G has symmetric γ-sets, none of which are primitive, and G has a dominating set $X = X_1 \cup X_2 \cup X_3$ with the following properties:*

(a) $X_i \succ V(G) - X$, $i = 1, 2, 3$,

(b) *for each $i = 1, 2, 3$, the sets $\{N(x) - X\}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,*

(c) *the sets* X_i *are disjoint and* $|X| = |X_1| + |X_2| + |X_3| = \gamma(G) + 1$,

(d) $|X_2| = |X_3| = 1$.

Proof. Let G be a consistent fixer. Then by Theorem 1, G has a symmetric γ-set $D = D_1 \cup D_2$. Suppose $|D_1|, |D_2| \geq 2$ for any such set D. We show that (ii) holds.

Since G is also a Cartesian 3-fixer, there exists a minimum dominating set $W = W_1 \cup W_2 \cup W_3$ of $G \,\square\, K_3$ of cardinality $\gamma(G)+1$. Let $X_i = p(W_i)$, $i = 1, 2, 3$, $X = X_1 \cup X_2 \cup X_3$ and $Y = V(G) - X$.

Then $X \subseteq V(G)$ is a dominating set of G of cardinality at most $\gamma(G)+1$, i.e., $\gamma(G) \leq |X| \leq \gamma(G) + 1$. If $Y = \emptyset$, then $|V(G)| = |X| \leq \gamma(G) + 1$, contradicting the statement $3 < |V(G)| - \gamma(G) + 2$. Therefore $Y \neq \emptyset$, and so to dominate $p^{-1}(Y)$, $W_i \neq \emptyset$ for each i. Hence $X_i \neq \emptyset$ and, moreover, $X_i \succ Y$ for each $i = 1, 2, 3$. Thus (a) holds.
Without loss of generality, assume that $|X_1| \geq |X_2| \geq |X_3|$ and that W has been chosen so that $|X_1|$ is as large as possible. Since $\gamma(G) \leq |X| \leq \gamma(G) + 1$,

(1) at most one vertex of X occurs in more than one set X_i.

Similarly, no vertex occurs in all three X_i, i.e.,

(2) $$X_1 \cap X_2 \cap X_3 = \emptyset.$$

We now prove the following statement:

(3) Each vertex in $X_2 \cup X_3$ is adjacent to some vertex in Y.

Suppose there exists $x \in X_2$ that is not adjacent to any vertex in Y, and w_2 is a vertex of W_2 such that $p(w_2) = x$. (The argument is the same if $x \in X_3$.) If $x \in X_1$ and w_1 is a vertex of W_1 such that $p(w_1) = x$, then $W - \{w_1\}$ is a dominating set of $G \,\square\, K_3$ of cardinality $\gamma(G)$, which is impossible by Corollary 2. Thus $x \notin X_1$. But then $W' = (W_1 \cup \{w_1\}) \cup (W_2 - \{w_2\}) \cup W_3$ is a minimum dominating set of $G \,\square\, K_3$ such that $X_1' = p(W_1 \cup \{w_1\}) = X_1 \cup \{x\}$ has larger cardinality than X_1, contradicting the choice of W. Thus (3) holds.

(b) Suppose two distinct vertices $u, v \in X_i$ are both adjacent to some vertex $y \in Y$. By (a), y is adjacent to a vertex in each X_i. By (1) and (2), at least one X_j, $j \neq i$, contains a neighbour w of y such that $w \notin \{u, v\}$. But $X_k \succ Y$, $k \neq i, j$, so $(X - \{u, v, w\}) \cup \{y\}$ is a dominating set of G that has cardinality at most $\gamma(G) - 1$, a contradiction. Hence each vertex $y \in Y$ is dominated by exactly one vertex from X_i, and (b) follows.

(c) We only prove that $X_2 \cap X_3 = \emptyset$; the proofs that $X_1 \cap X_2 = \emptyset$ and $X_1 \cap X_3 = \emptyset$ are similar. It will follow that $|X| = |X_1| + |X_2| + |X_3| = \gamma(G) + 1$. Suppose there exists a vertex $z \in X_2 \cap X_3$. Then $|X| = \gamma(G)$ and, by (1) and (2), $X_1 \cap (X_2 \cup X_3) = \emptyset$, so that $X = X_1 \cup (X_2 \cup X_3)$ is a symmetric γ-set of G.

