THE $k$-RAINBOW DOMATIC NUMBER OF A GRAPH

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Abstract

For a positive integer $k$, a $k$-rainbow dominating function of a graph $G$ is a function $f$ from the vertex set $V(G)$ to the set of all subsets of the set \{1, 2, \ldots, k\} such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition \( \bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\} \) is fulfilled, where $N(v)$ is the neighborhood of $v$. The 1-rainbow domination is the same as the ordinary domination. A set \( \{f_1, f_2, \ldots, f_d\} \) of $k$-rainbow dominating functions on $G$ with the property that \( \sum_{i=1}^{d} |f_i(v)| \leq k \) for each $v \in V(G)$, is called a $k$-rainbow dominating family (of functions) on $G$. The maximum number of functions in a $k$-rainbow dominating family on $G$ is the $k$-rainbow domatic number of $G$, denoted by $d_{rk}(G)$. Note that $d_{r1}(G)$ is the classical domatic number $d(G)$. In this paper we initiate the study of the $k$-rainbow domatic number in graphs and we present some bounds for $d_{rk}(G)$. Many of the known bounds of $d(G)$ are immediate consequences of our results.

Keywords: $k$-rainbow dominating function, $k$-rainbow domination number, $k$-rainbow domatic number.

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1. Introduction

In this paper, $G$ is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $d(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of $S$ is the set $N[S] = N(S) \cup S$. The complement of a graph $G$ is denoted by $\overline{G}$. We write $K_n$ for the complete graph of order $n$, $C_n$ for a cycle of length $n$ and $P_n$ for a path of order $n$.

A subset $S$ of vertices of $G$ is a dominating set if $N[S] = V$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A domatic partition is a partition of $V$ into dominating sets, and the domatic number $d(G)$ is the largest number of sets in a domatic partition. The domatic number was introduced by Cockayne and Hedetniemi [7]. In their paper, they showed that

\begin{equation}
\gamma(G) \cdot d(G) \leq n.
\end{equation}

For a positive integer $k$, a $k$-rainbow dominating function (kRDF) of a graph $G$ is a function $f$ from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The weight of a kRDF $f$ is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The $k$-rainbow domination number of a graph $G$, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of $G$. A $\gamma_{rk}(G)$-function is a $k$-rainbow dominating function of $G$ with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The $k$-rainbow domination number was introduced by Brešar, Henning, and Rall [2] and has been studied by several authors (see for example [3, 4, 5, 12]). Rainbow domination of a graph $G$ coincides with ordinary domination of the Cartesian product of $G$ with the complete graph, in particular, $\gamma_{rk}(G) = \gamma(G \Box K_k)$ for any graph $G$ [2]. This implies (cf. [4]) that

\begin{equation}
\gamma_{r1}(G) \leq \gamma_{r2}(G) \leq \cdots \leq \gamma_{rk}(G) \leq n \text{ for any graph } G \text{ of order } n.
\end{equation}

Furthermore, it was proved in [8] that

\[
\min\{|V(G)|, \gamma(G) + k - 2\} \leq \gamma_{rk}(G) \leq k\gamma(G) \text{ for any } k \geq 2 \text{ and any graph } G.
\]

A set $\{f_1, f_2, \ldots, f_d\}$ of $k$-rainbow dominating functions of $G$ with the property that $\sum_{i=1}^{d} |f_i(v)| \leq k$ for each $v \in V(G)$, is called a $k$-rainbow dominating family (of functions) on $G$. The maximum number of functions in a $k$-rainbow dominating family (kRDF family) on $G$ is the $k$-rainbow domatic number of $G$, denoted by
The \( d_{rk}(G) \). The \( k \)-rainbow domatic number is well-defined and

\[
d_{rk}(G) \geq k, \text{ for all graphs } G
\]

since the set consisting of the function \( f_i : V(G) \rightarrow \mathcal{P}\{1,2,\ldots,k\} \) defined by \( f_i(v) = \{i\} \) for each \( v \in V(G) \) and each \( i \in \{1,2,\ldots,k\} \), forms a kRD family on \( G \).

Our purpose in this paper is to initiate the study of the \( k \)-rainbow domatic number in graphs. We first study basic properties and bounds for the \( k \)-rainbow domatic number of a graph. In addition, we determine the 2-rainbow domatic number of some classes of graphs.

