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INDEPENDENT TRANSVERSAL DOMINATION IN GRAPHS

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Abstract

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex in $V - S$ is adjacent to a vertex in S. A dominating set which intersects every maximum independent set in G is called an independent transversal dominating set. The minimum cardinality of an independent transversal dominating set is called the independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. In this paper we begin an investigation of this parameter.

Keywords: dominating set, independent set, independent transversal dominating set.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book by Chartrand and Lesniak [1]. All graphs in this paper are assumed to be non-trivial.

One of the fastest growing areas within graph theory is the study of domination and related subset problems such as independence, covering and matching. In fact, there are scores of graph theoretic concepts involving domination, covering and independence. The bibliography in domination maintained by Haynes *et al.* [3] currently has over 1200 entries; Hedetniemi and Laskar [5] edited a recent issue of *Discrete Mathematics* devoted entirely to domination, and a survey of advanced topics in domination is given in the book by Haynes *et al.* [4].

Nevertheless, despite the many variations possible, we can so far identify only a limited number of basic domination parameters; "basic" in the sense that they are defined for every non-trivial connected graph. For instance independent domination, connected domination, total domination, global domination and acyclic domination are some basic domination parameters. In this sequence, we introduce another basic domination parameter namely independent transversal domination and initiate the study of this new domination parameter.

2. Definitions and Notations

In a graph $G = (V, E)$, the *open neighbourhood* of a vertex $v \in V$ is $N(v) = \{x \in V : vx \in E\}$, the set of vertices adjacent to v. The *closed neighbourhood* is $N[v] = N(v) \cup \{v\}$. A *clique* in a graph G is a complete subgraph of G. The maximum order of clique in G is called the *clique number* and is denoted by $\omega(G)$ and a clique of order $\omega(G)$ is called a *maximum clique*. The subgraph induced by a set $S \subseteq V$ is denoted $\langle S \rangle$. If G is a graph, then G^+ is the graph obtained from G by attaching a pendant edge at every vertex of G.

A set $S \subseteq V$ is a *dominating set* if every vertex in $V - S$ is adjacent to a vertex of S and the minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. A minimum dominating set of a graph G is called a *γ-set* of G. An *independent dominating set* S is a dominating set S such that S is an independent set. The *independent domination number* $i(G)$ is the minimum cardinality of an independent dominating set. The maximum cardinality of an independent set is called the *independence number* and is denoted by $\beta_0(G)$. A maximum independent set is called a β_0-set. A dominating set S such that $\langle S \rangle$ is connected is called a *connected dominating* set. A *total dominating set* is a dominating set S such that $\langle S \rangle$ has no isolates. The minimum cardinality of a connected (total) dominating set is called the *connected* (*total*) *domination number* and is denoted by $\gamma_c(G)(\gamma_t(G))$. These two parameters can respectively be seen in [6] and [2]. A set $S \subseteq V$ which dominates both G and $\bar{G}$ is called a *global dominating set* and the minimum order of a global dominating set is called the *global domination number*, denoted by $\gamma_g(G)$.

We need the following theorems.

Theorem 2.1 [3]**.** *If a graph G of order n has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

Theorem 2.2 [3]**.** *For any graph G with even order n having no isolated vertices $\gamma(G) = \frac{n}{2}$ if and only if the components of G are C_4 or H^+ for any connected graph H.*

3. Independent Transversal Domination Number

In this section, we determine the value of independent transversal domination number for some standard families of graphs such as paths, cycles and wheels. Also we determine $\gamma_{it}(G)$ for disconnected graphs.

Definition. A dominating set $S \subseteq V$ of a graph G is said to be an independent transversal dominating set if S intersects every maximum independent set of G. The minimum cardinality of an independent transversal dominating set of G is called the independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. An independent transversal dominating set S of G with $|S| = \gamma_{it}(G)$ is called a γ_{it}-set.

Example 3.1. (i) If G is a complete multipartite graph having r maximum independent sets, then

$$\gamma_{it}(G) = \begin{cases} 2 & \text{if } r = 1, \\ r & \text{otherwise.} \end{cases}$$

In particular, $\gamma_{it}(K_n) = n$ and $\gamma_{it}(K_{m,n}) = 2$.

(ii) For any star, the value of γ_{it} is two and for bistars it is three.

(iii) If G is a connected graph on n vertices, then $\gamma_{it}(G^+) = n$. For, if $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $u_i (1 \leq i \leq n)$ is the pendant vertex in G^+ adjacent to v_i, then $S = \{u_1, \ldots, u_n\}$ is a γ-set of G^+ and since $\beta_0(G^+) = n$ it follows that S intersects every β_0-set of G^+ so that S is an independent transversal dominating set of G^+ of minimum order.