If $|X_3| = 1$, then $X_3 = \{z\} \subseteq X_2$ and $X = X_1 \cup X_2$. By (a), z dominates all of Y. But $z \in X_2$, and so (b) implies that $X_2 = \{z\}$, i.e., $|X_2| = 1$. Then X is a primitive symmetric γ-set, which is not the case under consideration. Therefore $|X_3| \geq 2$; say $w, z \in X_3$. By (1), $w \notin X_1 \cup X_2$, and by (3), w is adjacent to some vertex in Y. Since $X_2 \succ Y$, there exists $v \in X_2$ such that v and w have a common neighbour in Y. This contradicts Observation 3(i)(c) for the symmetric γ-set $X = X_1 \cup (X_2 \cup X_3)$. Therefore $X_2 \cap X_3 = \emptyset$.

(d) Suppose that $|X_2| \geq 2$. Then $|X_1| \geq 2$. Let $y_1 \in Y$ and choose $x_1 \in X_1$, $x_2 \in X_2$ such that x_1 and x_2 are both adjacent to y_1. Since $X_3 \succ Y$, the set $X' = (X - \{x_1, x_2\}) \cup \{y_1\}$ is a dominating set of G of cardinality $\gamma(G)$, i.e., a γ-set of G. We show that

$$\{x_1, x_2\} \succ Y. \tag{4}$$

Suppose to the contrary that $y \in Y$ is not adjacent to either x_1 or x_2. Then there exist $x_1' \in X_1 - \{x_1\}$ and $x_2' \in X_2 - \{x_2\}$ adjacent to y, so that $(X' - \{x_1', x_2'\}) \cup \{y\}$ is a dominating set of G of cardinality $\gamma(G) - 1$, which is impossible.

Let $v \in X_2 - \{x_2\}$. By (3) there exists a vertex $y_2 \in Y$ adjacent to v. By (b) y_2 is not adjacent to x_2 and so, by (4), y_2 is adjacent to x_1. It follows similar to (4) that $\{x_1, v\} \succ Y$. But then any vertex in Y not adjacent to x_1 is adjacent to both x_2 and v, which is impossible by (b). Thus $x_1 \succ Y$, and (b) implies that $|X_1| = 1$, a contradiction. Therefore $|X_2| = 1$ which, by the choice of the X_i, also implies that $|X_3| = 1$.

Conversely, let G be a graph that satisfies the conditions of the statement, $2 \leq n < |V(G)| - \gamma(G) + 2$ and $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. If G has a symmetric γ-set $D = D_1 \cup D_2$ with $D_2 = \{x\}$, then the set $W = \{(v, u_1) : v \in D_1\} \cup \{(x, u_i) : i = 2, 3, \ldots, n\}$ is a dominating set of $G \,\square\, K_n$ of cardinality $\gamma(G) + n - 2$, as illustrated in Figure 2.

Suppose that $|D_1|, |D_2| \geq 2$ and that G has a set $X = X_1 \cup X_2 \cup X_3$ with the stated properties. Let $X_2 = \{x_2\}$ and $X_3 = \{x_3\}$. Then the set

$$W = \{(v, u_1) : v \in X_1\} \cup \{(x_2, u_2)\} \cup \{(x_3, u_i) : i = 3, 4, \ldots, n\}$$

is a dominating set of $G \,\square\, K_n$ of cardinality $\gamma(G) + n - 2$. ■

The dominating set $X = X_1 \cup X_2 \cup X_3$ in Theorem 5(ii) has the following additional properties.

Proposition 6. *Let G be a connected graph of order at least 3. If G is a consistent fixer with no primitive symmetric γ-set, then the dominating set $X = X_1 \cup X_2 \cup X_3$ in Theorem 5(ii) has the following properties:*

(i) $X_1 \cup X_2$ *and* $X_1 \cup X_3$ *are independent sets,*

(ii) $\gamma(G[N(x)]) \geq 2$ *for every* $x \in X_1$,

(iii) *for some* $x \in X_1$, $G[N(x)]$ *has a* γ*-set,* $\{y_1, y_2\}$ *say, such that for every* $x' \in X_1 - \{x\}$,

  (a) $y_1 \succ N(x')$ *and* $N(y_2) \cap N(x') = \emptyset$, *or*

  (b) $y_2 \succ N(x')$ *and* $N(y_1) \cap N(x') = \emptyset$.

