CHARACTERIZING CARTESIAN FIXERS AND MULTIPLIERS

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Abstract

Let $G \Box H$ denote the Cartesian product of the graphs $G$ and $H$. In 2004, Hartnell and Rall [On dominating the Cartesian product of a graph and $K_2$, Discuss. Math. Graph Theory 24(3) (2004), 389–402] characterized prism fixers, i.e., graphs $G$ for which $\gamma(G \Box K_2) = \gamma(G)$, and noted that $\gamma(G \Box K_n) \geq \min\{|V(G)|, \gamma(G) + n - 2\}$. We call a graph $G$ a consistent fixer if $\gamma(G \Box K_n) = \gamma(G) + n - 2$ for each $n$ such that $2 < n < |V(G)| - \gamma(G) + 2$, and characterize this class of graphs.

Also in 2004, Burger, Mynhardt and Weakley [On the domination number of prisms of graphs, Discuss. Math. Graph Theory 24(2) (2004), 303–318] characterized prism doublers, i.e., graphs $G$ for which $\gamma(G \Box K_2) = 2\gamma(G)$. In general $\gamma(G \Box K_n) \leq n\gamma(G)$ for any $n \geq 2$. We call a graph attaining equality in this bound a Cartesian $n$-multiplier and also characterize this class of graphs.

Keywords: Cartesian product, prism fixer, Cartesian fixer, prism doubler, Cartesian multiplier, domination number.

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1. Introduction

We generally follow the notation and terminology of [5]. For two graphs $G$ and $H$, the Cartesian product $G \square H$ is the graph with vertex set $V(G) \times V(H)$ and vertex $(v_i, u_j)$ adjacent to $(v_k, u_l)$ if and only if (a) $v_iv_k \in E(G)$ and $u_j = u_l$, or (b) $v_i = v_k$ and $u_j, u_l \in E(H)$. The graph $G \square K_2$ is called the prism of $G$.

As usual $\gamma(G)$ denotes the domination number of $G$. A set $D \subseteq V(G)$ is called a $\gamma$-set if it is a dominating set with $|D| = \gamma(G)$. The domination number $\gamma(G \square K_2)$ of the prism of $G$ lies between $\gamma(G)$ and $2\gamma(G)$. The edgeless graph $G = \overline{K_m}$ attains equality in the lower bound, whereas $\gamma(K_m \square K_2) = 2\gamma(K_m)$.

In 2004, Hartnell and Rall [4] characterized graphs, called prism fixers, for which $\gamma(G \square K_2) = \gamma(G)$. A $\gamma$-set $D$ of $G$ is called a symmetric $\gamma$-set if $D$ can be partitioned into two nonempty subsets $D_1$ and $D_2$ such that $V(G) = N[D_1] = D_2$ and $V(G) - N[D_2] = D_1$. We write $D = D_1 \cup D_2$ for convenience. A symmetric $\gamma$-set $D = D_1 \cup D_2$ is called primitive if $|D_i| = 1$ for at least one $i$.

**Theorem 1** [4]. A connected graph $G$ is a prism fixer if and only if $G$ has a symmetric $\gamma$-set.

Hartnell and Rall generalized the lower bound for $\gamma(G \square K_2)$ to $\gamma(G \square K_n)$ by utilizing one of their results in [3]. They confirmed that the lower bound is sharp by providing a family of graphs attaining equality.

**Corollary 2** [4]. For any graph $G$ and $n \geq 2$, $\gamma(G \square K_n) \geq \min\{|V(G)|, \gamma(G) + n - 2\}$.

Note that $\gamma(G \square K_n) = |V(G)|$ for the edgeless graph $G = \overline{K_m}$. Also, if $n \geq |V(G)| - \gamma(G) + 2$, then $\min\{|V(G)|, \gamma(G) + n - 2\} = |V(G)|$. A minimum domination strategy is to take all vertices in a single copy of $G$ as a dominating set, hence $\gamma(G \square K_n) = |V(G)|$.

For $2 \leq n < |V(G)| - \gamma(G) + 2$, Corollary 2 gives a nontrivial lower bound, and a graph $G$ is called a Cartesian $n$-fixer if $\gamma(G \square K_n) = \gamma(G) + n - 2$. We henceforth simply refer to a Cartesian $n$-fixer as an $n$-fixer. Furthermore, if $G$ is an $n$-fixer for each $n$ such that $2 \leq n < |V(G)| - \gamma(G) + 2$, then $G$ is called a consistent $n$-fixer. We characterize these graphs in Section 2. In Section 3 we discuss graphs that are $n$-fixers for only some values of $n$ in the range $2 \leq n < |V(G)| - \gamma(G) + 2$. In 2004, Burger, Mynhardt and Weakley [1] characterized prism doublers, i.e., graphs $G$ for which $\gamma(G \square K_2) = 2\gamma(G)$. In general $\gamma(G \square K_n) \leq n\gamma(G)$ for any $n \geq 2$, and a graph attaining equality in this upper bound is called a Cartesian $n$-multiplier. Once again, we refer to such a graph simply as an $n$-multiplier. In Section 4 we follow a similar argument to that in [1] to characterize $n$-multipliers.

For $A, B \subseteq V(G)$, we abbreviate “$A$ dominates $B$” to “$A \triangleright B$”; if $B = V(G)$ we write $A \triangleright G$ and if $B = \{b\}$ we write $A \triangleright b$. Further, $N(v) = \{u \in V(G) :$
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$uv \in E(G)$ and $N[v] = N(v) \cup \{v\}$ denote the open and closed neighbourhoods, respectively, of a vertex $v$ of $G$. The closed neighbourhood of $S \subseteq V(G)$ is the set $N[S] = \bigcup_{s \in S} N[s]$, the open neighbourhood of $S$ is $N(S) = \bigcup_{s \in S} N(s)$, while $N\{S\}$ denotes the set $N(S) - S$.