Theorem 3.2. *For any path P_n of order n, we have*

$$\gamma_{it}(P_n) = \begin{cases} 2 & \text{if } n = 2, 3, \\ 3 & \text{if } n = 6, \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$. Clearly $\gamma_{it}(P_2) = \gamma(P_3) = 2$. Suppose $n = 6$. Then $S = \{v_1, v_2, v_5\}$ is an independent transversal dominating set of P_6 so that $\gamma_{it}(P_6) \leq 3$. Also, since $D = \{v_2, v_5\}$ is the only γ-set of P_6 and $V - D = \{v_1, v_3, v_6\}$ is a β_0-set, it follows that $\gamma_{it}(P_6) \geq \gamma + 1 = 3$. Thus $\gamma_{it}(P_6) = 3$.

Assume $n \notin \{2, 3, 6\}$. If $n \equiv 0 (\text{mod } 3)$, then $S = \{v_{3i-1} : 1 \leq i \leq \frac{n}{3}\}$ is the γ-set of P_n. Further $\langle V - S \rangle = (\lfloor \frac{n-2}{3} \rfloor) K_2 \cup 2K_1$ and hence every independent set in $V - S$ contains at most $\lfloor \frac{n-2}{3} \rfloor + 2$ vertices. Now, since $\lfloor \frac{n-2}{3} \rfloor + 2 < \lceil \frac{n}{2} \rceil = \beta_0(P_n)$, it follows that $V - S$ contains no β_0-set and hence $\gamma_{it}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.

If $n \equiv 1 (\text{mod } 3)$, then $S = \{v_{3i+1} : 0 \leq i \leq \frac{n-1}{3}\}$ is a γ-set of P_n. Further, since $\langle V - S \rangle = \lfloor \frac{n}{3} \rfloor K_2$ it follows that every γ-set in $V - S$ contains at most $\lfloor \frac{n}{3} \rfloor$ vertices and hence $V - S$ contains no β_0-set. Thus $\gamma_{it}(p_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.

If $n \equiv 2 \pmod 3$, then $S = \{v_{3i+1} : 0 \leq i \leq \frac{n-2}{3}\}$ is a γ-set of P_n. Further, since $\langle V - S\rangle = \frac{n-2}{3}K_2 \cup K_1$, every γ-set in $V - S$ contains at most $\frac{n+1}{3}$ vertices and hence $V - S$ contains no β_0-set so that $\gamma_{it}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$. ■

Theorem 3.3. *For any cycle C_n of order n, we have*

$$\gamma_{it}(C_n) = \begin{cases} 3 & \text{if } n = 3, 5, \\ \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Clearly $\gamma_{it}(C_3) = 3$. Suppose $n = 5$. Then $S = \{v_1, v_2, v_3\}$ is an independent transversal dominating set of C_5 so that $\gamma_{it}(C_5) \leq 3$. Further since for every γ-set D of $C_5, V - D$ contains a maximum independent set it follows that $\gamma_{it}(C_5) > \gamma(C_5) = 2$. Thus $\gamma_{it}(C_5) = 3$.

Assume $n \notin \{3, 5\}$. If $n \equiv 0 \pmod 3$, then $S = \{v_{3i+1} : 0 \leq i \leq \frac{n-3}{3}\}$ is a γ-set of C_n. Now, since $\langle V - S\rangle = (\frac{n}{3})K_2$, every independent set in $V - S$ contains at most $\frac{n}{3}$ vertices and hence $V - S$ contains no β_0-set so that $\gamma_{it}(C_n) = \gamma(C_n) = \frac{n}{3}$.

If $n \equiv 1 \pmod 3$, then $S = \{v_{3i+1} : 0 \leq i \leq \frac{n-1}{3}\}$ is a γ-set of C_n and $\langle V - S\rangle = (\frac{n-1}{3})K_2$. Hence every independent set in $V - S$ contains at most $\frac{n-1}{3}$ vertices so that $V - S$ contains no β_0-set. Thus $\gamma_{it}(C_n) = \gamma(C_n) = \frac{n}{3}$.

If $n \equiv 2 \pmod 3$, then $S = \{v_{3i+1} : 0 \leq i \leq \frac{n-2}{3}\}$ is a γ-set of C_n and $\langle V - S\rangle = (\frac{n-2}{3})K_2 \cup K_1$. Hence every independent set in $V - S$ contains at most $\frac{n-1}{3} + 1$ vertices so that $V - S$ contains no β_0-set. Thus $\gamma_{it}(C_n) = \gamma(C_n) = \frac{n}{3}$. ■

Next, we just state the value of γ_{it} for wheels without proof being staightforward to determine.