2. Properties of the \( k \)-rainbow Domatic Number

In this section we mainly present basic properties of \( d_{rk}(G) \) and bounds on the \( k \)-rainbow domatic number of a graph. However, we start with a lower and an upper bound on the \( k \)-rainbow domination number.

**Observation 1.** If \( G \) is a graph of order \( n \), then \( \gamma_{rk}(G) \leq n - \Delta(G) + k - 1 \).

**Proof.** Let \( v \) be a vertex of maximum degree \( \Delta(G) \). Define \( f : V(G) \rightarrow \mathcal{P}\{1,2,\ldots,k\} \) by \( f(v) = \{1,2,\ldots,k\} \) and

\[
f(x) = \begin{cases} 
\emptyset & \text{if } x \in N(v), \\
\{1\} & \text{if } x \in V(G) - N[v].
\end{cases}
\]

It is easy to see that \( f \) is a \( k \)-rainbow dominating function on \( G \) and so \( \gamma_{rk}(G) \leq n - \Delta(G) + k - 1 \).

Let \( k \geq 1 \) be an integer, and let \( G \) be a graph of order \( n \geq k \) and maximum degree \( \Delta(G) = n - 1 \). Since \( n \geq k \), we observe that \( \gamma_{rk}(G) \geq k \). If \( v \) is a vertex of maximum degree \( \Delta(G) \), then define \( f : V(G) \rightarrow \mathcal{P}\{1,2,\ldots,k\} \) by \( f(v) = \{1,2,\ldots,k\} \), \( f(x) = \emptyset \) if \( x \in V(G) \setminus \{v\} \). Because of \( d(v) = \Delta(G) = n - 1 \), \( f \) is a \( k \)-rainbow dominating function on \( G \) and thus \( \gamma_{rk}(G) \leq k \). It follows that \( \gamma_{rk}(G) = k = n - \Delta(G) + k - 1 \). This example shows that Observation 1 is sharp. The case \( k = 1 \) in Observation 1 is attributed to Berge [1]. In 1979, Walikar, Acharya and Sampathkumar [10] proved \( \gamma(G) \geq \lceil n/(\Delta(G) + 1) \rceil \) for each graph of order \( n \). Next we will give an analogous lower bound for \( \gamma_{rk}(G) \) when \( k \geq 2 \).

**Theorem 2.** If \( G \) is a graph of order \( n \) and maximum degree \( \Delta \), then

\[
\gamma_{r2}(G) \geq \left\lceil \frac{2n}{\Delta + 2} \right\rceil.
\]
Proof. Let $f$ be a $\gamma_{r_2}(G)$-function and let $V_i = \{v \mid |f(v)| = i\}$ for $i = 0, 1, 2$. Then $\gamma_{r_2}(G) = |V_1| + 2|V_2|$ and $n = |V_0| + |V_1| + |V_2|$. Since each vertex of $V_0$ is adjacent to at least one vertex of $V_2$ or at least two vertices of $V_1$, we deduce that $|V_0| \leq \Delta|V_2| + \frac{1}{2}\Delta|V_1|$. This implies that 

$$(\Delta + 2)\gamma_{r_2}(G) = 2\gamma_{r_2}(G) + \Delta(|V_1| + 2|V_2|) \geq 2\gamma_{r_2}(G) + 2|V_0|$$

$$= 2|V_1| + 4|V_2| + 2|V_0| = 2n + 2|V_2| \geq 2n,$$

and this leads to the desired bound. $\blacksquare$

Using inequality (2) and Theorem 2, we obtain the next result immediately.

**Theorem 3.** If $k \geq 2$ is an integer, and $G$ is a graph of order $n$ and maximum degree $\Delta$, then

$$\gamma_{rk}(G) \geq \left\lceil \frac{2n}{\Delta + 2} \right\rceil.$$  

**Theorem 4.** If $G$ is a graph of order $n$, then $\gamma_{rk}(G) \cdot d_{rk}(G) \leq kn$. Moreover, if $\gamma_{rk}(G) \cdot d_{rk}(G) = kn$, then for each kRD family $\{f_1, f_2, \ldots, f_d\}$ on $G$ with $d = d_{rk}(G)$, each function $f_i$ is a $\gamma_{rk}(G)$-function and $\sum_{i=1}^{d} |f_i(v)| = k$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a kRD family on $G$ such that $d = d_{rk}(G)$. Then 