Proof. Say $X_2 = \{x_2\}$, $X_3 = \{x_3\}$, $Y = V(G) - X$, and note that

$$x_i \succ Y, \ i = 2, 3. \tag{5}$$

(i) Consider any symmetric γ-set $D = D_1 \cup D_2$ of G and recall that $|D_i| \geq 2$. Define $Y' = V(G) - D$. We compare D and X, and show that

$$\begin{aligned} &|D_i \cap Y| = 1 \text{ for } i = 1, 2, \quad |D \cap X_1| = \gamma(G) - 2 = |X_1| - 1, \\ &\text{and} \quad |X_1 \cap Y'| = 1. \end{aligned} \tag{6}$$

We begin by showing that $\{x_2, x_3\} \cap D = \emptyset$. Suppose $x_2 \in D$; without loss of generality say $x_2 \in D_2$. Then (5) and Observation 3(i)(b) imply that $Y \cap D = \emptyset$. Now if $x_3 \in D$, then Observation 3(i)(c) implies that $x_3 \in D_1$ and that the only vertices in $X_1 \cap D$ are vertices that are nonadjacent to all vertices in Y. But $|X| = \gamma(G) + 1$, $|X_1| = \gamma(G) - 1$ and $|D| = \gamma(G)$, so that $\gamma(G) - 2$ vertices in X_1 are in D. Therefore exactly one vertex in X_1, say x_1, is adjacent to vertices in Y. By Theorem 5(ii)(a), $x_1 \succ Y$. Furthermore, $x_1 \in Y'$ by Observation 3(i)(c). If there exists a $v \in X_1 - \{x_1\}$, then $v \in D$, hence v is adjacent to at least two vertices in Y' by Observation 3(i)(d). Since $Y' - \{x_1\} = Y$, this is a contradiction. So $X_1 = \{x_1\}$ and it follows that D is a primitive symmetric γ-set, a contradiction. Therefore $x_3 \notin D$ and so $D = X_1 \cup X_2$ and $V(G) - D = Y \cup \{x_3\}$.

Let $u \in D_2 - \{x_2\}$. By Observation 3(i)(d), u is adjacent to at least two vertices in Y', so u is adjacent to some $y \in Y$. But then y is adjacent to the two vertices $x_2, u \in D_2$, contradicting Observation 3(i)(c). Hence $x_2 \notin D$. Similarly, $x_3 \notin D$, i.e., $\{x_2, x_3\} \subseteq Y'$.

Since $|X_1| = \gamma(G) - 1$, it follows that $Y \cap D \neq \emptyset$. If $|D_i \cap Y| \geq 2$ for some i, then by (5), two vertices in D_i have $x_2 \in Y'$ as common neighbour, contrary to Observation 3(i)(c). Thus $|D_i \cap Y| \leq 1$ for each i, so $|Y \cap D| \leq 2$. If $Y \cap D = \{y\}$, then $D = X_1 \cup \{y\}$. But by Theorem 5(ii)(a), y is adjacent to some vertex in X_1, contradicting Observation 3(i)(b). Therefore $|Y \cap D| = 2$ and (6) follows.

Let $X_1 \cap Y' = \{x_1\}$ and $D_i \cap Y = \{y_i\}$, $i = 1, 2$. Then $X_1 - \{x_1\} \subseteq D$ and so $X_1 - \{x_1\}$ is independent (Observation 3(i)(b)).

Suppose x_1 is not adjacent to y_1. Since $X_1 \succ Y$, y_1 is adjacent to some $x' \in X_1 - \{x_1\} \subseteq D$. But $y_1 \in D$ and D is independent, a contradiction. Hence

x_1 is adjacent to y_1 and, similarly, to y_2. It now follows from Observation 3(i)(c) that x_1 is not adjacent to any vertex in X_1 and so X_1 is independent.