Theorem 3.4. *If W_n is a wheel on n vertices, then*

$$\gamma_{it}(W_n) = \begin{cases} 2 & \text{if } n = 5, \\ 3 & \text{if } n \geq 7 \text{ and is odd or } n = 6, \\ 4 & \text{otherwise.} \end{cases}$$

In the following theorem we determine the value of the independent transversal domination number for disconnected graphs.

Theorem 3.5. *If G is a disconnected graph with components $G_1, G_2, \ldots, G_r$, then $\gamma_{it}(G) = \min_{1\leq i\leq r}\{\gamma_{it}(G_i) + \sum_{j=1,j\neq i}^{r} \gamma(G_j)\}$.*

Proof. Assume that $\gamma_{it}(G_1)+\sum_{j=2}^{r}\gamma(G_j)=\min_{1\leq i\leq r} \{\gamma_{it}(G_i)+\sum_{j=1,j\neq i}^{r}\gamma(G_j)\}$. Let D be a γ_{it}-set of G_1 and let S_j be a γ-set of G_j, for all $j \geq 2$. Then $D\cup(\bigcup_{j=2}^{r} S_j)$ is an independent transversal dominating set of G so that $\gamma_{it}(G) \leq \gamma_{it}(G_1) + \sum_{j=2}^{r}\gamma(G_j) = \min_{1\leq i\leq r} \{\gamma_{it}(G_i) + \sum_{j=1,j\neq i}^{r}\gamma(G_j)\}$. Conversely, let S be any independent transversal dominating set of G. Then S must intersect the vertex set $V(G_j)$ of each component G_j of G and $S \cap V(G_j)$ is a dominating set of G_j for all $j \geq 1$. Further, for at least one j, the set $S \cap V(G_j)$ must be an independent transversal dominating set of G_j, for otherwise each component G_j will have a maximum independent set not intersecting the set $S \cap V(G_j)$ and hence union of these maximum independent sets form a maximum independent set of G not intersecting S. Hence $|S| \geq \min_{1 \leq i \leq r} \{\gamma_{it}(G_i) + \sum_{j=1, j\neq i}^{r} \gamma(G_j)\}$. ■

Corollary 3.6. *If G has an isolated vertex, then $\gamma_{it}(G) = \gamma(G)$.*

In view of the above theorem we can confine ourself to connected graphs at the study of independent transversal domination. So the graphs are connected in the rest of this paper if there is no specific explanation.

4. BOUNDS OF γ_{it}

In this section, we obtain some bounds for $\gamma_{it}(G)$ in terms of the order of graph, clique number, vertex covering number and some well-known basic domination parameters.

4.1. Bounds in terms of order and degree

Theorem 4.1.1. *For any graph G, we have $1 \leq \gamma_{it}(G) \leq n$. Further $\gamma_{it}(G) = n$ if and only if $G = K_n$.*

Proof. The inequalities are trivial. Suppose $\gamma_{it}(G) = 1$. Let $S = \{u\}$ be an independent transversal dominating set of G. Then $\deg u = n - 1$ and u belongs to every β_0-set of G. Hence $\{u\}$ is the only β_0-set of G so that $G = K_1$.

Now, suppose $n \geq 2$ and $\gamma_{it}(G) = n$. If $\beta_0(G) \geq 2$, then $V - \{v\}$, where $v \in V$, is an independent transversal dominating set of G and hence $\gamma_{it}(G) \leq n-1$, which is a contradiction. Thus $\beta_0(G) = 1$ so that $G = K_n$. Also $\gamma_{it}(K_n) = n$. ■

Theorem 4.1.2. *Let G be a graph on n vertices. Then $\gamma_{it}(G) = n - 1$ if and only if $G = P_3$.*

Proof. Suppose $\gamma_{it}(G) = n - 1$. Then it follows from the above theorem that $\beta_0(G) \geq 2$. If there exist two adjacent vertices u and v of degree at least two, then $S = V - \{u, v\}$ is an independent transversal dominating set of G and hence $\gamma_{it}(G) \leq n - 2$, which is a contradiction. Hence for any two adjacent vertices u and v in G either u or v is a pendant vertex so that $G = K_{1,n-1}$. Since $\gamma_{it}(K_{1,n-1}) = 2$, it follows that $n = 3$ so that $G = P_3$.

The converse is obvious. ■

Theorem 4.1.3. *Let G be a non-complete connected graph with $\beta_0(G) \geq \frac{n}{2}$. Then $\gamma_{it}(G) \leq \frac{n}{2}$.*

Proof. Let S be a β_0-set in G. Suppose $\beta_0(G) > \frac{n}{2}$. Then $D = (V - S) \cup \{u\}$, where $u \in S$, is a dominating set of G which intersects every β_0-set of G. Hence $\gamma_{it}(G) \leq |D| = \lfloor \frac{n}{2} \rfloor$.

