LIST COLORING OF COMPLETE MULTIPARTITE GRAPHS

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Abstract

The choice number of a graph $G$ is the smallest integer $k$ such that for every assignment of a list $L(v)$ of $k$ colors to each vertex $v$ of $G$, there is a proper coloring of $G$ that assigns to each vertex $v$ a color from $L(v)$. We present upper and lower bounds on the choice number of complete multipartite graphs with partite classes of equal sizes and complete $r$-partite graphs with $r-1$ partite classes of order two.

Keywords: list coloring, choice number, complete multipartite graph.

2010 Mathematics Subject Classification: 05C15.

1. Introduction

All graphs considered here are finite, undirected, without loops and multiple edges. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A list assignment to the vertices of a graph $G$ is the assignment of a list $L(v)$ of colors $C$ to every vertex $v \in V(G)$. A $k$-list assignment is a list assignment such that $|L(v)| \geq k$ for every vertex $v$. An $L$-coloring of $G$ is a function $f : V(G) \to C$ such that $f(v) \in L(v)$ for all $v \in V(G)$ and $f(v) \neq f(w)$ for each edge $vw \in E(G)$. If $G$ has an $L$-coloring, then $G$ is said to be $L$-colorable. If for any $k$-list assignment $L$ there exists an $L$-coloring, then $G$ is $k$-choosable. The choice number $Ch(G)$ of a graph $G$ is the minimum integer $k$ such that $G$ is $k$-choosable.

The study of choice numbers of graphs was initiated by Vizing [7] and by Erdős, Rubin and Taylor [3]. For a survey about the list coloring problem we refer to [6] and [8]. In this paper we focus on the choice numbers of complete multipartite graphs.
2. Complete Multipartite Graphs with Partite Classes of Different Sizes

Let $K_{n_1, n_2, \ldots, n_r}$ be the complete $r$-partite graph with the partite classes of order $n_1, n_2, \ldots, n_r$. A well-known result of Erdős, Rubin and Taylor [3] says that the choice number of the complete $r$-partite graph $K_{2,2,\ldots,2}$ is $r$. Gravier and Maffray [4] proved that also $\text{Ch}(K_{3,3,2,\ldots,2}) = r$ for $r \geq 3$. Enomoto et al. [2] showed that $\text{Ch}(K_{5,2,\ldots,2}) = r + 1$ and the choice number of the complete $r$-partite graph $K_{4,2,\ldots,2}$ is equal to $r$ if $r$ is odd, and $r + 1$ if $r$ is even.

Motivated by these results we study the value $\text{Ch}(K_{n,2,\ldots,2})$ for any positive integer $n$. In the proof of Theorem 1 we write $L(S)$ for the union $\bigcup_{v \in S} L(v)$ where $S \subseteq V(G)$. If $C$ is a set of colors, then $L \setminus C$ denotes the list assignment obtained from $L$ by removing the colors in $C$ from each $L(v)$ where $v \in V(G)$.

First, we show that the graph $K_{(t+2)(t+3)/2,2,\ldots,2}$ is $(r + t)$-choosable.

**Theorem 1.** Let $t$ be a positive integer and let $G$ be a complete $r$-partite graph with one partite class of order $(t + 2)(t + 3)/2$ and $r - 1$ partite classes of order two. Then $\text{Ch}(G) \leq r + t$.

**Proof.** Let $V_1$ be the partite class of order $(t + 2)(t + 3)/2$ and let $V_i = \{v_i, w_i\}$, $2 \leq i \leq r$, be the partite classes of order two. Let $L_1$ be any $(r + t)$-list assignment to the vertices of $G$. We prove that $G$ is $L_1$-colorable. We distinguish three cases:

**Case 1.** $t \geq r - 1$.
We can color the vertices of $V_2, V_3, \ldots, V_r$ with $2r - 2$ different colors. Since $|L_1(v)| \geq 2r - 1$ for every vertex $v \in V_1$, we can color the vertices of $V_1$ as well.