Consider two graphs $G$ and $H$, with vertex sets labelled $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ respectively. Vertices $(v_i, u_j)$ of the Cartesian product $G \square H$ are labelled $v_{i,j}$ for convenience. The subgraph induced by all vertices that differ from a given vertex $v_{i,j}$ only in the first [second] coordinate, is known as the (Cartesian) $G$-layer [$H$-layer] through $v_{i,j}$.

We often consider projections $p_G : V(G \square H) \to V(G)$ and $p_H : V(G \square H) \to V(H)$. A general vertex $v_{i,j}$ of $G \square H$ has as first coordinate the vertex $p_G(v_{i,j}) = v_i \in V(G)$ and second coordinate $p_H(v_{i,j}) = u_j \in V(H)$. The preimage $p_G^{-1}(v)$ of a vertex $v_i$ in $G$ is the set of vertices in $G \square H$ that have $v_i$ as first coordinate, that is, the vertex set of the $H$-layer through $v_{i,j}$ for any $j$. The preimage of $A \subseteq V(G)$ is the set $p_G^{-1}(A) = \bigcup_{v \in A} p_G^{-1}(v)$. The projection $p_G$ and preimage $p_G^{-1}$ are abbreviated to $p$ and $p^{-1}$ respectively.

As an example, consider the graph $P_4 \square P_4$ in Figure 1. For this graph we have $p(\{v_{1,3}, v_{3,2}\}) = \{v_1, v_3\}$, while $p^{-1}(\{v_1, v_3\}) = \{v_{i,j} : i = 1, 3, j = 1, 2, 3, 4\}$.

Lastly, a dominating set $W$ of $G \square H$ can be partitioned into sets $W_1, W_2, \ldots, W_n$, where $W_i$ is a subset of vertices in the $i^{th}$ $G$-layer. We write $W = W_1 \cup W_2 \cup \cdots \cup W_n$ when this partition is clear from the context.

2. Consistent Fixers

Hartnell and Rall [4] provided examples of graphs that show that the lower bound in Corollary 2 is sharp. Let $G_k$ be the graph with vertex set $V(G_k) = \{v\} \cup \{x_i, y_i, z_i : i = 1, 2, \ldots, k\}$, and edge set $\{vx_i, x_iy_i, y_iz_i, z_iv : i = 1, 2, \ldots, k\}$. 
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(The 4-cycles $G_k$ share a common vertex $v$, $i = 1, 2, \ldots, k$.) Then $\gamma(G_k) = k + 1$ and $D = \{(y_i, u_1) : i = 1, 2, \ldots, k\} \cup \{(v, u_j) : j = 2, 3, \ldots, n\}$ is a dominating set of $G_k \square K_n$ of cardinality $k + n - 1 = \gamma(G_k) + n - 2$. The graph $G_3$ is illustrated in Figure 2. If $k > \frac{n-2}{2}$, then $|V(G_k)| = 3k + 1 > k + n - 1$ and hence $\gamma(G_k \square K_n) = \gamma(G_k) + n - 2$.

For the graph $G_3$ in Figure 2, let $D_1 = \{y_1, y_2, y_3\}$ and $D_2 = \{v\}$, and note that $D = D_1 \cup D_2$ is a primitive symmetric $\gamma$-set of $G_3$. In general, any graph $G$ that has a primitive symmetric $\gamma$-set satisfies $\gamma(G \square K_n) = \gamma(G) + n - 2$ for any $2 \leq n < |V(G)| - \gamma(G) + 2$:

Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $D = D_1 \cup D_2$ be a primitive symmetric $\gamma$-set of $G$ with $D_2 = \{x\}$. Figure 3 illustrates the dominating set $W = \{(v, u_1) : v \in D_1\} \cup \{(x, u_i) : i = 2, 3, \ldots, n\}$ of $G \square K_n$ of cardinality $\gamma(G) + n - 2$. In the first $G$-layer, the set $Y = V(G) - D$ is dominated by $\{(v, u_1) : v \in D_1\}$, and in the $i$th $G$-layer $Y$ is dominated by $(x, u_i)$, $i \geq 2$.

The question now arises whether graphs with primitive symmetric $\gamma$-sets are the only $n$-fixers. Our characterization will show that this is not the case.

We first state some useful properties of a graph having a symmetric $\gamma$-set.

Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $D = D_1 \cup D_2$ be a primitive symmetric $\gamma$-set of $G$ with $D_2 = \{x\}$. Figure 3 illustrates the dominating set $W = \{(v, u_1) : v \in D_1\} \cup \{(x, u_i) : i = 2, 3, \ldots, n\}$ of $G \square K_n$ of cardinality $\gamma(G) + n - 2$. In the first $G$-layer, the set $Y = V(G) - D$ is dominated by $\{(v, u_1) : v \in D_1\}$, and in the $i$th $G$-layer $Y$ is dominated by $(x, u_i)$, $i \geq 2$.

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Figure 2. The graph $G_3$.

Figure 3. A domination strategy for $G \square K_n$ if $G$ has a primitive symmetric $\gamma$-set.
Observation 3 [4].

(i) Let $G$ be a connected graph with symmetric $\gamma$-set $D = D_1 \cup D_2$ and let $Y = V(G) - D$. Then
   (a) $N[D_i] = D_i \cup Y$, $i = 1, 2$,
   (b) $D$ is an independent set,
   (c) the sets $\{N(x) - X\}_{x \in D}$ are disjoint, and these sets form a partition of $Y$,
   (d) each vertex in $D$ is adjacent to at least two vertices in $Y$.

(ii) Let $G$ be a graph with at least one symmetric $\gamma$-set, but no primitive symmetric $\gamma$-set, and let $Y = V(G) - D$. Then $\gamma(G[Y]) > 1$.

(iii) If $G$ is a 2-fixer and $W = W_1 \cup W_2$ is a $\gamma$-set of $G \Box K_2$, then $p(W_1) \cup p(W_2)$ is a symmetric $\gamma$-set of $G$.

Suppose $G$ is a 2-fixer with no primitive symmetric $\gamma$-set and $\gamma(G \Box K_3) = \gamma(G) + 1$. Then a minimum domination strategy for the Cartesian product $G \Box K_3$ will never be to take a $\gamma$-set of $G \Box K_2$ and select one vertex in the third $G$-layer, as we show next.