Suppose $\beta_0(G) = \frac{n}{2}$. If $V - S$ is not independent, then S is an independent transversal dominating set of G and hence $\gamma_{it}(G) \leq |S| = \frac{n}{2}$. If $V - S$ is independent, then G is a bipartite graph with the bipartition $(S, V - S)$. Now, Suppose $\delta(G) \geq 2$. Then $D = (S - \{u\}) \cup \{v\}$, where $u \in S$ and $v \in N(u)$, is a dominating set of G. Clearly D intersects every β_0-set of G and hence $\gamma_{it}(G) \leq |D| = |S| = \frac{n}{2}$. Suppose $\delta(G) = 1$. Let u be a pendant vertex. Assume without loss of generality that $u \in S$. Let $v \in N(u)$. Since G is connected it follows that $\deg v \geq 2$. Now, if there exists a pendant vertex $w \neq u$ in S which is adjacent to v, then $[V - (S \cup \{v\})] \cup \{u, w\}$ is an independent dominating set of cardinality greater than $\frac{n}{2}$, which is a contradiction and hence every neighbour of v other than u has degree at least two. Hence $D = [V - (S \cup \{v\})] \cup \{u\}$ is a dominating set of G. Now, since $\deg v \geq 2$, it follows that $V - D$ contains no β_0-set of G and hence D is an independent transversal dominating set of G so that $\gamma_{it}(G) \leq \frac{n}{2}$. ■

Corollary 4.1.4. *If G is bipartite, then $\gamma_{it}(G) \leq \frac{n}{2}$.*

Proof. Since $\beta_0(G) \geq \frac{n}{2}$ for a bipartite graph G, the result follows from Theorem 4.1.3. ■

Remark 4.1.5. The bound given in Theorem 4.1.3 is sharp. For the graph $G = H^+$, where H is a connected graph on n vertices, we have $\gamma_{it}(G) = n$. Further, if H is a bipartite graph, then G also is bipartite and thus there is an infinite family of bipartite graphs with γ_{it} being half of their order.

Theorem 4.1.6. *Let a and b be two positive integers with $b \geq 2a - 1$. Then there exists a graph G on b vertices such that $\gamma_{it}(G) = a$.*

Proof. Let $b = 2a + r$, $r \geq -1$ and let H be any connected graph on a vertices. Let $V(H) = \{v_1, v_2, \ldots, v_a\}$. Let G be the graph obtained from H by attaching $r + 1$ pendant edges at v_1 and one pendant edge at each v_i, for $i \geq 2$. Let $u_i (i \geq 2)$ be the pendant vertex in G adjacent to v_i. Clearly $\gamma(G) = a$ and $S = \{v_1, u_2, u_3, \ldots, u_a\}$ is a γ-set of G. Further every maximum independent set of G intersects S and hence $\gamma_{it}(G) = a$. Also $|V(G)| = b$. ■

We have proved in Theorem 4.1.3 that the value of $\gamma_{it}(G)$ is bounded by $\frac{n}{2}$ for all non-complete connected graphs G of order n with independence number at least $\frac{n}{2}$. However, this upper bound for $\gamma_{it}(G)$ is either retained or increased just by one for all graphs that we have come across, even if we relax the condition being $\beta_0(G) \geq \frac{n}{2}$ in the theorem. For example, for the graph G given in Figure 1, we have $\gamma_{it}(G) = 3 = \lceil \frac{n}{2} \rceil$ and $\beta_0(G) = 2 < \frac{n}{2}$.

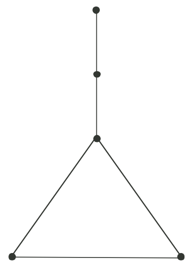

Figure 1

Motivated by this observation and by Theorem 4.1.6, we take risk of posing the following conjecture.

Conjecture 4.1.7. *If G is a non-complete connected graph on n vertices, then* $\gamma_{it}(G) \leq \lceil \frac{n}{2} \rceil$.

Theorem 4.1.8. *For any graph G, we have $\gamma(G) \leq \gamma_{it}(G) \leq \gamma(G) + \delta(G)$.*

Proof. Since an independent transversal dominating set of G is a dominating set, it follows that $\gamma(G) \leq \gamma_{it}(G)$. Now, let u be a vertex in G with $\deg u = \delta(G)$ and let S be a γ-set in G. Then every maximum independent set of G contains a vertex of $N[u]$ so that $S \cup N[u]$ is an independent transversal dominating set of G. Also, since S intersects $N[u]$, it follows that $|S \cup N[u]| \leq \gamma(G) + \delta(G)$ and hence the right inequality follows. ■

Corollary 4.1.9. *The Conjecture* 4.1.7 *is true for all connected graphs with* $\delta(G) = 1$.