By (5), x_2 and x_3 are adjacent to y_1 and y_2, hence as in the case of x_1, neither x_2 nor x_3 is adjacent to any vertex in $X_1 - \{x_1\}$. Since G is connected, each vertex in $X_1 - \{x_1\}$ is therefore adjacent to a vertex in Y; since D is independent this vertex is necessarily in $Y - \{y_1, y_2\}$. Since $|D_1| \geq 2$, there exists $x_4 \in D_1 - \{y_1\}$; necessarily $x_4 \subseteq X_1 - \{x_1\}$. Let $y_4 \in Y - \{y_1, y_2\}$ be adjacent to x_4 and consider the set $X' = (X - \{x_1, x_3, x_4\}) \cup \{y_4\}$. Then $x_2 \succ Y$, $y_4 \succ x_4$ and $y_4 \succ x_3$ by (5). Therefore $X' \succ G - x_1$. But $|X'| < \gamma(G)$ and so $X' \not\succ G$, i.e., $X' \not\succ x_1$. In particular, x_2 is not adjacent to x_1. Similarly, x_3 is not adjacent to x_1, and the proof of (i) is complete.

(ii) Since $\gamma(G) \geq 4$, $|X_1| \geq 3$. Say $X_1 = \{x_1, x_4, x_5, \ldots, x_k\}$ and define $Y_i = N(x_i)$, $i = 1, 4, 5, \ldots, k$. By (i), no vertex in X_1 is adjacent to any vertex in X, so $Y_i \subseteq Y$ for each i, and since G is connected, $Y_i \neq \emptyset$. By Theorem 5(ii)(a) and (b), the sets $Y_1, Y_4, \ldots, Y_k$ partition Y. Suppose that for some i there exists a vertex $y \in Y_i$ that is adjacent to all other vertices in Y_i and consider $X' = (X - \{x_i, x_2, x_3\}) \cup \{y\}$. Then by (5), $y \succ Y_i \cup \{x_i, x_2, x_3\}$, while $X_1 - \{x_i\} \succ Y - Y_i$, so that $X' \succ G$. But $|X'| = \gamma(G) - 1$, which is impossible. This proves (ii).

(iii) As shown above, $D = \{y_1, y_2, x_4, \ldots, x_k\}$ and $Y' = \{x_1, x_2, x_3\} \cup (Y - \{y_1, y_2\})$. By Observation 3(i)(c), each vertex in Y' is adjacent to exactly one vertex in each D_i. In particular, since X_1 is independent, x_1 is adjacent to y_1 and y_2. Since the Y_i partition Y, no vertex in Y is adjacent to two vertices in X_1. But for each $i = 4, \ldots, k$, x_i is in exactly one of D_1 or D_2, so if $x_i \in D_1 - \{y_1\}$, then each vertex in $Y_i = N(x_i)$ is also adjacent to y_2 but not to y_1, and if $x_i \in D_2 - \{y_2\}$, then each vertex in Y_i is also adjacent to y_1 but not to y_2. Moreover, $\{y_1, y_2\} \succ Y \supseteq Y_1 = N(x_1)$ and so, by (ii), $\{y_1, y_2\}$ is a γ-set of $N(x_1)$. Therefore (iii) holds with $x = x_1$. ■

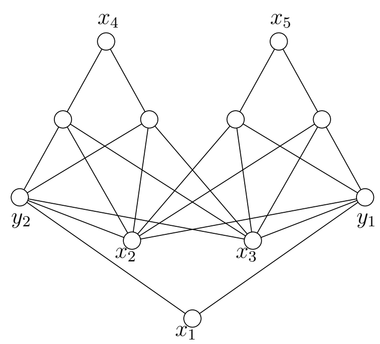

Figure 4. A consistent fixer with no primitive symmetric γ-set.

The properties of the dominating set $X = X_1 \cup X_2 \cup X_3$ given in Theorem 5 and Proposition 6 allow us to easily construct consistent fixers without primitive symmetric γ-sets. Figure 4 shows a consistent fixer G that has a symmetric γ-set $D = D_1 \cup D_2$ with $|D_1| = |D_2| = 2$. In this example, $D_1 = \{y_1, x_4\}$, $D_2 = \{y_2, x_5\}$, $X_1 = \{x_1, x_4, x_5\}$, $X_2 = \{x_2\}$ and $X_3 = \{x_3\}$. Since $\Delta(G) = 6$, G has no primitive symmetric γ-set.