**Case 2.** There exists a color $c \in L_1(v_i) \cap L_1(w_i)$ for some $i \in \{2, 3, \ldots, r\}$.
It is easy to show by induction on $r$ that $G$ is $L_1$-colorable. The step $r = 1$ is trivial. For the induction step, assign $c$ to both $v_i$ and $w_i$, and remove $c$ from the lists of the remaining vertices. By the induction hypothesis, the remaining vertices can be colored with colors from the revised lists.

**Case 3.** $t \leq r - 2$ and $L_1(v_i) \cap L_1(w_i) = \emptyset$ for every $i \in \{2, 3, \ldots, r\}$.
We prove by contradiction that $G$ is $L_1$-colorable. Assume that $G$ is not $L_1$-colorable. Let $L$ be an $(r + t)$-list assignment such that $G$ is not $L$-colorable. Let $X_j$, $j = 1, 2, \ldots, t$, be the largest subset of $V_1 \setminus (\bigcup_{j=1}^{t} X_j)$ with $\bigcap_{v \in X_j} L(v) \neq \emptyset$. Set $|X_j| = x_j$ and choose a color $c_j \in \bigcap_{v \in X_j} L(v)$. Define $L' = L \setminus \{c_1, c_2, \ldots, c_t\}$ and $G' = G \setminus (\bigcup_{j=1}^{t} X_j)$. Note that $|L'(v)| = r + t$ for each $v \in V(G') \cap V_1$ and $|L'(v_i)|, |L'(w_i)| \geq r$ for any $i \in \{2, 3, \ldots, r\}$. Since $G$ is not $L$-colorable, $G'$ is not $L'$-colorable. It follows that there exists a set of vertices $T \subseteq V(G')$ such that $|L'(T)| < |T|$, i.e., $L'$ does not satisfy Hall’s condition. Let $S$ denote a maximal subset of $V(G')$ such that $|L'(S)| < |S|$. We consider two subcases:
Case 3a. \(|S \cap V_i| \leq 1\) for every \(i \in \{2, 3, \ldots, r\}\).

Since \(|L'(v_i)|, |L'(w_i)| \geq r\) and \(|S \setminus V_1| \leq r - 1\), \(S \setminus V_1\) can be colored from the list \(L'\). Further, \(|L'(v)| = r + t\) for \(v \in S \cap V_1\), therefore we can also color the vertices in \(S \cap V_1\).

Let \(L'' = L' \setminus L'(S)\). We show that \(G' \setminus S\) is \(L''\)-colorable. If \(G' \setminus S\) is not \(L''\)-colorable, we have a nonempty subset \(S' \subset V(G') \setminus S\) with \(|L''(S')| < |S'|\). Then \(|L'(S \cup S')| = |L'(S)| + |L''(S')| < |S| + |S'|\), which contradicts the maximality of \(S\).

Case 3b. Both \(v_i, w_i \in S\) for some \(i \in \{2, 3, \ldots, r\}\).

Then \(|S| > |L'(S)| \geq |L'(v_i)| + |L'(w_i)| \geq 2(r + t) - t\). Set \(|S| = 2r + t + 1 + \epsilon\) where \(\epsilon \geq 0\). Clearly, \(|L'(S)| \leq 2r + t + \epsilon\). Let \(S_1 = S \cap V_1\). We have \(|S_1| \geq |S| - (2r - 2) = t + 3 + \epsilon\). By the maximality of \(X_t\), every color in \(L'(S)\) appears in the lists of at most \(x_t\) vertices of \(S_1\). It means that

\[
(1) \quad (r + t)|S_1| = \sum_{v \in S_1} |L'(v)| \leq x_t|L'(S)|.
\]

It is evident that \(\sum_{i=1}^{t} x_i + |S_1| \leq |V_1| = (t + 2)(t + 3)/2\). Hence, \(tx_t + |S_1| \leq (t + 2)(t + 3)/2\), or equivalently

\[
(2) \quad x_t \leq [(t + 2)(t + 3)/2 - |S_1|]/t.
\]

By (1) and (2), we have \((r + t)|S_1| \leq [(t + 2)(t + 3)/2 - |S_1|]|L'(S)|/t\). Since \(|S_1| \geq t + 3 + \epsilon\) and \(|L'(S)| \leq 2r + t + \epsilon\), we have \((r + t)(t + 3 + \epsilon) \leq [(t + 2)(t + 3)/2 - (t + 3 + \epsilon)](2r + t + \epsilon)/t\) which yields \(\frac{t^3}{2} + (3 + \epsilon)\frac{t^2}{2} + (r - \frac{1}{2})t + (2r + \epsilon)\epsilon \leq 0\), a contradiction. This finishes the proof.