Proof. If G is a connected graph with $\delta(G) = 1$, then it follows from Theorem 4.1.8 that $\gamma_{it}(G) \leq \gamma(G) + 1$. Also, by Theorem 2.1, we have $\gamma(G) \leq \frac{n}{2}$. So, obviously $\gamma_{it}(G) \leq \lceil \frac{n}{2} \rceil$ when $\gamma(G) < \frac{n}{2}$. If $\gamma(G) = \frac{n}{2}$, by Theorem 2.2, G is either C_4 or H^+ for any connected graph H; in either case $\gamma_{it}(G) = \frac{n}{2}$. ■

Corollary 4.1.10. *If T is a tree, then $\gamma_{it}(T)$ is either $\gamma(T)$ or $\gamma(T) + 1$.*

In view of Corollary 4.1.10 we can split the family of all trees into two classes. Let us say a tree T is of *class* 1 or *class* 2 according as $\gamma_{it}(T)$ is $\gamma(T)$ or $\gamma(T) + 1$. For example, stars are of class 2 and subdivision graph of stars are of class 1 so that both classes of trees are non-empty and thus we have the following problem.

Problem 4.1.11. Characterize the class 1 trees.

Theorem 4.1.12. *If G is a graph with* $\operatorname{diam} G = 2$, *then* $\gamma_{it}(G) \leq \delta(G) + 1$.

Proof. Let u be a vertex with $\deg u = \delta(G)$. Then $N[u]$ is a dominating set of G, because diam $G = 2$. Now, it follows from the fact that every maximum independent set contains a vertex of $N[u]$, this closed neighbourhood itself is an independent transversal dominating set so that $\gamma_{it}(G) \leq \delta(G) + 1$. ■

4.2. Bounds in terms of covering and cliques

We now establish an upper bound for $\gamma_{it}(G)$ in terms of the vertex covering number $\alpha_0(G)$ and the clique number $\omega(G)$.

Theorem 4.2.1. *If G has no isolates, then $\gamma_{it}(G) \leq \alpha_0(G) + 1$ and the bound is sharp.*

Proof. Let S be a minimum vertex cover of G. Then S is a dominating set and $V - S$ is a maximum independent set. Hence $S \cup \{u\}$, where $u \in V - S$ is an independent transversal dominating set of G so that $\gamma_{it}(G) \leq \alpha_0(G) + 1$. The bound is attained for complete graphs and stars. ■

Corollary 4.2.2. *Let G be a graph on n vertices without isolates. Then $\gamma(G) + \gamma_{it}(G) \leq n + 1$ and $i(G) + \gamma_{it}(G) \leq n + 1$.*

Proof. Since $\alpha_0(G)+\beta_0(G) = n$ and $\gamma(G) \leq i(G) \leq \beta_0(G)$, the corollary follows. ■

Theorem 4.2.3. *For any non-complete graph G with $\delta(G) \geq 2$, we have $\gamma_{it}(G) \leq \alpha_0(G)$.*

Proof. Let S be a β_0-set of G. Then $V - S$ is a dominating set of G. Since $G \neq K_2$ and $\delta(G) \geq 2$, there exists a vertex v in $V - S$ such that $|N(v) \cap S| \geq 2$. Let u and w be two neighbours of v in S. Since $\delta(G) \geq 2$, it follows that every neighbour of v in S is adjacent to at least one vertex other than v in $V - S$ and hence $D = (V - S) - \{v\}$ is a dominating set of $G - \{v\}$. Then $D \cup \{w\}$ is an independent transversal dominating set of G. This is because $(S - \{w\}) \cup \{v\}$ is the only set in the complement of $D \cup \{w\}$, which is not an independent set, and hence $\gamma_{it}(G) \leq n - \beta_0(G) = \alpha_0(G)$. ■

Corollary 4.2.4. *If G is a non-complete graph with $\gamma_{it}(G) = \alpha_0(G) + 1$, then $\gamma(G) = \alpha_0(G)$.*

Proof. Suppose $\gamma_{it}(G) = \alpha_0(G)+1$. It follows from Theorem 4.2.3 and Theorem 4.1.8 that $\gamma_{it}(G) \leq \gamma(G) + 1$ and hence $\alpha_0(G) \leq \gamma(G)$. Also, since it is always true that $\gamma(G) \leq \alpha_0(G)$, we have $\gamma(G) = \alpha_0(G)$. ■

In the following we present an upper bound for $\gamma_{it}(G)$ involving the clique number $\omega(G)$ and characterize the graphs attaining the bound.

Theorem 4.2.5. *For any non-complete graph G with clique number ω, $\gamma_{it}(G) \leq n - \omega + 1$. Further equality holds if and only if the following are satisfied.*

(i) $\beta_0(G) = 2$.