Lemma 4. Let $G$ be a connected 3-fixer with symmetric $\gamma$-set $D = D_1 \cup D_2$, but no primitive symmetric $\gamma$-set. Then no $\gamma$-set $W = W_1 \cup W_2 \cup W_3$ of $G \Box K_3$ has $p(W_1) = D_1$, $p(W_2) = D_2$ and $|W_3| = 1$.

Proof. Let $D = D_1 \cup D_2$ be a symmetric $\gamma$-set of $G$ with $|D_1|, |D_2| \geq 2$ and let $Y = V(G) - D$. Suppose $W = W_1 \cup W_2 \cup W_3$ is a $\gamma$-set of $G \Box K_3$, with $p(W_1) = D_1$, $p(W_2) = D_2$ and $W_3 = \{(x, u_3)\}$. Then $x \succ Y$. If $x \notin D$, then $x \in Y$ and so $\gamma(G[Y]) = 1$, contradicting Observation 3(ii). So assume $x \in D$, say $x \in D_2$, and let $z \in D_2 - \{x\}$. Then $z$ is adjacent to some vertex in $Y$, hence $x$ and $z$ have a common neighbour in $Y$, contradicting Observation 3(i)(c).}

We now provide a characterization of consistent fixers. We only consider connected graphs and also require $G$ to have at least three vertices; since $\gamma(G) \leq \frac{1}{2}|V(G)|$ for any connected graph $G$, this requirement ensures that a value $n \geq 3$ is included in the range $2 \leq n < |V(G)| - \gamma(G) + 2$.

Theorem 5. Let $G$ be a connected graph of order at least 3. Then $G$ is a consistent fixer if and only if

(i) $G$ has a primitive symmetric $\gamma$-set, or

(ii) $G$ has symmetric $\gamma$-sets, none of which are primitive, and $G$ has a dominating set $X = X_1 \cup X_2 \cup X_3$ with the following properties:
   (a) $X_i \succ V(G) - X$, $i = 1, 2, 3$,
   (b) for each $i = 1, 2, 3$, the sets $\{N(x) - X\}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,}
(c) the sets $X_i$ are disjoint and $|X| = |X_1| + |X_2| + |X_3| = \gamma(G) + 1$.

(d) $|X_2| = |X_3| = 1$.

**Proof.** Let $G$ be a consistent fixer. Then by Theorem 1, $G$ has a symmetric \( \gamma \)-set \( D = D_1 \cup D_2 \). Suppose \(|D_1|, |D_2| \geq 2\) for any such set \( D \). We show that (ii) holds.

Since $G$ is also a Cartesian 3-fixer, there exists a minimum dominating set \( W = W_1 \cup W_2 \cup W_3 \) of \( G \sqcap K_3 \) of cardinality \( \gamma(G) + 1 \). Let \( X_i = p(W_i), i = 1, 2, 3 \), \( X = X_1 \cup X_2 \cup X_3 \) and \( Y = V(G) - X \).

Then \( X \subseteq V(G) \) is a dominating set of \( G \) of cardinality at most \( \gamma(G) + 1 \), i.e., \( \gamma(G) \leq |X| \leq \gamma(G) + 1 \). If \( Y = \emptyset \), then \( |V(G)| = |X| \leq \gamma(G) + 1 \), contradicting the statement \( 3 < |V(G)| - \gamma(G) + 2 \). Therefore \( Y \neq \emptyset \), and so to dominate \( p^{-1}(Y) \), \( W_i \neq \emptyset \) for each \( i \). Hence \( X_i \neq \emptyset \) and, moreover, \( X_i > Y \) for each \( i = 1, 2, 3 \). Thus (a) holds.

Without loss of generality, assume that \( |X_1| \geq |X_2| \geq |X_3| \) and that \( W \) has been chosen so that \( |X_1| \) is as large as possible. Since \( \gamma(G) \leq |X| \leq \gamma(G) + 1 \),

\[
\text{(1) at most one vertex of } X \text{ occurs in more than one set } X_i.
\]

Similarly, no vertex occurs in all three \( X_i \), i.e.,

\[
\text{(2) } X_1 \cap X_2 \cap X_3 = \emptyset.
\]

We now prove the following statement:

\[
\text{(3) Each vertex in } X_2 \cup X_3 \text{ is adjacent to some vertex in } Y.
\]

Suppose there exists \( x \in X_2 \) that is not adjacent to any vertex in \( Y \), and \( w_2 \) is a vertex of \( W_2 \) such that \( p(w_2) = x \). (The argument is the same if \( x \in X_3 \).)

If \( x \in X_1 \) and \( w_1 \) is a vertex of \( W_1 \) such that \( p(w_1) = x \), then \( W - \{w_1\} \) is a dominating set of \( G \sqcap K_3 \) of cardinality \( \gamma(G) \), which is impossible by Corollary 2. Thus \( x \notin X_1 \). But then \( W' = (W_1 \cup \{w_1\}) \cup (W_2 - \{w_2\}) \cup W_3 \) is a minimum dominating set of \( G \sqcap K_3 \) such that \( X_1' = p(W_1 \cup \{w_1\}) = X_1 \cup \{x\} \) has larger cardinality than \( X_1 \), contradicting the choice of \( W \). Thus (3) holds.

(b) Suppose two distinct vertices \( u, v \in X_i \) are both adjacent to some vertex \( y \in Y \). By (a), \( y \) is adjacent to a vertex in each \( X_i \). By (1) and (2), at least one \( X_j \), \( j \neq i \), contains a neighbour \( w \) of \( y \) such that \( w \notin \{u, v\} \). But \( X_k > Y \), \( k \neq i, j \), so \( (X - \{u, v, w\}) \cup \{y\} \) is a dominating set of \( G \) that has cardinality at most \( \gamma(G) - 1 \), a contradiction. Hence each vertex \( y \in Y \) is dominated by exactly one vertex from \( X_i \), and (b) follows.