$$d \cdot \gamma_{rk}(G) = \sum_{i=1}^{d} \gamma_{rk}(G) \leq \sum_{i=1}^{d} \sum_{v \in V} |f_i(v)|$$

$$= \sum_{v \in V} \sum_{i=1}^{d} |f_i(v)| \leq \sum_{v \in V} k = kn.$$  

If $\gamma_{rk}(G) \cdot d_{rk}(G) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the kRD family $\{f_1, f_2, \ldots, f_d\}$ on $G$ and for each $i$, $\sum_{v \in V} |f_i(v)| = \gamma_{rk}(G)$. Thus each function $f_i$ is a $\gamma_{rk}(G)$-function, and $\sum_{i=1}^{d} |f_i(v)| = k$ for all $v \in V$. $\blacksquare$

The case $k = 1$ in Theorem 4 leads to the well-known inequality $\gamma(G) \cdot d(G) \leq n$, given by Cockayne and Hedetniemi [7] in 1977.

**Corollary 5.** If $k$ is a positive integer, and $G$ is a graph of order $n \geq k$, then 

$$d_{rk}(G) \leq n.$$  

Proof. The hypothesis $n \geq k$ leads to $\gamma_{rk}(G) \geq k$. Therefore it follows from Theorem 4 that 

$$d_{rk}(G) \leq \frac{kn}{\gamma_{rk}(G)} \leq \frac{kn}{k} = n,$$

and this is the desired inequality. $\blacksquare$
Corollary 6. If \( k \) is a positive integer, and \( G \) is isomorphic to the complete graph \( K_n \) of order \( n \geq k \), then \( d_{rk}(G) = n \).

Proof. In view of Corollary 5, we have \( d_{rk}(G) \leq n \). If \( \{v_1, v_2, \ldots, v_n\} \) is the vertex set of \( G \), then we define the function \( f_i : V(G) \to P(\{1, 2, \ldots, k\}) \) by \( f_i(v_j) = \{1, 2, \ldots, k\} \) for \( i = j \) and \( f_i(v_j) = \emptyset \) for \( i \neq j \), where \( i, j \in \{1, 2, \ldots, n\} \). Then \( \{f_1, f_2, \ldots, f_n\} \) is a kRD family on \( G \) and thus \( d_{rk}(G) = n \). 

Theorem 7. If \( G \) is a graph of order \( n \geq k \), then

\[
\gamma_{rk}(G) + d_{rk}(G) \leq n + k.
\]

Proof. Applying Theorem 4, we obtain

\[
\gamma_{rk}(G) + d_{rk}(G) \leq \frac{kn}{d_{rk}(G)} + d_{rk}(G).
\]

Note that \( d_{rk}(G) \geq k \), by inequality (3), and that Corollary 5 implies that \( d_{rk}(G) \leq n \). Using these inequalities, and the fact that the function \( g(x) = x + (kn)/x \) is decreasing for \( k \leq x \leq \sqrt{kn} \) and increasing for \( \sqrt{kn} \leq x \leq n \), we obtain

\[
\gamma_{rk}(G) + d_{rk}(G) \leq \max \left\{ \frac{kn}{k} + k, \frac{kn}{n} + n \right\} = n + k,
\]

and this is the desired bound.

If \( G \) is isomorphic to the complete graph of order \( n \geq k \), then \( \gamma_{rk}(G) = k \) and \( d_{rk}(G) = n \) by Corollary 6. Thus \( \gamma_{rk}(K_n) \cdot d_{rk}(K_n) = nk \) and \( \gamma_{rk}(K_n) + d_{rk}(K_n) = n + k \) when \( n \geq k \). This example shows that Theorems 4 and 7 are sharp.