If G is a consistent fixer, then $G \,\square\, K_n$, $n \geq 3$, has a minimum dominating set that contains exactly one vertex in all but one of the G-layers of $G \,\square\, K_n$, as stated in the following corollary.

Corollary 7. *If G is a consistent fixer and $3 \leq n < |V(G)| - \gamma(G) + 2$, then $G \,\square\, K_n$ has a γ-set $X = X_1 \cup \cdots \cup X_n$ with $|X_i| = 1$ for $i = 2, \ldots, n$, where X_i lies in the i^{th} G-layer of $G \,\square\, K_n$, $i = 1, \ldots, n$.*

3. Other Fixers

For any integer $t \geq 4$ there exist graphs that are 2-fixers and n-fixers for $t \leq n < |V(G)| - \gamma(G) + 2$, but not for $2 < n < t$. Figure 5 shows a graph G that is a 2-fixer and a 4-fixer, but not a 3-fixer. Each vertex x_2, x_3 and x_6 is adjacent only to the vertices y_1, y_2, a, b, c and d, but these edges are omitted in the figure for the sake of clarity. The graph has a symmetric γ-set $D = D_1 \cup D_2$ with $D_1 = \{x_4, y_1\}$ and $D_2 = \{x_5, y_2\}$. Since $\Delta(G) = 6$, G does not have a primitive symmetric γ-set. Furthermore, it is easy to verify that G does not have a set $X = X_1 \cup X_2 \cup X_3$ with the properties stated in Theorem 5, and therefore is not a 3-fixer. However, for $n \geq 4$, the set

$$W = \{(x_1, u_1), (x_4, u_1), (x_5, u_1), (x_2, u_2), (x_3, u_3)\} \cup \{(x_6, u_i) : i \geq 4\}$$

is a dominating set of $G \,\square\, K_n$ of cardinality $\gamma(G) + n - 2$, so that G is an n-fixer.

The characterization of these n-fixers is similar to that of Theorem 5 and the proof is therefore omitted.

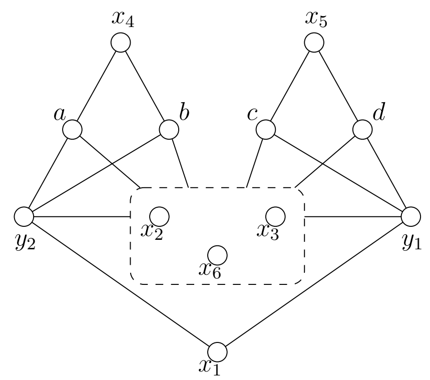

Figure 5. A graph that is a 2-fixer and a 4-fixer, but not a 3-fixer.

Theorem 8. *Let G be a connected graph and $t \geq 4$. Then G is a 2-fixer and an n-fixer for $n \geq t$, but not for $2 < n < t$, if and only if*

(i) *G has symmetric γ-sets, none of which are primitive, and*

(ii) *t is the smallest integer such that G has a dominating set $X = X_1 \cup \cdots \cup X_t$ with the following properties:*

 (a) *$X_i \succ V(G) - X$, $i = 1, 2, \ldots, t$,*

 (b) *for each $i = 1, 2, \ldots, t$, the sets $\{N(x) - X\}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,*

 (c) *the sets X_i are disjoint and $|X| = \sum_{i=1}^{t} |X_i| = \gamma(G) + t - 2$,*

 (d) *$|X_i| = 1$ for $i \geq 2$.*

Similar to Proposition 6, the set $X = X_1 \cup \cdots \cup X_t$ has the following additional properties.