(c) We only prove that \( X_2 \cap X_3 = \emptyset \); the proofs that \( X_1 \cap X_2 = \emptyset \) and \( X_1 \cap X_3 = \emptyset \) are similar. It will follow that \( |X| = |X_1| + |X_2| + |X_3| = \gamma(G) + 1 \). Suppose
there exists a vertex \( z \in X_2 \cap X_3 \). Then \( |X| = \gamma(G) \) and, by (1) and (2), \( X_1 \cap (X_2 \cup X_3) = \emptyset \), so that \( X = X_1 \cup (X_2 \cup X_3) \) is a symmetric \( \gamma \)-set of \( G \).

If \( |X_3| = 1 \), then \( X_3 = \{z\} \subseteq X_2 \) and \( X = X_1 \cup X_2 \). By (a), \( z \) dominates all of \( Y \). But \( z \in X_2 \), and so (b) implies that \( X_2 = \{z\} \), i.e., \( |X_2| = 1 \). Then \( X \) is a primitive symmetric \( \gamma \)-set, which is not the case under consideration. Therefore \( |X_3| \geq 2 \); say \( w, z \in X_3 \). By (1), \( w \notin X_1 \cup X_2 \), and by (3), \( w \) is adjacent to some vertex in \( Y \). Since \( X_2 \nearrow Y \), there exists \( v \in X_2 \) such that \( v \) and \( w \) have a common neighbour in \( Y \). This contradicts Observation 3(i)(c) for the symmetric \( \gamma \)-set \( X = X_1 \cup (X_2 \cup X_3) \). Therefore \( X_2 \cap X_3 = \emptyset \).

(d) Suppose that \( |X_2| \geq 2 \). Then \( |X_1| \geq 2 \). Let \( y_1 \in Y \) and choose \( x_1 \in X_1 \), \( x_2 \in X_2 \) such that \( x_1 \) and \( x_2 \) are both adjacent to \( y_1 \). Since \( X_3 \nearrow Y \), the set \( X' = (X - \{x_1, x_2\}) \cup \{y_1\} \) is a dominating set of \( G \) of cardinality \( \gamma(G) \), i.e., a \( \gamma \)-set of \( G \). We show that

\[
(x_1, x_2) \succ Y.
\]

Suppose to the contrary that \( y \in Y \) is not adjacent to either \( x_1 \) or \( x_2 \). Then there exist \( x'_1 \in X_1 - \{x_1\} \) and \( x'_2 \in X_2 - \{x_2\} \) adjacent to \( y \), so that \( (X' - \{x'_1, x'_2\}) \cup \{y\} \) is a dominating set of \( G \) of cardinality \( \gamma(G) - 1 \), which is impossible.

Let \( v \in X_2 - \{x_2\} \). By (3) there exists a vertex \( y_2 \in Y \) adjacent to \( v \). By (b) \( y_2 \) is not adjacent to \( x_2 \) and so, by (4), \( y_2 \) is adjacent to \( x_1 \). It follows similar to (4) that \( \{x_1, v\} \succ Y \). But then any vertex in \( Y \) not adjacent to \( x_1 \) is adjacent to both \( x_2 \) and \( v \), which is impossible by (b). Thus \( x_1 \succ Y \), and (b) implies that \( |X_1| = 1 \), a contradiction. Therefore \( |X_2| = 1 \) which, by the choice of the \( X_i \), also implies that \( |X_3| = 1 \).

Conversely, let \( G \) be a graph that satisfies the conditions of the statement, \( 2 \leq n < |V(G)| - \gamma(G) + 2 \) and \( V(K_n) = \{u_1, u_2, \ldots, u_n\} \). If \( G \) has a symmetric \( \gamma \)-set \( D = D_1 \cup D_2 \) with \( D_2 = \{x\} \), then the set \( W = \{(v, u_1) : v \in D_1 \} \cup \{(x, u_i) : i = 2, 3, \ldots, n\} \) is a dominating set of \( G \triangle K_n \) of cardinality \( \gamma(G) + n - 2 \), as illustrated in Figure 2.

Suppose that \( |D_1|, |D_2| \geq 2 \) and that \( G \) has a set \( X = X_1 \cup X_2 \cup X_3 \) with the stated properties. Let \( X_2 = \{x_2\} \) and \( X_3 = \{x_3\} \). Then the set

\[
W = \{(v, u_1) : v \in X_1 \} \cup \{(x_2, u_2)\} \cup \{(x_3, u_i) : i = 3, 4, \ldots, n\}
\]

is a dominating set of \( G \triangle K_n \) of cardinality \( \gamma(G) + n - 2 \).

The dominating set \( X = X_1 \cup X_2 \cup X_3 \) in Theorem 5(ii) has the following additional properties.

**Proposition 6.** Let \( G \) be a connected graph of order at least 3. If \( G \) is a consistent fixer with no primitive symmetric \( \gamma \)-set, then the dominating set \( X = X_1 \cup X_2 \cup X_3 \) in Theorem 5(ii) has the following properties:
(i) $X_1 \cup X_2$ and $X_1 \cup X_3$ are independent sets,
(ii) $\gamma(G[N(x)]) \geq 2$ for every $x \in X_1$,
(iii) for some $x \in X_1$, $G[N(x)]$ has a $\gamma$-set, \{y_1, y_2\} say, such that for every $x' \in X_1 - \{x\}$,
   (a) $y_1 \succ N(x')$ and $N(y_2) \cap N(x') = \emptyset$, or
   (b) $y_2 \succ N(x')$ and $N(y_1) \cap N(x') = \emptyset$.