Corollary 8 (Cockayne and Hedetniemi, [7], 1977). If \( G \) is a graph of order \( n \geq 1 \), then \( \gamma(G) + d(G) \leq n + 1 \)

Theorem 9. For every graph \( G \),

\[
d_{rk}(G) \leq \delta(G) + k.
\]

Proof. Let \( \{f_1, f_2, \ldots, f_d\} \) be a kRD family on \( G \) such that \( d = d_{rk}(G) \), and let \( v \) be a vertex of minimum degree \( \delta(G) \). Since \( \sum_{u \in N[v]} |f_i(u)| \geq 1 \) for all \( i \in \{1, 2, \ldots, d\} \) and \( \sum_{u \in N[v]} |f_i(u)| < k \) for at most \( k \) indices \( i \in \{1, 2, \ldots, d\} \), we obtain

\[
k d - k(k - 1) \leq \sum_{i=1}^{d} \sum_{u \in N[v]} |f_i(u)| = \sum_{u \in N[v]} \sum_{i=1}^{d} |f_i(u)| \leq \sum_{u \in N[v]} k = k(\delta(G) + 1),
\]

and this leads to the desired bound.
To prove sharpness of Theorem 9, let \( p \geq 2 \) be an integer, and let \( G_i \) be a copy of \( K_{p+k+1} \) with vertex set \( V(G_i) = \{ v_1^i, v_2^i, \ldots, v_{p+k+1}^i \} \) for \( 1 \leq i \leq p \). Now let \( G \) be the graph obtained from \( \bigcup_{i=1}^p G_i \) by adding a new vertex \( v \) and joining \( v \) to each \( v_1^i \). Define the \( k \)-rainbow dominating functions \( f_1, f_2, \ldots, f_{p+k} \) as follows: for \( 1 \leq i \leq p \) and \( 1 \leq s \leq k \),

- \( f_i(v_1^i) = \{ 1, 2, \ldots, k \} \),
- \( f_i(v_{j+1}^i) = \{ 1, 2, \ldots, k \} \) if \( j \in \{ 1, 2, \ldots, p \} - \{ i \} \) and \( f(x) = \emptyset \) otherwise,
- \( f_{p+s}(v) = \{ 1 \} \),
- \( f_{p+s}(v_{j+1}^i) = \{ 1, 2, \ldots, k \} \) if \( j \in \{ 1, 2, \ldots, p \} \) and \( f(x) = \emptyset \) otherwise.

It is straightforward to verify that \( f_i \) is a \( k \)-rainbow dominating function on \( G \) for each \( i \) and \( \{ f_1, f_2, \ldots, f_{p+k} \} \) is a \( k \)-rainbow dominating family on \( G \). Since \( \delta(G) = p \), we have \( d_{r_k}(G) = \delta(G) + k \).

The special case \( k = 1 \) in Theorem 9 was done by Cockayne and Hedetniemi [7]. As an application of Theorem 9, we will prove the following Nordhaus-Gaddum type result.

**Theorem 10.** For every graph \( G \) of order \( n \),

\[
d_{rk}(G) + d_{rk}(\overline{G}) \leq n + 2k - 1.
\]

If \( d_{rk}(G) + d_{rk}(\overline{G}) = n + 2k - 1 \), then \( G \) is regular.

**Proof.** It follows from Theorem 9 that

\[
d_{rk}(G) + d_{rk}(\overline{G}) \leq (\delta(G) + k) + (\delta(\overline{G}) + k)
= (\delta(G) + k) + (n - \Delta(G) - 1 + k) \leq n + 2k - 1.
\]

If \( G \) is not regular, then \( \Delta(G) - \delta(G) \geq 1 \), and this inequality chain leads to the better bound \( d_{rk}(G) + d_{rk}(\overline{G}) \leq n + 2k - 2 \), and the proof is complete. \( \square \)

**Corollary 11** (Cockayne and Hedetniemi [7] 1977). If \( G \) is a graph of order \( n \geq 1 \), then \( d(G) + d(\overline{G}) \leq n + 1 \).

3. Properties of the 2-rainbow Domatic Number

Let \( A_1 \cup A_2 \cup \cdots \cup A_d \) be a domatic partition of \( V(G) \) into dominating sets such that \( d = d(G) \). Then the set of functions \( \{ f_1, f_2, \ldots, f_d \} \) with \( f_i(v) = \{ 1, 2 \} \) if \( v \in A_i \) and \( f_i(v) = \emptyset \), otherwise for \( 1 \leq i \leq d \) is a 2RD family on \( G \). This shows that \( d(G) \leq d_{r2}(G) \) for every graph \( G \).