Proposition 9. *Let G be a connected graph of order at least 3, and $t \geq 3$. If G is a 2-fixer and an n-fixer, $n \geq t$, that has no primitive symmetric γ-set, then the dominating set $X = X_1 \cup \cdots \cup X_t$ in Theorem 8(ii) has the following properties:*

(i) *$X_1 \cup X_i$ is an independent set, $i = 2, \ldots, t$,*

(ii) *$\gamma(G[N(x)]) \geq 2$ for every $x \in X_1$,*

(iii) *for some $x \in X_1$, $G[N(x)]$ has a γ-set, $\{y_1, y_2\}$ say, such that for every $x' \in X_1 - \{x\}$,*

 (a) *$y_1 \succ N(x')$ and $N(x') \cap N(y_2) = \emptyset$, or*

 (b) *$y_2 \succ N(x')$ and $N(x') \cap N(y_1) = \emptyset$.*

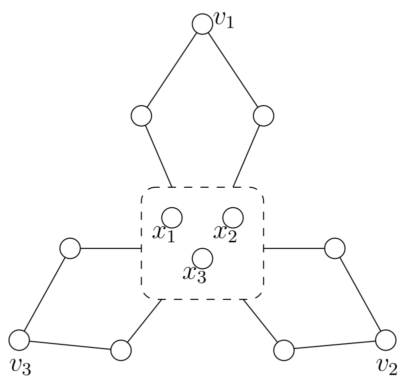

Figure 6. An n-fixer only for $n \geq 4$.

Lastly, we consider graphs that are n-fixers for $n \geq t \geq 3$, but not for $n < t$. As an example, Figure 6 shows a graph G that is an n-fixer for $n \geq 4$ only. In this graph, each vertex x_1, x_2 and x_3 is adjacent only to the neighbours of v_1, v_2 and v_3. It is easy to verify that $\gamma(G) = 4$, the graph does not have a symmetric γ-set, and that it is not a 3-fixer.

The following characterization describes such fixers. The proof is also similar to that of Theorem 5 and is omitted.

Theorem 10. *Let G be a connected graph and $t \geq 3$. Then G is an n-fixer for $n \geq t$, but not for $2 < n < t$, if and only if G does not have a symmetric γ-set, and t is the smallest integer such that G has a dominating set $X = X_1 \cup \cdots \cup X_t$ with the following properties:*

(a) $X_i \succ V(G) - X$, $i = 1, 2, \ldots, t$,

(b) *for each $i = 1, 2, \ldots, t$, the sets $\{N(x) - X\}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,*

(c) *the sets X_i are disjoint and $|X| = \sum_{i=1}^{t} |X_i| = \gamma(G) + t - 2$,*

(d) $|X_i| = 1$ *for $i \geq 2$.*

4. Cartesian n-multipliers

Consider n such that $\gamma(G) + n - 2 < |V(G)|$ and recall that $\gamma(G) + n - 2 \leq \gamma(G \,\square\, K_n) \leq n\gamma(G)$. We observe that, for any positive integer m and for any $0 \leq i \leq (m-1)(n-1) + 1$, there exists a graph G such that $\gamma(G) = m$ and $\gamma(G \,\square\, K_n) = m + n - 2 + i$. (The upper bound on i ensures that $\gamma(G) + n - 2 + i \leq n\gamma(G)$.) Consider the complete bipartite graph $G = K_{l,k}$ with $l \leq k$ and let $x_1, x_2, \ldots, x_l$ be the vertices in the smaller partite set. With notation as in Theorem 8, let $X_i = \{x_i\}$ and $X = \{x_1, x_2, \ldots, x_l\}$. If $l = 2$, then X is a primitive symmetric γ-set of G, which is a consistent fixer by Theorem 5. If $l = n \geq 3$, then X satisfies the conditions in Theorem 10, so G is an n-fixer. If $l = n + i$, then $\gamma(G \,\square\, K_n) = \gamma(G) + n - 2 + i$, up to values of i for which $\gamma(G \,\square\, K_n) = n\gamma(G)$, in which case G is an n-multiplier (or a prism doubler if $n = 2$).

Burger, Mynhardt and Weakley [1] characterized prism doublers as follows.