**Proof.** Say $X_2 = \{x_2\}, X_3 = \{x_3\}, Y = V(G) - X$, and note that

$$x_i \succ Y, \ i = 2, 3.$$ (5)

(i) Consider any symmetric $\gamma$-set $D = D_1 \cup D_2$ of $G$ and recall that $|D_i| \geq 2$. Define $Y' = V(G) - D$. We compare $D$ and $X$, and show that

$$|D_i \cap Y| = 1 \text{ for } i = 1, 2, \ |D \cap X_1| = \gamma(G) - 2 = |X_1| - 1,$$

and $|X_1 \cap Y'| = 1$. (6)

We begin by showing that $\{x_2, x_3\} \cap D = \emptyset$. Suppose $x_2 \in D$; without loss of generality say $x_2 \in D_2$. Then (5) and Observation 3(i)(b) imply that $Y \cap D = \emptyset$. Now if $x_3 \in D$, then Observation 3(i)(c) implies that $x_3 \in D_1$ and that the only vertices in $X_1 \cap D$ are vertices that are nonadjacent to all vertices in $Y$. But $|X| = \gamma(G) + 1$, $|X_1| = \gamma(G) - 1$ and $|D| = \gamma(G)$, so that $\gamma(G) - 2$ vertices in $X_1$ are in $D$. Therefore exactly one vertex in $X_1$, say $x_1$, is adjacent to vertices in $Y$. By Theorem 5(ii)(a), $x_1 \succ Y$. Furthermore, $x_1 \in Y'$ by Observation 3(i)(c). If there exists a $v \in X_1 - \{x_1\}$, then $v \in D$, hence $v$ is adjacent to at least two vertices in $Y'$ by Observation 3(i)(d). Since $Y' - \{x_1\} = Y$, this is a contradiction. So $X_1 = \{x_1\}$ and it follows that $D$ is a primitive symmetric $\gamma$-set, a contradiction. Therefore $x_3 \notin D$ and so $D = X_1 \cup X_2$ and $V(G) - D = Y \cup \{x_3\}$.

Let $u \in D_2 - \{x_2\}$. By Observation 3(i)(d), $u$ is adjacent to at least two vertices in $Y'$, so $u$ is adjacent to some $y \in Y$. But then $y$ is adjacent to the two vertices $x_2, u \in D_2$, contradicting Observation 3(i)(c). Hence $x_2 \notin D$. Similarly, $x_3 \notin D$, i.e., $\{x_2, x_3\} \subseteq Y'$.

Since $|X_1| = \gamma(G) - 1$, it follows that $Y \cap D \neq \emptyset$. If $|D_i \cap Y| \geq 2$ for some $i$, then by (5), two vertices in $D_i$ have $x_2 \in Y'$ as common neighbour, contrary to Observation 3(i)(c). Thus $|D_i \cap Y| \leq 1$ for each $i$, so $|Y \cap D| \leq 2$. If $Y \cap D = \{y\}$, then $D = X_1 \cup \{y\}$. But by Theorem 5(ii)(a), $y$ is adjacent to some vertex in $X_1$, contradicting Observation 3(i)(b). Therefore $|Y \cap D| = 2$ and (6) follows.

Let $X_1 \cap Y' = \{x_1\}$ and $D_i \cap Y = \{y_i\}, i = 1, 2$. Then $X_1 - \{x_1\} \subseteq D$ and so $X_1 - \{x_1\}$ is independent (Observation 3(i)(b)).

Suppose $x_1$ is not adjacent to $y_1$. Since $X_1 \succ Y$, $y_1$ is adjacent to some $x' \in X_1 - \{x_1\} \subseteq D$. But $y_1 \in D$ and $D$ is independent, a contradiction. Hence
$x_1$ is adjacent to $y_1$ and, similarly, to $y_2$. It now follows from Observation 3(i)(c) that $x_1$ is not adjacent to any vertex in $X_1$ and so $X_1$ is independent.

By (5), $x_2$ and $x_3$ are adjacent to $y_1$ and $y_2$, hence as in the case of $x_1$, neither $x_2$ nor $x_3$ is adjacent to any vertex in $X_1 - \{x_1\}$. Since $G$ is connected, each vertex in $X_1 - \{x_1\}$ is therefore adjacent to a vertex in $Y$; since $D$ is independent this vertex is necessarily in $Y - \{y_1, y_2\}$. Since $|D_1| \geq 2$, there exists $x_4 \in D_1 - \{y_1\}$; necessarily $x_4 \subseteq X_1 - \{x_1\}$. Let $y_4 \in Y - \{y_1, y_2\}$ be adjacent to $x_4$ and consider the set $X' = (X - \{x_1, x_3, x_4\}) \cup \{y_4\}$. Then $x_2 \succ Y$, $y_4 \succ x_4$ and $y_4 \succ x_3$ by (5). Therefore $X' \succ G - x_1$. But $|X'| < \gamma(G)$ and so $X' \not\succ G$, i.e., $X' \not\succ x_1$. In particular, $x_2$ is not adjacent to $x_1$. Similarly, $x_3$ is not adjacent to $x_1$, and the proof of (i) is complete.

(ii) Since $\gamma(G) \geq 4$, $|X_1| \geq 3$. Say $X_1 = \{x_1, x_4, x_5, \ldots, x_k\}$ and define $Y_i = N(x_i)$, $i = 1, 4, 5, \ldots, k$. By (i), no vertex in $X_1$ is adjacent to any vertex in $X$, so $Y_i \subseteq Y$ for each $i$, and since $G$ is connected, $Y_i \neq \emptyset$. By Theorem 5(ii)(a) and (b), the sets $Y_1, Y_4, \ldots, Y_k$ partition $Y$. Suppose that for some $i$ there exists a vertex $y \in Y_i$ that is adjacent to all other vertices in $Y_i$ and consider $X' = (X - \{x_i, x_2, x_3\}) \cup \{y\}$. Then by (5), $y \succ Y \cup \{x_i, x_2, x_3\}$, while $X_1 - \{x_1\} \succ Y - Y_i$, so that $X' \succ G$. But $|X'| = \gamma(G) - 1$, which is impossible. This proves (ii).