**Observation 12.** Let \( G \) be a graph of order \( n \geq 2 \). Then \( \gamma_{r2}(G) = n \) and \( d_{r2}(G) = 2 \) if and only if \( \Delta(G) \leq 1 \).
If $\gamma_{r2}(G) = n$, then, by Theorem 1, $\Delta(G) \leq 1$.

Conversely, let $\Delta(G) \leq 1$. If $\Delta(G) = 0$, then obviously $\gamma_{r2}(G) = n$ and $d_{r2}(G) = 2$. Let $\Delta(G) = 1$. Then $G = rK_1 \cup \frac{n-r}{2}K_2$ with $n-r \geq 2$ even, and we have

$$\gamma_{r2}(G) = r\gamma_{r2}(K_1) + \frac{n-r}{2}\gamma_{r2}(K_2) = r + (n-r) = n.$$

By (3) and Theorem 4, we obtain $d_{r2}(G) = 2$. This completes the proof. 

Using Theorem 9 and the following proposition, we determine the 2-rainbow domatic number of paths.

**Proposition A [3].** For $n \geq 2$,

$$\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

**Proposition 13.** For $n \geq 3$,

$$d_{r2}(P_n) = \begin{cases} 2 & \text{if } n = 4, \\ 3 & \text{otherwise}. \end{cases}$$

**Proof.** Let $G = P_n$. If $n = 4$, then Proposition 3 implies $\gamma_{r2}(G) = 3$, and the result follows from Theorem 4 and (3). Assume now that $n \neq 4$. By Theorem 4 and Proposition 3, we have $d_{r2}(G) \leq 3$. Consider four cases.

**Case 1.** $n \equiv 3 \pmod{4}$. Define the 2-rainbow dominating functions $f_1, f_2, f_3$ as follows:

$f_1(v_{4i+1}) = \{1\}, f_1(v_{4i+3}) = \{2\}$ for $0 \leq i \leq (n-3)/4$, and $f_1(x) = \emptyset$ otherwise,

$f_2(v_{4i+1}) = \{2\}, f_2(v_{4i+3}) = \{1\}$ for $0 \leq i \leq (n-3)/4$, and $f_2(x) = \emptyset$ otherwise,

$f_3(v_{2i+2}) = \{1, 2\}$ for $0 \leq i \leq (n-3)/2$, and $f_3(x) = \emptyset$ otherwise.

It is easy to see that $f_i$ is a 2-rainbow dominating function on $G$ for each $i$ and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on $G$.

**Case 2.** $n \equiv 1 \pmod{4}$. Define the 2-rainbow dominating functions $f_1, f_2, f_3$ as follows:

$f_1(v_n) = \{1\}, f_1(v_{4i+1}) = \{1\}, f_1(v_{4i+3}) = \{2\}$ for $0 \leq i \leq (n-1)/4 - 1$ and $f_1(x) = \emptyset$ otherwise,

$f_2(v_n) = \{2\}, f_2(v_{4i+1}) = \{2\}, f_2(v_{4i+3}) = \{1\}$ for $0 \leq i \leq (n-1)/4 - 1$ and $f_2(x) = \emptyset$ otherwise,

$f_3(v_{2i}) = \{1, 2\}$ for $1 \leq i \leq (n-1)/2$, and $f_3(x) = \emptyset$ otherwise.

Clearly, $f_i$ is a 2-rainbow dominating function on $G$ for each $i$ and $\{f_1, f_2, f_3\}$ is a 2-rainbow dominating family on $G$. 

Case 3. $n \equiv 0 \pmod{4}$. Define the 2-rainbow dominating functions $f_1, f_2, f_3$ as follows:

- $f_1(v_1) = f_1(v_n) = f_1(v_{4i+6}) = \{1\}$, $f_1(v_3) = f_1(v_4) = f_1(v_{4i+8}) = \{2\}$ for $0 \leq i \leq n/4 - 2$, and $f_1(x) = \emptyset$ otherwise,
- $f_2(v_1) = f_2(v_n) = f_2(v_{4i+6}) = \{2\}$, $f_2(v_3) = f_2(v_4) = f_2(v_{4i+8}) = \{1\}$ for $0 \leq i \leq n/4 - 2$, and $f_2(x) = \emptyset$ otherwise,
- $f_3(v_2) = f_3(v_{2i+1}) = \{1, 2\}$ for $2 \leq i \leq n/2 - 1$, and $f_3(x) = \emptyset$ otherwise.