If \(t = 1\), then \(Ch(K_{6,2,\ldots,2}) \leq r + 1\). This bound also comes from the result \(Ch(K_{3,3,2,\ldots,2}) = r\) of Gravier and Maffray [4], because the complete \(r\)-partite graph \(K_{6,2,\ldots,2}\) is a subgraph of the complete \((r + 1)\)-partite graph \(K_{3,3,2,\ldots,2}\). Since the choice number of the complete \(r\)-partite graph \(K_{5,2,\ldots,2}\) is equal to \(r + 1\), it is clear that \(Ch(K_{6,2,\ldots,2}) = r + 1\) as well.

Now we present a lower bound on the choice number of complete \(r\)-partite graphs with \(r - 1\) partite classes of order at most two.

**Theorem 2.** Let \(s, r, t\) be integers such that \(0 \leq s < r\) and \(t > 0\). Let \(G\) be a complete \(r\)-partite graph consisting of one partite class of order \(\binom{2t+s}{t}\), \(r - s - 1\) partite classes of order two, and \(s\) partite classes of order one. Then \(Ch(G) > \lfloor \frac{r+s-1}{2t+s} \rfloor (2t + s)\).

**Proof.** Let \(n = \binom{2t+s}{t}\) and \(m = \frac{r+s-1}{2t+s}\). Let \(G\) be a complete \(r\)-partite graph with the partite classes \(V_1, V_i = \{v_i, w_i\}, V_j = \{v_j\}\), where \(|V_i| = n; i = 2, 3, \ldots, r - s\) and \(j = r - s + 1, r - s + 2, \ldots, r\). Let \(A_1, A_2, \ldots, A_{2t+s}, B_1, B_2, \ldots, B_{2t+s}\) be
disjoint color sets of order \( \lfloor m \rfloor \) such that \( \bigcup_{i=1}^{2t+s} A_i = A, \bigcup_{i=1}^{2t+s} B_i = B \). We define a list assignment \( L \) to \( V(G) \) by the following way:

\[
L(v_j) = A, \ j = 2, 3, \ldots, r, \\
L(w_i) = B, \ i = 2, 3, \ldots, r - s.
\]

The lists of colors given to the vertices of \( V_1 \) consist of \( 2t + s \) different sets \( A_{x_1}, A_{x_2}, \ldots, A_{x_{t+s}}, B_{y_1}, B_{y_2}, \ldots, B_{y_{t+s}} \), where \( x_1, x_2, \ldots, x_{t+s}, y_1, y_2, \ldots, y_{t+s} \in \{1, 2, \ldots, 2t + s\} \). Since the number of vertices in \( V_1 \) is \( n = \binom{2t+s}{t} \), we are able to assign to any two vertices in \( V_1 \) different lists.