Proposition 11 [1]**.** *A graph G is a prism doubler if and only if for each set $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$, either*

(i) $|Y| \geq 2\gamma(G) - |X|$, *or*

(ii) $|Y| = 2\gamma(G) - |X| - d$ *for some $1 \leq d \leq |X|$, and at least d vertices (necessarily in $N[X]$) are required to dominate $N\{X\} - N[Y]$.*

Following a similar argument to that used in [1], we provide a characterization of n-multipliers. In $G \,\square\, K_n$ we denote the i^{th} G-layer of G by G_i and $V(G_i)$

by V_i. For $S \subseteq V(G)$, let $\langle S\rangle_i$ denote the counterpart of S in G_i. Note that if $|V(G)| < n\gamma(G)$, then G is not an n-multiplier since V_1 is a dominating set of $G \,\square\, K_n$. Thus we only consider graphs G of order at least $n\gamma(G)$.

Proposition 12. *A graph G is an n-multiplier if and only if for each set $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$, either*

(i) $|Y| \geq n\gamma(G) - |X|$, *or*

(ii) $|Y| = n\gamma(G) - |X| - d$ *for some* $1 \leq d \leq (n-1)|X|$, *and for any partition* $Y_2, Y_3, \ldots, Y_n$ *of* Y, *the subgraph of* $G \,\square\, K_n$ *induced by* $\bigcup_{i=2}^{n} \langle N\{X\} - N[Y_i]\rangle_i$ *has domination number at least* d.

Proof. Suppose G is an n-multiplier and consider any set $X \subseteq V(G)$, where $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$.

If $|Y| \geq n\gamma(G) - |X|$, then (i) holds. If $|Y| < n\gamma(G) - n|X|$, then $(\bigcup_{i=1}^{n}\langle X\rangle_i) \cup \langle Y\rangle_1$ is a dominating set of $G \,\square\, K_n$ of cardinality $n|X| + |Y| < n\gamma(G)$ — a contradiction.
Hence we assume that $|Y| = n\gamma(G) - |X| - d$ for some $1 \leq d \leq (n-1)|X|$. Suppose there exists a partition $Y_2, Y_3, \ldots, Y_n$ of Y such that the subgraph of $G \,\square\, K_n$ induced by $\bigcup_{i=2}^{n} \langle N\{X\} - N[Y_i]\rangle_i$ is dominated by some set D of cardinality less than d. Then $\langle X\rangle_1 \cup (\bigcup_{i=2}^{n}\langle Y_i\rangle_i) \cup D$ is a dominating set of $G \,\square\, K_n$ of cardinality less than $|X| + |Y| + d = n\gamma(G)$ — a contradiction.

Conversely, suppose that $\gamma(G \,\square\, K_n) < n\gamma(G)$, and consider any minimum dominating set $D = D_1 \cup \cdots \cup D_n$ of $G \,\square\, K_n$. Let $B_i = p(D_i)$, $i = 1, \ldots, n$. Then $|B_i| < \gamma(G)$ for some i; without loss of generality assume $|B_1| < \gamma(G)$. Then $|B_1| > 0$, otherwise at least $|V(G)|$ vertices are needed to dominate G_1 in $G \,\square\, K_n$. But then $|V(G)| \leq |D| < n\gamma(G)$ and these graphs are not considered. Thus $0 < |B_1| < \gamma(G)$. We show that neither (i) nor (ii) holds for the set $X = B_1$.

Let $B = B_1 \cup B_2 \cup \cdots \cup B_n$ and $Y = V(G) - N[B_1]$. In the layer G_1, $V_1 - N[D_1]$ is dominated by $D_2 \cup \cdots \cup D_n$. Therefore in G, $Y \subseteq \bigcup_{i=2}^{n} B_i$ and so $|Y| \leq |B| - |B_1| < n\gamma(G) - |B_1|$. Thus (i) does not hold. If $|Y| < n\gamma(G) - n|B_1|$, then (ii) does not hold either and we are done. Hence we assume that $|Y| = n\gamma(G) - |B_1| - d$ for some $1 \leq d \leq (n-1)|B_1|$.