(ii) *If S is a dominating set such that $\langle V - S \rangle$ is complete, then $|V - S| \leq \omega - 1$.*

Proof. Let H be a maximum clique in G. Let $u \in V(H)$. Then $S = V(G) - V(H - u)$ is a dominating set of G. Since $\beta_0(G) \geq 2$ and H is a maximum clique, it follows that every maximum independent set of G intersects S. Hence S is an independent transversal dominating set so that $\gamma_{it}(G) \leq n - \omega + 1$.

Suppose $\gamma_{it}(G) = n - \omega + 1$. Let H be a maximum clique in G. Let u and v be two adjacent vertices such that $u \in V(H)$ and $v \in V(G) - V(H)$. Then $D = \{u\} \cup [V(G) - V(H \cup \{u\}]$ is a dominating set of G with $|D| = n - \omega$. Since $\gamma_{it}(G) = n - \omega + 1$, there exists a β_0-set S in G such that $D \cap S = \emptyset$. Hence S consists of the vertex u and a vertex $w \neq u$ in H so that $\beta_0(G) = 2$.

Suppose (ii) is not true. Then there exists a dominating set S such that $\langle V_S \rangle$ is a clique of size ω. Since $\beta_0(G) = 2$, every β_0-set intersects S so that $\gamma_{it}(G) \leq n - \omega$, which is a contradiction. Hence (ii) is true.

Conversely suppose (i) and (ii) hold. Let S be an independent transversal dominating set of G. Since $\beta_0(G) = 2$ we have $\langle V - S \rangle$ is a clique in G. Hence it follows from (ii) that $|V - S| < \omega$ so that $|S| \geq n - \omega + 1$. Thus $\gamma_{it}(G) = n - \omega + 1$. ■

4.3. γ_{it} for bipartite graphs

It would be of some interest discussing the parameter γ_{it} for bipartite graphs; we begin with a small observation.

Observation 4.3.1. *Let G be a bipartite graph. If S is a γ-set of G, then any two (three) β_0-sets of G in $V - S$ intersect.*

Proof. Suppose G is a bipartite graph with the bipartition (X, Y) such that $|X| \leq |Y|$ and S is a γ-set of G. If there exist two disjoint β_0-sets of G in $V - S$, then $2\beta_0(G) + \gamma(G) \leq |X| + |Y|$ so that $\gamma(G) \leq |X| + |Y| - 2\beta_0(G)$ and since $\beta_0(G) \geq |Y|$, we have $\gamma(G) \leq |X| - |Y| \leq 0$, which is absurd and hence any two β_0-sets of G in $V - S$ intersect.

Now, suppose there exist three β_0-sets of G, say D_1, D_2 and D_3, in $V - S$ such that $D_1 \cap D_2 \cap D_3 = \emptyset$. As discussed above, any two of these sets intersect, and let $D_1 \cap D_2 = S_1, D_1 \cap D_3 = S_2$ and $D_2 \cap D_3 = S_3$. Then $|X| + |Y| \geq \gamma(G) + |D_1| + |D_2| - |S_1| + |D_3| - |S_2| - |S_3| = \gamma(G) + 3\beta_0(G) - (|S_1| + |S_2| + |S_3|)$. Obviously, $S_1 \cup S_2 \cup S_3$ is an independent set and hence $|S_1| + |S_2| + |S_3| \leq \beta_0(G)$ so that $|X| + |Y| \geq \gamma(G) + 2\beta_0(G) \geq \gamma(G) + 2|Y|$, which is absurd. Thus any three β_0-sets of G in $V - S$ intersect. ■

It follows from Observation 4.3.1 that, if G is a bipartite graph with a γ-set S such that there exist at most three β_0-sets not intersecting S, then we can have an independent transversal dominating set of G by adjoining S with a vertex in the intersection of those β_0-sets and thus $\gamma_{it}(G)$ is at most $\gamma(G)+1$. But we do not know that whether this can be extended for any bipartite graph even when it has more than three β_0-sets outside S. However, we have a feeling that whether or not all the β_0-sets of a bipartite graph G outside S intersect, the value of $\gamma_{it}(G)$ is bounded by $\gamma(G)+1$. Also, see that Corollary 4.1.10 supports this and thus we are forced to pose the following conjecture.

Conjecture 4.3.2. *If G is a connected bipartite graph, then $\gamma_{it}(G)$ is either $\gamma(G)$ or $\gamma(G)+1$.*

Obviously, Conjecture 4.3.2 is true for a bipartite graph with the bipartition (X, Y) such that $|X| \leq |Y|$ and $\gamma(G) = |X|$. We now characterize such bipartite graphs for which $\gamma_{it}(G) = \gamma(G)+1$.