We show by contradiction that \( G \) cannot be colored from the list \( L \). Suppose that \( G \) can be colored from \( L \). We use \( r - 1 \) different colors of \( A \) to color the vertices \( v_2, v_3, \ldots, v_r \) and \( r - s - 1 \) different colors of \( B \) to color \( w_2, w_3, \ldots, w_{r-s} \). Since \( |A| = |B| = \lfloor m \rfloor (2t + s) \leq r + t - 1 \), the number of colors in \( A \) (in \( B \)) not used to color \( V_2, V_3, \ldots, V_r \) is at most \( t \) (at most \( t + s \)). It follows that there are at most \( 2t + s \) sets \( A_{x'_1}, A_{x'_2}, \ldots, A_{x'_t}, B_{y'_1}, B_{y'_2}, \ldots, B_{y'_{t+s}} \), where \( x'_1, x'_2, \ldots, x'_t, y'_1, y'_2, \ldots, y'_{t+s} \in \{1, 2, \ldots, 2t + s\} \) containing colors that were not employed in coloring \( V_2, V_3, \ldots, V_r \). Try to color \( V_1 \) with these colors. According to the assignment of color sets to the vertices of \( V_1 \), there exists a vertex \( v \in V_1 \) having none of the sets \( A_{x'_1}, A_{x'_2}, \ldots, A_{x'_t}, B_{y'_1}, B_{y'_2}, \ldots, B_{y'_{t+s}} \) in its list, a contradiction. Hence, \( G \) is not \( L \)-colorable.

Note that we get the bound \( Ch(K_{(\binom{2t+s}{t})^2}) \geq r + t \) if \( s = 0 \) and \( r = pt + 1 \) for some odd integer \( p \).

3. Complete Multipartite Graphs with Partite Classes of Equal Sizes

Let \( K_{nsr} \) denote the complete multipartite graph with \( r \) partite classes of order \( n \). The problem is to determine the value of the choice number \( Ch(K_{nsr}) \). If \( n = 1 \), then \( K_{nsr} \) is a clique on \( r \) vertices and hence, obviously, \( Ch(K_{1sr}) = r \). In the previous section we mentioned that \( Ch(K_{2sr}) = r \) as well. Alon [1] established the general bounds \( c_1 r \log n \leq Ch(K_{nsr}) \leq c_2 r \log n \) for every \( r, n \geq 2 \), where \( c_1, c_2 \) are two positive constants. Later, Kierstead [5] solved the problem in the case \( n = 3 \). He showed that \( Ch(K_{3sr}) = \lceil \frac{4r-1}{3} \rceil \). Yang [9] studied the value of \( Ch(K_{4sr}) \) and obtained the bounds \( \lceil \frac{1}{2}r \rceil \leq Ch(K_{4sr}) \leq \lceil \frac{2}{3}r \rceil \). We present results giving exact bounds on \( Ch(K_{nsr}) \) for large \( n \). In the proof of Theorem 3 we use the following lemma proved in [5].

**Lemma 1.** A graph \( G \) is \( k \)-choosable if \( G \) is \( L \)-colorable for every \( k \)-list assignment \( L \) such that \( |\bigcup_{v \in V(G)} L(v)| < |V(G)| \).
Let us derive an upper bound on the choice number of complete multipartite graphs with partite classes of equal sizes.

**Theorem 3.** Let $0 < \alpha < n$ and let $x_j = \lfloor (\alpha - \frac{n}{r} \sum_{i=1}^{j-1} x_i) \rfloor + 1$, $j = 1, 2, \ldots, \lfloor \alpha \rfloor$. If $n \leq \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$, then $Ch(K_{n \star r}) \leq \lceil \alpha r \rceil$.

**Proof.** Let $V_i$, $i = 1, 2, \ldots, r$, be the $i$-th partite class of $K_{n \star r}$. We prove the result by induction on $r$. The case $r = 1$ is trivial. For the induction step consider an $[\alpha r]$-list assignment $L$ to the vertices of $K_{n \star r}$. We prove that if $n \leq \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$, then any partite class $V_i$ can be colored with at most $\lfloor \alpha \rfloor$ colors.

Assume that $n = \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$. In this paragraph we show by induction on $j$ ($j = 1, 2, \ldots, \lfloor \alpha \rfloor$), that there exists a color $c_j$ which can be used for coloring $x_j$ vertices of $V_i$ that have not been colored by $c_1, c_2, \ldots, c_{j-1}$. Note that $c_1, c_j$, where $l, l' \in \{1, 2, \ldots, \lfloor \alpha \rfloor\}$, $l \neq l'$, do not have to be different.