It is easy to see that $f_i$ is a 2-rainbow dominating function on $G$ for each $i$ and \{f_1, f_2, f_3\} is a 2-rainbow dominating family on $G$.

Case 4. $n \equiv 2 \pmod{4}$. Define the 2-rainbow dominating functions $f_1, f_2, f_3$ as follows:

- $f_1(v_1) = f_1(v_n) = f_1(v_{4i+6}) = \{1\}$, $f_1(v_3) = f_1(v_4) = f_1(v_{4i+8}) = \{2\}$ for $0 \leq i \leq (n-2)/4-2$, and $f_1(x) = \emptyset$ otherwise,
- $f_2(v_1) = f_2(v_n) = f_2(v_{4i+6}) = \{2\}$, $f_2(v_3) = f_2(v_4) = f_2(v_{4i+8}) = \{1\}$ for $0 \leq i \leq (n-2)/4-2$, and $f_2(x) = \emptyset$ otherwise,
- $f_3(v_2) = f_3(v_{2i+1}) = \{1, 2\}$ for $2 \leq i \leq n/2 - 1$, and $f_3(x) = \emptyset$ otherwise.

Clearly $f_i$ is a 2-rainbow dominating function on $G$ for each $i$ and \{f_1, f_2, f_3\} is a 2-rainbow dominating family on $G$. This completes the proof. ■

Using Theorem 4 and the following proposition, we determine the 2-rainbow dominant number of cycles.

**Proposition B [3]**. For $n \geq 3$,

$$\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.$$

**Proposition 14**. If $C_n$ is the cycle on $n \geq 4$ vertices, then

$$d_{r2}(C_n) = \begin{cases} 4 & n \equiv 0 \pmod{4}, \\ 3 & \text{otherwise}. \end{cases}$$

**Proof**. Let $C_n = (v_1, v_2, \ldots, v_n)$. Consider four cases.

Case 1. $n \equiv 0 \pmod{4}$. Define the 2-rainbow dominating functions $f_1, f_2, f_3, f_4$ as follows:

- $f_1(v_{4i-1+1}) = \{1\}$, $f_1(v_{4i-1+3}) = \{2\}$ for $0 \leq i \leq n/4 - 1$, and $f_1(x) = \emptyset$ otherwise,
- $f_2(v_{4i-1+1}) = \{2\}$, $f_2(v_{4i-1+3}) = \{1\}$ for $0 \leq i \leq n/4 - 1$, and $f_2(x) = \emptyset$ otherwise,
- $f_3(v_{4i-1+2}) = \{1\}$, $f_3(v_{4i-1+4}) = \{2\}$ for $0 \leq i \leq n/4 - 1$, and $f_3(x) = \emptyset$ otherwise,
Clearly, $f_i$ is a 2-rainbow dominating function on $G$ for each $i$ and \{ $f_1, f_2, f_3, f_4$ \} is a 2-rainbow dominating family on $G$. Thus $d_{r2}(C_n) = 4$.

**Case 2.** $n \equiv 1 \pmod{4}$. Then by Theorem 4 and Proposition 3, $d_{r2}(C_n) \leq 3$.

Define the 2-rainbow dominating functions $f_1, f_2, f_3$ as follows:

$f_1(v_{4(i-1)+1}) = \{1\}, f_1(v_{4(i-1)+3}) = \{2\},$ for $0 \leq i \leq (n-1)/4 - 1$, and $f_1(x) = \emptyset$ otherwise,

$f_2(v_{4(i-1)+1}) = \{2\}, f_2(v_{4(i-1)+3}) = \{1\},$ for $0 \leq i \leq (n-1)/4 - 1$, and $f_2(x) = \emptyset$ otherwise,

$f_3(v_{4(i-1)+2}) = f_3(v_{4(i-1)+4}) = \{1, 2\}$ for $0 \leq i \leq (n-1)/4 - 1$, and $f_3(x) = 0$ otherwise.

Clearly, $f_i$ is a 2-rainbow dominating function on $G$ for each $i$ and \{ $f_1, f_2, f_3$ \} is a 2-rainbow dominating family on $G$. Thus $d_{r2}(C_n) = 3$.