(iii) As shown above, $D = \{y_1, y_2, x_4, \ldots, x_k\}$ and $Y' = \{x_1, x_2, x_3\} \cup (Y - \{y_1, y_2\})$. By Observation 3(i)(c), each vertex in $Y'$ is adjacent to exactly one vertex in each $D_i$. In particular, since $X_1$ is independent, $x_1$ is adjacent to $y_1$ and $y_2$. Since the $Y_i$ partition $Y$, no vertex in $Y$ is adjacent to two vertices in $X_1$. But for each $i = 4, \ldots, k$, $x_i$ is in exactly one of $D_1$ or $D_2$, so if $x_i \in D_1 - \{y_1\}$, then each vertex in $Y_i = N(x_i)$ is also adjacent to $y_2$ but not to $y_1$, and if $x_i \in D_2 - \{y_2\}$, then each vertex in $Y_i$ is also adjacent to $y_1$ but not to $y_2$. Moreover, $\{y_1, y_2\} \succ Y \supseteq Y_1 = N(x_1)$ and so, by (ii), $\{y_1, y_2\}$ is a $\gamma$-set of $N(x_1)$. Therefore (iii) holds with $x = x_1$. ■

![Figure 4. A consistent fixer with no primitive symmetric $\gamma$-set.](image)

The properties of the dominating set $X = X_1 \cup X_2 \cup X_3$ given in Theorem 5 and Proposition 6 allow us to easily construct consistent fixers without primitive

symmetric $\gamma$-sets. Figure 4 shows a consistent fixer $G$ that has a symmetric $\gamma$-set $D = D_1 \cup D_2$ with $|D_1| = |D_2| = 2$. In this example, $D_1 = \{y_1, x_4\}$, $D_2 = \{y_2, x_5\}$, $X_1 = \{x_1, x_4, x_5\}$, $X_2 = \{x_2\}$ and $X_3 = \{x_3\}$. Since $\Delta(G) = 6$, $G$ has no primitive symmetric $\gamma$-set.

If $G$ is a consistent fixer, then $G \Box K_n$, $n \geq 3$, has a minimum dominating set that contains exactly one vertex in all but one of the $G$-layers of $G \Box K_n$, as stated in the following corollary.

**Corollary 7.** If $G$ is a consistent fixer and $3 \leq n < |V(G)| - \gamma(G) + 2$, then $G \Box K_n$ has a $\gamma$-set $X = X_1 \cup \cdots \cup X_n$ with $|X_i| = 1$ for $i = 2, \ldots, n$, where $X_i$ lies in the $i^{th}$ $G$-layer of $G \Box K_n$, $i = 1, \ldots, n$.

### 3. Other Fixers

For any integer $t \geq 4$ there exist graphs that are 2-fixers and $n$-fixers for $t \leq n < |V(G)| - \gamma(G) + 2$, but not for $2 < n < t$. Figure 5 shows a graph $G$ that is a 2-fixer and a 4-fixer, but not a 3-fixer. Each vertex $x_2$, $x_3$ and $x_6$ is adjacent only to the vertices $y_1$, $y_2$, $a$, $b$, $c$ and $d$, but these edges are omitted in the figure for the sake of clarity. The graph has a symmetric $\gamma$-set $D = D_1 \cup D_2$ with $D_1 = \{x_4, y_1\}$ and $D_2 = \{x_5, y_2\}$. Since $\Delta(G) = 6$, $G$ does not have a primitive symmetric $\gamma$-set. Furthermore, it is easy to verify that $G$ does not have a set $X = X_1 \cup X_2 \cup X_3$ with the properties stated in Theorem 5, and therefore is not a 3-fixer. However, for $n \geq 4$, the set

$$W = \{(x_1, u_1), (x_4, u_1), (x_5, u_1), (x_2, u_2), (x_3, u_3)\} \cup \{(x_6, u_i) : i \geq 4\}$$

is a dominating set of $G \Box K_n$ of cardinality $\gamma(G) + n - 2$, so that $G$ is an $n$-fixer.

The characterization of these $n$-fixers is similar to that of Theorem 5 and the proof is therefore omitted.

![Figure 5](image-url)
**Theorem 8.** Let $G$ be a connected graph and $t \geq 4$. Then $G$ is a 2-fixer and an $n$-fixer for $n \geq t$, but not for $2 < n < t$, if and only if

(i) $G$ has symmetric $\gamma$-sets, none of which are primitive, and

(ii) $t$ is the smallest integer such that $G$ has a dominating set $X = X_1 \cup \cdots \cup X_t$ with the following properties:

(a) $X_i \succ V(G) - X$, $i = 1, 2, \ldots, t$,

(b) for each $i = 1, 2, \ldots, t$, the sets $\{N(x) - X\}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,

(c) the sets $X_i$ are disjoint and $|X| = \sum_{i=1}^t |X_i| = \gamma(G) + t - 2$,

(d) $|X_i| = 1$ for $i \geq 2$.

Similar to Proposition 6, the set $X = X_1 \cup \cdots \cup X_t$ has the following additional properties.

**Proposition 9.** Let $G$ be a connected graph of order at least 3, and $t \geq 3$. If $G$ is a 2-fixer and an $n$-fixer, $n \geq t$, that has no primitive symmetric $\gamma$-set, then the dominating set $X = X_1 \cup \cdots \cup X_t$ in Theorem 8(ii) has the following properties:

(i) $X_1 \cup X_i$ is an independent set, $i = 2, \ldots, t$,

(ii) $\gamma(G[N(x)]) \geq 2$ for every $x \in X_1$,

(iii) for some $x \in X_1$, $G[N(x)]$ has a $\gamma$-set, $\{y_1, y_2\}$ say, such that for every $x' \in X_1 - \{x\}$,

(a) $y_1 \succ N(x')$ and $N(x') \cap N(y_2) = \emptyset$, or

(b) $y_2 \succ N(x')$ and $N(x') \cap N(y_1) = \emptyset$.

Lastly, we consider graphs that are $n$-fixers for $n \geq t \geq 3$, but not for $n < t$. As an example, Figure 6 shows a graph $G$ that is an $n$-fixer for $n \geq 4$ only. In this

![Figure 6. An $n$-fixer only for $n \geq 4$.](image)
graph, each vertex $x_1$, $x_2$ and $x_3$ is adjacent only to the neighbours of $v_1$, $v_2$ and $v_3$. It is easy to verify that $\gamma(G) = 4$, the graph does not have a symmetric $\gamma$-set, and that it is not a 3-fixer.