Theorem 4.3.3. *Let G be a bipartite graph with bipartition (X, Y) such that $|X| \leq |Y|$ and $\gamma(G) = |X|$. Then $\gamma_{it}(G) = \gamma(G)+1$ if and only if every vertex in X is adjacent to at least two pendant vertices.*

Proof. We first claim that $\delta(G) = 1$. Suppose $\delta(G) \geq 2$. Since $\gamma(G) = |X|$, X is a γ-set. Also, since $\gamma_{it}(G) = \gamma(G)+1$ it follows that $\beta_0(G) = |Y|$. Now, let $u \in X$ and $v \in N(u)$. Since $\delta(G) \geq 2$, it follows that $S = (X - \{u\}) \cup \{v\}$ is a dominating set of G. Now since $\beta_0(G) = |Y|$ and $\delta(G) \geq 2$, every β_0- set contains either the vertex v or a vertex $w \neq u$ in X. Hence S intersects every β_0- set so that $\gamma_{it}(G) = |X| = \gamma(G)$, which is a contradiction. Thus $\delta(G) = 1$.

Further suppose there exists a vertex u in X such that $N(u)$ contains at most one pendant vertex. Then $S = (X - \{u\}) \cup \{v\}$, where $v \in N(u)$, and v is chosen to be a pendant vertex if it exists, is a dominating set of G. Also since $\beta_0(G) = |Y|$, it follows that S intersects every β_0-set of G and hence $\gamma_{it}(G) \leq |X| = \gamma(G)$, which is a contradiction. Thus every vertex in X is adjacent to at least two pendant vertices.

Conversely, if every vertex in X is adjacent to at least two pendant vertices, then X is the only γ-set of G so that $\gamma_{it}(G) = \gamma(G)+1$. ■

4.4. Relationship with other parameters

In this section we establish relations connecting the parameter γ_{it} and some domination parameters.

Remark 4.4.1. We have observed in Theorem 4.1.8 that $\gamma(G) \leq \gamma_{it}(G)$ for any graph G. Furtehr, the difference between the parameters γ_{it} and γ can be made arbitrarily large as $\gamma_{it}(K_n) = n$ and $\gamma(K_n) = 1$. Moreover, these parameters can assume arbitrary values. That is, if a and b are given positive integers with $1 \leq a \leq b$, there exists a graph G such that $\gamma(G) = a$ and $\gamma_{it}(G) = b$. For, if $a = b$, let $G = H^+$ for some connected graph H of order a and if $b \geq a+1$, let G be the graph obtained from a path $P = (v_1, v_2, \ldots, v_a)$ on a vertices by attaching a copy of K_{b-a+1} at v_1 and a copy of K_b at each other vertex v_i of the path P. Then in either of the cases $\gamma(G) = a$ and $\gamma_{it}(G) = b$.

Theorem 4.4.2. *If G is a graph with $\chi(G) = k = \frac{n}{\beta_0(G)}$, then $\gamma_g(G) \leq \gamma_{it}(G)$.*

Proof. Let $\{V_1, V_2, \ldots, V_k\}$ be a k-coloring of G. Then $V_i, 1 \leq i \leq k$, is a maximum independent set of G. Let S be a γ_{it}-set of G. Then $S \cap V_i \neq \emptyset$, for all $i = 1, 2, \ldots, k$. Since $\langle V_i \rangle, 1 \leq i \leq k$, is a clique in $\overline{G}$ it follows that S is a dominating set of $\overline{G}$. Hence S is a global dominating set of G. Thus $\gamma_g(G) \leq \gamma_{it}(G)$. ■

Remark 4.4.3. Let S be a minimum independent transversal dominating set of G. Then $\langle S \rangle$ is connected in G or $\overline{G}$. Hence S is a connected dominating set of G or $\overline{G}$. Thus for any graph G, either $\gamma_c(G) \leq \gamma_{it}(G)$ or $\gamma_c(\overline{G}) \leq \gamma_{it}(G)$.

Remark 4.4.4. For the complete graph K_n $(n \geq 2)$, $\gamma_{it}(K_n) = n$ and $\beta_0(K_n) = 1$. For the complete bipartite graph $K_{m,n}$ with $m \leq n$ and $n \geq 3, \gamma_{it}(K_{m,n}) = 2$ and $\beta_0(K_{m,n}) = n$. For the corona G^+ of a graph G on n vertices, $\gamma_{it}(G^+) = \beta_0(G^+) = n$. Hence there is no relation between γ_{it} and β_0.