**Case 3.** $n \equiv 3 \pmod{4}$. Then by Theorem 4 and Proposition 3, $d_{r2}(C_n) \leq 3$.

Define the 2-rainbow dominating functions $f_1, f_2, f_3$ as follows:

$f_1(v_{4(i-1)+1}) = \{1\}, f_1(v_{4(i-1)+3}) = \{2\},$ for $0 \leq i \leq (n+1)/4 - 1$, and $f_1(x) = \emptyset$ otherwise,

$f_2(v_{4(i-1)+1}) = \{2\}, f_2(v_{4(i-1)+3}) = \{1\},$ for $0 \leq i \leq (n+1)/4 - 1$, and $f_2(x) = \emptyset$ otherwise,

$f_3(v_{4(i-1)+2}) = f_3(v_{4(i-1)+4}) = \{1, 2\}$ for $0 \leq i \leq (n-3)/4 - 1$, $f_3(x) = 1$ and $f_3(x) = 0$ otherwise.

Clearly, $f_i$ is a 2-rainbow dominating function on $G$ for each $i$ and \{ $f_1, f_2, f_3$ \} is a 2-rainbow dominating family on $G$. Thus $d_{r2}(C_n) = 3$.

**Case 4.** $n \equiv 2 \pmod{4}$. Then by Theorem 4 and Proposition 3, $d_{r2}(C_n) \leq 3$.

Define the 2-rainbow dominating functions $f_1, f_2, f_3$ as follows:

$f_1(v_1) = f_1(v_2) = f_1(v_{4i+3}) = \{1\}, f_1(v_4) = f_1(v_5) = f_1(v_{4i+5}) = \{2\}$ for $1 \leq i \leq \frac{n-6}{4}$ and $f_1(x) = \emptyset$ otherwise,

$f_2(v_1) = f_2(v_2) = f_2(v_{4i+3}) = \{2\}, f_2(v_4) = f_2(v_5) = f_2(v_{4i+5}) = \{1\}$ for $1 \leq i \leq \frac{n-6}{4}$ and $f_2(x) = \emptyset$ otherwise,

$f_3(v_3) = f_3(v_{4i+2}) = \{1, 2\}$ for $1 \leq i \leq \frac{n-2}{4}$ and $f_3(x) = \emptyset$ otherwise.

Clearly, $f_i$ is a 2-rainbow dominating function on $G$ for each $i$ and \{ $f_1, f_2, f_3$ \} is a 2-rainbow dominating family on $G$. Thus $d_{r2}(C_n) = 3$.

Theorem 2 and its proof lead immediately to the next result.
Corollary 15. Let $G$ be a graph of order $n$ and maximum degree $\Delta$. Then

$$ \gamma_r(G) \geq \begin{cases} \left\lceil \frac{2n+2}{\Delta+2} \right\rceil & \text{if there is a } \gamma_r(G) \text{-function } f \text{ with } V_2 \neq \emptyset, \\ \left\lceil \frac{2n+2}{\Delta+2} \right\rceil & \text{otherwise.} \end{cases} $$

Using Corollary 15, we will improve the upper bound on $d_r(G)$ given in Theorem 9 for some regular graphs.

**Theorem 16.** If $G$ is a $\delta$-regular graph of order $n$ with $\delta \geq 1$ and a $\gamma_r(G)$-function $f$ such that $V_2 \neq \emptyset$ or $2n \not\equiv 0 \pmod{\delta+2}$, then

$$ d_r(G) \leq \delta + 1. $$

**Proof.** Let $\{f_1, f_2, \ldots, f_d\}$ be a 2RD family on $G$ such that $d = d_r(G)$. It follows that

$$ \sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V} |f_i(v)| = \sum_{v \in V} \sum_{i=1}^{d} |f_i(v)| \leq \sum_{v \in V} 2 = 2n. \quad (4) $$

Suppose to the contrary that $d \geq \delta + 2$. If $V_2 \neq \emptyset$, then Corollary 15 leads to

$$ \sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_r(G) \geq d \left\lceil \frac{2n+2}{\delta+2} \right\rceil \geq (\delta + 2) \left( \frac{2n+2}{\delta+2} \right) > 2n, $$

a contradiction to the inequality (4). If $2n \not\equiv 0 \pmod{\delta+2}$, then it follows from Corollary 15 that

$$ \sum_{i=1}^{d} \omega(f_i) \geq \sum_{i=1}^{d} \gamma_r(G) \geq d \left\lceil \frac{2n}{\delta+2} \right\rceil > (\delta + 2) \left( \frac{2n}{\delta+2} \right) = 2n, $$

a contradiction to (4) again. Therefore $d \leq \delta + 1$ and the proof is complete. \(\blacksquare\)

By Theorem 14, $d_r(C_4) = 4$ and therefore $d_r(C_4) = \delta(C_4) + 2$. This 2-regular graph demonstrates that the bound in Theorem 16 is not valid in general in the case that $2n \equiv 0 \pmod{\delta+2}$.