If $j = 1$, we have $x_1 = \lfloor \alpha \rfloor + 1$. Since $\sum_{v \in V_i} |L(v)| = \lfloor \alpha r \rfloor n$ and by Lemma 1, $|\bigcup_{v \in V_i} L(v)| < rn$, there exists a color $c_1$ which appears in the lists of at least $\lfloor \alpha \rfloor + 1$ vertices of $V_i$. Color these vertices with $c_1$. Suppose $j \geq 2$. We can color $\sum_{i=1}^{j-1} x_i$ vertices with $c_1, c_2, \ldots, c_{j-1}$. The sum of the numbers of colors in the lists of the remaining $n - \sum_{i=1}^{j-1} x_i$ vertices of $V_i$ is $(n - \sum_{i=1}^{j-1} x_i)[\lfloor \alpha r \rfloor]$. Since $|\bigcup_{v \in V_i} L(v)| < rn$, there is a color $c_j$ that appears in the lists of other $\lfloor (n - \sum_{i=1}^{j-1} x_i) \frac{3}{n} \rfloor + 1 = x_j$ vertices. Hence, we can color these vertices with $c_j$. It follows that it is possible to color $n = \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$ vertices of $V_i$ with at most $\lfloor \alpha \rfloor$ different colors.

Clearly, if $n < \sum_{i=1}^{\lfloor \alpha \rfloor} x_i$, all the vertices of $V_i$ can be colored with at most $\lfloor \alpha \rfloor$ colors too. Let us remove the colors that were employed in coloring $V_i$ from the lists given to the vertices in $V(K_{n \star r}) \setminus V_i$. We have at least $\lfloor \alpha r \rfloor - \lfloor \alpha \rfloor$ colors. Since $\lfloor \alpha r \rfloor - \lfloor \alpha \rfloor \geq \lfloor \alpha (r-1) \rfloor$, by applying the induction hypothesis, $r-1$ partite classes can be colored with $\lfloor \alpha (r-1) \rfloor$ colors, i.e., there exists a proper coloring of the vertices in $V(K_{n \star r}) \setminus V_i$ with colors from the revised lists.

Unfortunately, the result presented in Theorem 3 cannot be bounded from above by $c \log n$, where $c$ is a constant. Theorem 3, for example, yields the upper bounds $Ch(K_{5 \star r}) \leq \lceil \frac{3}{2} r \rceil$, $Ch(K_{15 \star r}) \leq 5r$, $Ch(K_{40 \star r}) \leq 10r$, $Ch(K_{75 \star r}) \leq 15r$ and $Ch(K_{121 \star r}) \leq 20r$. One can check that $10r \approx 6.24r \log 40$, $15r \approx 8r \log 75$ and $20r \approx 9.6r \log 121$.

The following result gives a lower bound on $Ch(K_{n \star r})$.

**Theorem 4.** Let $x, t, r$ be integers such that $x, t, r \geq 2$, $x \geq t$ and $n = \lceil x - t + 1 \rceil$. Then $Ch(K_{n \star r}) > (x - t + 1) \lfloor \frac{r-1}{x} \rfloor$.

**Proof.** Let $x, t, r \geq 2$, $x \geq t$, $n = \lceil x - t + 1 \rceil$ and let $k = (x - t + 1) \lfloor \frac{r-1}{x} \rfloor$. We show that there exists a $k$-list assignment $L$ of $K_{n \star r}$ such that $K_{n \star r}$ is not $L$-colorable.
Let $V_i, i = 1, 2, \ldots, r$, be the $i$-th partite class of $K_{n^*r}$. Let $A_1, A_2, \ldots, A_x$ be a family of disjoint color sets such that $|A_j| = |A_1|$ or $|A_j| = |A_1| + 1$, $j = 2, 3, \ldots, x$, and $|\bigcup_{j=1}^x A_j| = tr - 1$. Obviously, $|A_j| \geq \lceil \frac{tr-1}{x} \rceil$ for any $j \in \{1, 2, \ldots, x\}$.