The following characterization describes such fixers. The proof is also similar to that of Theorem 5 and is omitted.

**Theorem 10.** Let $G$ be a connected graph and $t \geq 3$. Then $G$ is an $n$-fixer for $n \geq t$, but not for $2 < n < t$, if and only if $G$ does not have a symmetric $\gamma$-set, and $t$ is the smallest integer such that $G$ has a dominating set $X = X_1 \cup \cdots \cup X_t$ with the following properties:

(a) $X_i \supset V(G) - X$, $i = 1, 2, \ldots, t$,
(b) for each $i = 1, 2, \ldots, t$, the sets $\{N(x) - X \}_{x \in X_i}$ are disjoint and form a partition of $V(G) - X$,
(c) the sets $X_i$ are disjoint and $|X_i| = \sum_{i=1}^{t} |X_i| = \gamma(G) + t - 2$,
(d) $|X_i| = 1$ for $i \geq 2$.

4. Cartesian $n$-multipliers

Consider $n$ such that $\gamma(G) + n - 2 < |V(G)|$ and recall that $\gamma(G) + n - 2 \leq \gamma(G \square K_n) \leq n \gamma(G)$. We observe that, for any positive integer $m$ and for any $0 \leq i \leq (m-1)(n-1) + 1$, there exists a graph $G$ such that $\gamma(G) = m$ and $\gamma(G \square K_n) = m + n - 2 + i$. (The upper bound on $i$ ensures that $\gamma(G) + n - 2 + i \leq n \gamma(G)$.) Consider the complete bipartite graph $G = K_{l,k}$ with $l \leq k$ and let $x_1, x_2, \ldots, x_l$ be the vertices in the smaller partite set. With notation as in Theorem 8, let $X_i = \{x_i\}$ and $X = \{x_1, x_2, \ldots, x_l\}$. If $l = 2$, then $X$ is a primitive symmetric $\gamma$-set of $G$, which is a consistent fixer by Theorem 5. If $l = n \geq 3$, then $X$ satisfies the conditions in Theorem 10, so $G$ is an $n$-fixer. If $l = n + i$, then $\gamma(G \square K_n) = \gamma(G) + n - 2 + i$, up to values of $i$ for which $\gamma(G \square K_n) = n \gamma(G)$, in which case $G$ is an $n$-multiplier (or a prism doubler if $n = 2$).

Burger, Mynhardt and Weakley [1] characterized prism doublers as follows.

**Proposition 11** [1]. A graph $G$ is a prism doubler if and only if for each set $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$, either

(i) $|Y| \geq 2 \gamma(G) - |X|$, or
(ii) $|Y| = 2 \gamma(G) - |X| - d$ for some $1 \leq d \leq |X|$, and at least $d$ vertices (necessarily in $N[X]$) are required to dominate $N\{X\} - N[Y]$.

Following a similar argument to that used in [1], we provide a characterization of $n$-multipliers. In $G \square K_n$ we denote the $i$th $G$-layer of $G$ by $G_i$ and $V(G_i)$
by $V_i$. For $S \subseteq V(G)$, let $\langle S \rangle_i$ denote the counterpart of $S$ in $G_i$. Note that if $|V(G)| < n\gamma(G)$, then $G$ is not an $n$-multiplier since $V_1$ is a dominating set of $G \boxtimes K_n$. Thus we only consider graphs $G$ of order at least $n\gamma(G)$.

**Proposition 12.** A graph $G$ is an $n$-multiplier if and only if for each set $X \subseteq V(G)$ with $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$, either

(i) $|Y| \geq n\gamma(G) - |X|$, or

(ii) $|Y| = n\gamma(G) - |X| - d$ for some $1 \leq d \leq (n-1)|X|$, and for any partition $Y_2, Y_3, \ldots, Y_n$ of $Y$, the subgraph of $G \boxtimes K_n$ induced by $\bigcup_{i=2}^{n}\langle N\{X\} - N[Y_i]\rangle_i$ has domination number $\geq 1$.

**Proof.** Suppose $G$ is an $n$-multiplier and consider any set $X \subseteq V(G)$, where $0 < |X| < \gamma(G)$, and $Y = V(G) - N[X]$.

If $|Y| \geq n\gamma(G) - |X|$, then (i) holds. If $|Y| < n\gamma(G) - |X|$, then $(\bigcup_{i=1}^{n}\langle X_i\rangle_i) \cup \langle Y_1\rangle_1$ is a dominating set of $G \boxtimes K_n$ of cardinality $n|X| + |Y| < n\gamma(G)$ — a contradiction.

Hence we assume that $|Y| = n\gamma(G) - |X| - d$ for some $1 \leq d \leq (n-1)|X|$. Suppose there exists a partition $Y_2, Y_3, \ldots, Y_n$ of $Y$ such that the subgraph of $G \boxtimes K_n$ induced by $\bigcup_{i=2}^{n}\langle N\{X\} - N[Y_i]\rangle_i$ is dominated by some set $D$ of cardinality less than $d$. Then $(X_1) \cup (\bigcup_{i=2}^{n}\langle Y_i\rangle_i) \cup D$ is a dominating set of $G \boxtimes K_n$ of cardinality less than $|X| + |Y| + d = n\gamma(G)$ — a contradiction.

Conversely, suppose that $\gamma(G \boxtimes K_n) < n\gamma(G)$, and consider any minimum dominating set $D = D_1 \cup \cdots \cup D_n$ of $G \boxtimes K_n$. Let $B_i = p(D_i)$, $i = 1, \ldots, n$. Then $|B_i| < \gamma(G)$ for some $i$; without loss of generality assume $|B_1| < \gamma(G)$. Then $|B_1| > 0$, otherwise at least $\gamma(G)$ vertices are needed to dominate $G_1$ in $G \boxtimes K_n$. But then $|V(G)| \leq |D| < n\gamma(G)$ and these graphs are not considered. Thus $0 < |B_1| < \gamma(G)$. We show that neither (i) nor (ii) holds for the set $X = B_1$.