Using Theorems 9, 10 and 16, we will improve the upper bound given in Theorem 10 in the case that $k = 2$.

**Theorem 17.** If $G$ is a graph of order $n$, then

$$ d_r(G) + d_r(G) \leq n + 2. $$
Proof. If $G$ is not regular, then Theorem 10 implies the desired result. Now let $G$ be $\delta$-regular.

Assume that $G$ has a $\gamma_{r2}(G)$-function $f$ such that $V_2 \neq \emptyset$ or $V_2 = \emptyset$ and $2|V_0| < \delta|V_1|$. Then we deduce from Theorem 16 that $d_{r2}(G) \leq \delta + 1$. Using Theorem 9, we obtain the desired result as follows

$$d_{r2}(G) + d_{r2}(\overline{G}) \leq (\delta(G) + 1) + (\delta(\overline{G}) + 2)$$

$$= (\delta(G) + 1) + (n - \delta(G) - 1 + 2) = n + 2.$$ 

It remains the case that $G$ has a $\gamma_{r2}(G)$-function $f$ such $V_2 = \emptyset$ and $2|V_0| = \delta|V_1|$. Note that $n = |V_0| + |V_1|$ and $|V_1| \geq 2$. Since $\delta(G) + \delta(G) = n - 1$, it follows that $\delta(G) \geq (n - 1)/2$ or $\delta(\overline{G}) \geq (n - 1)/2$. We assume, without loss of generality, that $\delta(G) \geq (n - 1)/2$.

If $|V_1| \geq 4$, then $2|V_0| = \delta|V_1| \geq 4\delta$ and thus $|V_0| \geq 2\delta$. This leads to the contradiction

$$n = |V_0| + |V_1| \geq 2\delta + 4 \geq n - 1 + 4 = n + 3.$$ 

In the case $|V_1| = 3$, we define $V_1' = \{v \mid f(v) = \{1\}\}$ and $V_1'' = \{v \mid f(v) = \{2\}\}$. We assume, without loss of generality, that $|V_1' = 1 < 2 = |V_1''|$. Since each vertex of $V_0$ is adjacent to at least one vertex of $V_1'$, we deduce that $|V_0| \leq \delta < 2\delta$. This implies that

$$2|V_0| = |V_0| + |V_0| < \delta + 2\delta = \delta|V_1'| + \delta|V_1''| = \delta|V_1|,$$ 

a contradiction to the assumption $2|V_0| = \delta|V_1|$. 

If $|V_1| = 2$, then $|V_0| = \delta$ and so $n = \delta + 2$. Hence $\delta(\overline{G}) = n - \delta - 1 = 1$ and so $d_{r2}(\overline{G}) = 2$. Now Theorem 9 implies that

$$d_{r2}(G) + d_{r2}(\overline{G}) \leq (\delta(G) + 2) + 2 = n + 2,$$ 

the desired bound. Since we have discussed all possible cases, the proof is complete.

If $G$ is isomorphic to the complete graph $K_n$ with $n \geq 2$, then Corollary 6 implies $d_{r2}(G) = n$. Since $d_{r2}(\overline{G}) = 2$, we obtain $d_{r2}(G) + d_{r2}(\overline{G}) = n + 2$. This example demonstrates that Theorem 17 is sharp.

We conclude this paper with a conjecture.

Conjecture 18. For every integer $k \geq 2$ and every graph $G$ of order $n$,

$$d_{r_k}(G) + d_{r_k}(\overline{G}) \leq n + 2k - 2.$$ 

Note that Theorem 17 shows that this conjecture is valid for $k = 2$. In addition, the complete graph $K_n$ demonstrates that Conjecture 1 does not hold for $k = 1$. 


References


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