Let $Y_2, Y_3, \ldots, Y_n$ be a partition of Y such that $Y_i \subseteq B_i$, $i = 2, 3, \ldots, n$, and let $Z_i = B_i - Y_i$. Then the set $D' = \bigcup_{i=2}^{n}\langle Z_i\rangle_i$ dominates the subgraph of $G \,\square\, K_n$ induced by $\bigcup_{i=2}^{n}\langle N\{B_1\} - N[Y_i]\rangle_i$. But

$$|D'| \leq \sum_{i=2}^{n} |B_i| - \sum_{i=2}^{n} |Y_i| < n\gamma(G) - |B_1| - |Y| = d.$$

Therefore (ii) does not hold. ■

We construct a family of multipliers with domination number 2. Let $n \geq 2$ and consider disjoint complete graphs K_{n+1} and K_{2n}, with vertex sets $A =$

$\{v_1, v_2, \ldots, v_{n+1}\}$ and $B = \{w_1, w_2, \ldots, w_{2n}\}$, respectively. Let G_n be the graph obtained by adding the edges $v_i w_i$, $i = 1, \ldots, n+1$. We use Proposition 12 to show that G_n is an n-multiplier. Since $\gamma(G) = 2$, we only consider sets X of cardinality 1. There are three possibilities for X.

- If $X = \{v_i\}$, then $Y = B - \{w_i\}$ and $|Y| = 2n - 1 = n\gamma(G_n) - |X|$.
- If $X = \{w_i\}$ with $i \le n+1$, then $Y = A - \{v_i\}$ and $|Y| = n = n\gamma(G_n) - |X| - d$ with $d = n - 1$. For any $Y' \subseteq Y$, $N(w_i) - N[Y']$ contains the vertices $w_{n+2}, \ldots, w_{2n}$. Thus, for any partition $Y_2, Y_3, \ldots, Y_n$ of Y, the subgraph of $G_n \square K_n$ induced by $\bigcup_{j=2}^{n} \langle N(w_i) - N[Y_j] \rangle_j$ has a subgraph isomorphic to $K_{n-1} \square K_{n-1}$, which has domination number $d = n-1$. Hence Proposition 12(ii) holds.
- If $X = \{w_i\}$, $i > n+1$, a similar argument shows that Proposition 12(ii) also holds.

It follows that G is an n-multiplier.

5. Conclusion

We conclude with open problems for future research. Let G and H be graphs of order m and n respectively. The Cartesian product $G \square H$ possesses a so-called *layer-partition* property, in that its vertex set allows two partitions $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_m\}$ such that (a) each $P_i \in \mathcal{P}$ induces a copy of G, called a *G-layer*, (b) each $Q_j \in \mathcal{Q}$ induces a copy of H, called an *H-layer*, (c) any P_i and Q_j intersect in exactly one vertex, and (d) any edge in the product is in either exactly one G-layer or exactly one H-layer.

In 1967, Chartrand and Harary [2] defined the *generalized prism* πG of G as the graph consisting of two copies of G, with edges between the copies determined by a permutation π acting on $V(G)$. For any permutation π, $\gamma(G) \le (\pi G) \le 2\gamma(G)$.

We now define a *generalized Cartesian product* $G \boxed{\pi} H$ that corresponds to $G \square H$ when π is the identity, πG when H is the graph K_2, and that retains a layer-partition property. For two labelled graphs G and H and permutation π acting on $V(G)$, the product $G \boxed{\pi} H$ is the graph with vertex set $V(G) \times V(H)$, and vertex (v_i, u_j) is adjacent to (v_k, u_l), $j \le l$, if and only if (a) $v_i v_k \in E(G)$ and $u_j = u_l$, or (b) $v_k = \pi^{l-j}(v_i)$ and $u_j u_l \in E(H)$.

Note that $\gamma(G) \le \gamma(G \boxed{\pi} H) \le \gamma(G)|V(H)|$ for any G, H and permutation π. Burger, Mynhardt and Weakley [1] investigated graphs G for which $\gamma(\pi G) = 2\gamma(G)$ for any π.

Question 1. *For some graph H of order n, is it possible to characterize graphs G for which $\gamma(G \boxed{\pi} H) = n\gamma(G)$ for every π?*

In 2006, Mynhardt and Xu [6] investigated graphs G for which $\gamma(\pi G) = \gamma(G)$ for any π, and conjectured that only the edgeless graphs have this property.

Question 2. *For some graph H of order n, does there exist a nontrivial graph G such that $\gamma(G \boxed{\pi} H) = \gamma(G) + n - 2$ for every π?*

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