Let $B = B_1 \cup B_2 \cup \cdots \cup B_n$ and $Y = V(G) - N[B_1]$. In the layer $G_1$, $V_1 - N[D_1]$ is dominated by $D_2 \cup \cdots \cup D_n$. Therefore in $G_i, Y \subseteq \bigcup_{i=2}^{n}B_i$ and so $|Y| \leq |B| - |B_1| < n\gamma(G) - |B_1|$. Thus (i) does not hold. If $|Y| < n\gamma(G) - |B_1|$, then (ii) does not hold either and we are done. Hence we assume that $|Y| = n\gamma(G) - |B_1| - d$ for some $1 \leq d \leq (n-1)|B_1|$. Let $Y_2, Y_3, \ldots, Y_n$ be a partition of $Y$ such that $Y_i \subseteq B_i$, $i = 2, 3, \ldots, n$, and let $Z_i = B_i - Y_i$. Then the set $D' = \bigcup_{i=2}^{n}\langle Z_i\rangle_i$ dominates the subgraph of $G \boxtimes K_n$ induced by $\bigcup_{i=2}^{n}\langle N\{B_1\} - N[Y_i]\rangle_i$. But

$$|D'| \leq \sum_{i=2}^{n}|B_i| - \sum_{i=2}^{n}|Y_i| < n\gamma(G) - |B_1| - |Y| = d.$$ 

Therefore (ii) does not hold.

We construct a family of multipliers with domination number $2$. Let $n \geq 2$ and consider disjoint complete graphs $K_{n+1}$ and $K_{2n}$, with vertex sets $A = \ldots$
\(\{v_1, v_2, \ldots, v_{n+1}\}\) and \(B = \{w_1, w_2, \ldots, w_{2n}\}\), respectively. Let \(G_n\) be the graph obtained by adding the edges \(v_iw_i, i = 1, \ldots, n+1\). We use Proposition 12 to show that \(G_n\) is an \(n\)-multiplier. Since \(\gamma(G) = 2\), we only consider sets \(X\) of cardinality 1. There are three possibilities for \(X\).

- If \(X = \{v_i\}\), then \(Y = B - \{w_i\}\) and \(|Y| = 2n - 1 = n\gamma(G_n) - |X|\).
- If \(X = \{w_i\}\) with \(i \leq n+1\), then \(Y = A - \{v_i\}\) and \(|Y| = n = n\gamma(G_n) - |X| - d\) with \(d = n - 1\). For any \(Y' \subseteq Y\), \(N(w_i) - N[Y']\) contains the vertices \(w_{n+2}, \ldots, w_{2n}\). Thus, for any partition \(Y_2, Y_3, \ldots, Y_n\) of \(Y\), the subgraph of \(G_n \square K_n\) induced by \(\bigcup_{j=2}^{n}(N(w_i) - N[Y_j])\) has a subgraph isomorphic to \(K_{n-1} \square K_{n-1}\), which has domination number \(d = n - 1\). Hence Proposition 12(ii) holds.
- If \(X = \{w_i\}\), \(i > n + 1\), a similar argument shows that Proposition 12(ii) also holds.

It follows that \(G\) is an \(n\)-multiplier.

5. Conclusion

We conclude with open problems for future research. Let \(G\) and \(H\) be graphs of order \(m\) and \(n\) respectively. The Cartesian product \(G \square H\) possesses a so-called layer-partition property, in that its vertex set allows two partitions \(P = \{P_1, P_2, \ldots, P_m\}\) and \(Q = \{Q_1, Q_2, \ldots, Q_m\}\) such that (a) each \(P_i \in P\) induces a copy of \(G\), called a \(G\)-layer, (b) each \(Q_j \in Q\) induces a copy of \(H\), called an \(H\)-layer, (c) any \(P_i\) and \(Q_j\) intersect in exactly one vertex, and (d) any edge in the product is in either exactly one \(G\)-layer or exactly one \(H\)-layer.

In 1967, Chartrand and Harary [2] defined the generalized prism \(\pi G\) of \(G\) as the graph consisting of two copies of \(G\), with edges between the copies determined by a permutation \(\pi\) acting on \(V(G)\). For any permutation \(\pi\), \(\gamma(G) \leq (\pi G) \leq 2\gamma(G)\).

We now define a generalized Cartesian product \(G \boxtimes H\) that corresponds to \(G \square H\) when \(\pi\) is the identity, \(\pi G\) when \(H\) is the graph \(K_2\), and that retains a layer-partition property. For two labelled graphs \(G\) and \(H\) and permutation \(\pi\) acting on \(V(G)\), the product \(G \boxtimes H\) is the graph with vertex set \(V(G) \times V(H)\), and vertex \((v_i, u_j)\) is adjacent to \((v_k, u_l)\), \(j \leq l\), if and only if (a) \(v_iv_k \in E(G)\) and \(u_j = u_l\), or (b) \(v_k = \pi^{-1}(v_i)\) and \(u_ju_l \in E(H)\).

Note that \(\gamma(G) \leq \gamma(G \boxtimes H) \leq \gamma(G)|V(H)|\) for any \(G, H\) and permutation \(\pi\). Burger, Mynhardt and Weakley [1] investigated graphs \(G\) for which \(\gamma(\pi G) = 2\gamma(G)\) for any \(\pi\).

**Question 1.** For some graph \(H\) of order \(n\), is it possible to characterize graphs \(G\) for which \(\gamma(G \boxtimes H) = n\gamma(G)\) for every \(\pi\)?
In 2006, Mynhardt and Xu [6] investigated graphs $G$ for which $\gamma(\pi G) = \gamma(G)$ for any $\pi$, and conjectured that only the edgeless graphs have this property.

**Question 2.** For some graph $H$ of order $n$, does there exist a nontrivial graph $G$ such that $\gamma(G \Box H) = \gamma(G) + n - 2$ for every $\pi$?

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