Define a list assignment $L$ as follows: Let the lists given to the $n$ vertices of every partite class $V_i$ consist of $x - t + 1$ different sets $A_{y_1}, A_{y_2}, \ldots, A_{y_{x-t+1}}$, $y_1, y_2, \ldots, y_{x-t+1} \in \{1, 2, \ldots, x\}$, where any two vertices in the same part have different lists. Note that $|L(v)| \geq (x - t + 1)\lceil \frac{tr-1}{x} \rceil$ for each vertex $v \in V(K_{n^*r})$. Then for any partite class $V_i$ and any $t - 1$ colors $a_j \in A_{y_j}$, $j = 1, 2, \ldots, t - 1$; $y_j \in \{1, 2, \ldots, x\}$ there is a vertex $v \in V_i$ having none of the sets $A_{y_j}$ in its list. Therefore, in any coloring from these lists, we must use at least $t$ colors on each partite class. Since the number of colors in $\bigcup_{j=1}^x A_j$ is less than $tr$, $K_{n^*r}$ is not $L$-colorable. 

Theorem 4 says that if, for instance $t = 2$, then $n = x$ and $Ch(K_{n*5r}) > (n - 1)\lceil \frac{3r-1}{n} \rceil$. In particular, for $n = 5$ we have $Ch(K_{5*5r}) > 4\lceil \frac{3r-1}{x} \rceil$. If $t = 3$, then $Ch(K_{n*5r}) > (x - 2)\lceil \frac{3r-1}{x} \rceil$. For example, in the case $x = 6$ we get $Ch(K_{15*5r}) > 4\lceil \frac{3r-1}{6} \rceil = 4\lceil \frac{r-1}{2} \rceil$.

Finally, we present a corollary of Theorem 4 which yields a lower bound in the form $cr \log n$.

**Corollary 1.** Let $r \geq 2$ and $n = \lceil \frac{x}{2} \rceil$ where $x \geq 5$. Then

$$Ch(K_{n*sr}) > \left\lceil \frac{x}{r} \right\rceil \left\lceil \frac{\log_{2.1} 3n}{2} \right\rceil.$$ 

**Proof.** For $x, t, r \geq 2$, $x \geq t$ and $n = \lceil \frac{x}{x-t+1} \rceil$, we have $Ch(K_{n*sr}) > (x - t + 1)\lceil \frac{tr-1}{x} \rceil$. Let $t = \left\lceil \frac{x}{2} \right\rceil + 1$. Then $Ch(K_{n*sr}) > \left\lceil \frac{x}{2} \right\rceil \left\lceil \frac{\left\lceil x/2 \right\rceil r + x - 1}{x} \right\rceil \geq \left\lceil \frac{x}{2} \right\rceil \left\lceil \frac{x}{2} \right\rceil$.

It is well-known that $\frac{x^x}{e^x} \leq x! \leq \left(\frac{x+1}{2}\right)^{x+1} e^{-x}$ for any $x$. For $x \geq 5$, the following inequalities also hold: $\frac{2x^x}{e^x} < x! < \frac{6e^{x+1}}{5e^x}$. Then $n = \left\lceil \frac{x}{2} \right\rceil \left\lceil \frac{x}{2} \right\rceil < \frac{6x^{x+1}/(5e^x)}{4\left\lceil \frac{x}{2} \right\rceil \left\lceil \frac{x}{2} \right\rceil e^{x-2} \leq \frac{3x^{x+1}}{10(x-1)e^x} \leq \frac{3x^2x^2}{10(x-1)e^x}$. Since $x2^x < 7.6(2.1)^x$ for any $x$ (note that $7.5(2.1)^x < x2^x$ for $19 \leq x \leq 22$ and $(\frac{x}{e^x})^x < 3.1$ for any $x \geq 5$, we have $n < \frac{7.68(2.1)^x}{e^x} < (2.1)^x$. Consequently, $\log_{2.1} 3n < x$, hence $Ch(K_{n*sr}) > \left\lceil \frac{x}{2} \right\rceil \left\lceil \frac{\log_{2.1} 3n}{2} \right\rceil$ for any $n = \lceil \frac{x}{2} \rceil$ where $x \geq 5$. 

**References**


Received 26 January 2009
Revised 11 January 2011
Accepted 11 January 2011