Remark 4.4.5. If G has a unique maximum independent set, then $\gamma_{it}(G) \leq \gamma(G)+1$. For if S is a γ-set and D is the maximum independent set then $S \cup \{u\}$, where $u \in D$, is an independent transversal dominating set. However, a graph G with $\gamma_{it}(G) \leq \gamma(G)+1$ can have more than one maximum independent sets. For example, for the graphs G_1 and G_2 given in Figure 2, we have $\gamma_{it}(G_1) = \gamma(G_1)$ and $\gamma_{it}(G_2) = \gamma(G_2)+1$. However each of G_1 and G_2 has two maximum independent sets.

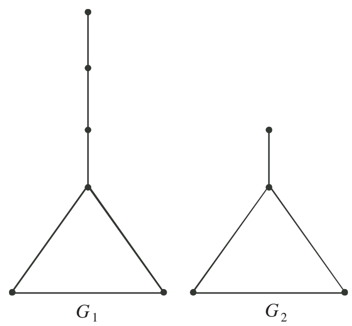

Figure 2

Conclusion and Scope

Theory of domination is an important as well as fastest growing area in graph theory. A number of variations of domination have been introduced by several authors. In this sequence we have introduced a new variation in domination namely, independent transversal domination. In this paper we have just initiated a study of this parameter. However, there is a wide scope for further research on this parameter and we here list some of them.

(A) The following are some interesting open problems.

1. Characterize non-complete connected graphs G on n vertices with $\beta_0(G) \geq \frac{n}{2}$ for which $\gamma_{it}(G) = \frac{n}{2}$. In particular, characterizing bipartite graphs G for which $\gamma_{it}(G) = \frac{n}{2}$ is worthy trying.
2. Characterize graphs for which (i) $\gamma_{it}(G) = \gamma(G)$, (ii) $\gamma_{it}(G) = \gamma(G) + \delta(G)$, (iii) $\gamma_{it}(G) = \alpha_0(G) + 1$.
3. Characterize graphs of diameter two for which $\gamma_{it}(G) = \delta(G) + 1$.
4. Characterize graphs G on n vertices for which (i) $\gamma(G) + \gamma_{it}(G) = n + 1$, (ii) $i(G) + \gamma_{it}(G) = n + 1$, (iii) $\gamma_{it}(G) = \gamma_c(G)$.
5. Given three positive integers a, b and c with $a \leq b \leq a + c$, does there exist a graph G such that $\gamma(G) = a, \gamma_{it}(G) = b$ and $\delta(G) = c$?

(B) For any graph theoretic parameter the effect of removal of a vertex or an edge on the parameters is of practical importance. As far as our parameter γ_{it} is concerned, removal of either a vertex or an edge may increase or decrease the value of γ_{it} or may remain unchanged. For example, for the star $K_{1,n-1} (n \geq 4)$, we have $\gamma_{it}(K_{1,n-1}) = 2$, whereas $\gamma_{it}(K_{1,n-1} - u) = 2$ where u is any pendant vertex of the star. Also $\gamma_{it}(W_7) = 3$, whereas $\gamma_{it}(W_7 - v) = 2$, where v is the centre vertex of the wheel. Hence it is possible to partition V into the sets V_0, V_+ and V_-, where

$$V_0 = \{v \in V : \gamma_{it}(G - v) = \gamma_{it}(G)\},$$
$$V_+ = \{v \in V : \gamma_{it}(G - v) > \gamma_{it}(G)\},$$
$$V_- = \{v \in V : \gamma_{it}(G - v) < \gamma_{it}(G)\}.$$

Similarly, one can partition the edge set E into the sets E_0, E_+ and E_-. Now, we can start investigating the properties of these sets.

(C) Since the value of γ_{it} for the complete graph K_n and the totally disconnected graph $\overline{K_n}$ on n vertices is n, it is always possible for a graph G to increase the value of $\gamma_{it}(G)$ by adding edges from the complement or by deleting the edges of G. Hence one can naturally ask for the minimum number of edges to be deleted (added) in order to increase the value of $\gamma_{it}(G)$; let us call the earlier as *independent transversal dombondage number* and the later one as *independent transversal domination reinforcement* number and denote them by $b_{it}(G)$ and $r_{it}(G)$ respectively. One can initiate a study of these parameters.

(D) Partitioning the vertex set V of a graph G into subsets of V having certain property is one direction of research in graph theory. For instance, one such partition is domatic partition which is a partition of V into dominating sets. Analogously, we can demand each set in the partition of V to have the property being independent transversal domination instead of just domination and call this partition an *independent transversal domatic partition*. Further, since $V(G)$ is always an independent transversal dominating set of G, such partition exists for all graphs so that asking the maximum order of such partition is reasonable; let us call this maximum order as the *independent transversal domatic number* and denote it by $d_{it}(G)$. Now, begin investigating the parameter d_{it}.

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