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CONNECTED GLOBAL OFFENSIVE k-ALLIANCES IN GRAPHS

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Abstract

We consider finite graphs G with vertex set $V(G)$. For a subset $S \subseteq V(G)$, we define by $G[S]$ the subgraph induced by S. By $n(G) = |V(G)|$ and $\delta(G)$ we denote the order and the minimum degree of G, respectively. Let k be a positive integer. A subset $S \subseteq V(G)$ is a *connected global offensive k-alliance* of the connected graph G, if $G[S]$ is connected and $|N(v) \cap S| \geq |N(v) - S| + k$ for every vertex $v \in V(G) - S$, where $N(v)$ is the neighborhood of v. The *connected global offensive k-alliance number* $\gamma_o^{k,c}(G)$ is the minimum cardinality of a connected global offensive k-alliance in G.

In this paper we characterize connected graphs G with $\gamma_o^{k,c}(G) = n(G)$. In the case that $\delta(G) \geq k \geq 2$, we also characterize the family of connected graphs G with $\gamma_o^{k,c}(G) = n(G) - 1$. Furthermore, we present different tight bounds of $\gamma_o^{k,c}(G)$.

Keywords: alliances in graphs, connected global offensive k-alliance, global offensive k-alliance, domination.

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1. Terminology and Introduction

We consider finite, undirected and simple graphs G with vertex set $V(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* and is denoted by $n = n(G)$. The *neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. By $\delta = \delta(G)$ and $\Delta = \Delta(G)$, we denote the *minimum degree* and the *maximum degree* of the graph G, respectively. For a subset $S \subseteq V(G)$, we define by $G[S]$ the subgraph induced by S.

The complete graph of order n is denoted by K_n, and $K_{s,t}$ is the complete bipartite graph with the two parts of cardinality s and t.

Two vertices that are not adjacent in a graph G are said to be *independent*. A set I of vertices is *independent* if every two vertices of I are independent. The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality among the independent sets of vertices of G.

A *vertex-cut* in a connected graph G is a set S of vertices of G such that $G - S$ is disconnected. The *connectivity* $\kappa(G)$ of a graph G is the minimum cardinality of a vertex-cut of G if G is not complete, and $\kappa(G) = n - 1$ if G is isomorphic to the complete graph K_n.

Kristiansen, Hedetniemi and Hedetniemi [9] introduced several types of alliances in graphs, including defensive and offensive alliances. As a generalization of the offensive alliance, Shafique and Dutton [11, 12] defined the global offensive k-alliance for a positive integer k as follows. A subset $S \subseteq V(G)$ is a *global offensive k-alliance* of the graph G if $|N(v) \cap S| \geq |N(v) - S| + k$ for every vertex $v \in V(G) - S$. The *global offensive k-alliance number* $\gamma_o^k(G)$ is the minimum cardinality of a global offensive k-alliance in G. A global offensive k-alliance set of the minimum cardinality of a graph G is called a $\gamma_o^k(G)$-set. Results on global offensive k-alliances were given, for example, by Bermudo, Rodríguez-Velázquez, Sigarreta and Yero [1], Chellali [2], Chellali, Haynes, Randerath and Volkmann [3] and Fernau, Rodríguez and Sigarreta [4].

In this paper, we are interested in connected global offensive k-alliances. Analogously to the definition above, a subset $S \subseteq V(G)$ is a *connected global offensive k-alliance* of the connected graph G, if $G[S]$ is connected and $|N(v) \cap S| \geq |N(v) - S| + k$ for every vertex $v \in V(G) - S$. The *connected global offensive k-alliance number* $\gamma_o^{k,c}(G)$ is the minimum cardinality of a connected global offensive k-alliance in G. A connected global offensive k-alliance set of the minimum cardinality of a connected graph G is called a $\gamma_o^{k,c}(G)$-set.

A subset $D \subseteq V(G)$ is a *k-dominating set* of the graph G if $|N_G(v) \cap D| \geq k$ for every $v \in V(G) - D$. The *k-domination number* $\gamma^k(G)$ is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number $\gamma^1(G)$ is the usual *domination number* $\gamma(G)$. A subset $D \subseteq V(G)$ is a connected k*-dominating set* of a connected graph G, if D is a k-dominating set of G and the induced subgraph $G[D]$ is connected. The *connected* k*-domination number* $\gamma^{k,c}(G)$ is the minimum cardinality among the connected k-dominating sets of G.

In [5, 6], Fink and Jacobson introduced the concept of k-domination. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi and Slater [7, 8].

In this paper we characterize the connected graphs G with $\gamma_o^{k,c}(G) = n(G)$. If G is a connected graph with $\delta(G) \geq k \geq 3$, then we show that $\gamma_o^{k,c}(G) = n(G) - 1$ if and only if G is isomorphic to the complete graph K_{k+1} or K_{k+2}. In addition, we derive different sharp bounds on $\gamma_o^{k,c}(G)$, as for example, $\gamma_o^{k,c}(G) \leq 2\gamma_o^k(G) - k + 1$.

2. Main Results

Observation 1. *If $k \geq 1$ is an integer, then $\gamma_o^{k,c}(G) \geq \gamma^{k,c}(G)$ for any connected graph G.*

Proof. If S is a $\gamma_o^{k,c}(G)$-set, then $G[S]$ is connected and every vertex of $V(G) - S$ has at least k neighbors in S. Thus S is a connected k-dominating set of G and so $\gamma^{k,c}(G) \leq |S| = \gamma_o^{k,c}(G)$. ■

In view of Observation 1, each lower bound of $\gamma^{k,c}(G)$ is also a lower bound of $\gamma_o^{k,c}(G)$. Now we characterize all connected graphs G with the property that $\gamma_o^{k,c}(G) = n(G)$.

Observation 2. *Let $k \geq 2$ be an integer, and let G be a connected graph of order $n \geq 2$. Then $\gamma_o^{k,c}(G) = n$ if and only if all vertices of G are either cut-vertices or vertices of degree less than k.*

Proof. If each vertex of G is either a cut-vertex or has degree less than k, then the definition of the connected global offensive k-alliance number leads to $\gamma_o^{k,c}(G) = n$ immediately.

Conversely, assume that $\gamma_o^{k,c}(G) = n$. Suppose to the contrary that G contains a non-cut-vertex u with $d_G(u) \geq k$. This implies that $G - u$ is a connected graph. Since $d_G(u) \geq k$, we deduce that $V(G - u)$ is a connected global offensive k-alliance of G. Therefore we obtain the contradiction $\gamma_o^{k,c}(G) \leq n - 1$, and the proof is complete. ■

Corollary 3. *Let $k \geq 2$ be an integer. If T is a tree, then $\gamma_o^{k,c}(T) = n(T)$.*

Corollary 4. *If $k \geq 2$ is an integer, and G is a connected graph with $\delta(G) \geq k$, then $\gamma_o^{k,c}(G) \leq n(G) - 1$.*

Next we derive a characterization of all connected graphs G with $\gamma_o^{k,c}(G) = n(G) - 1$ when $\delta(G) \geq k \geq 2$.

Theorem 5. *Let $k \geq 2$ be an integer, and let G be a connected graph of order n and minimum degree δ.*

(i) *If $\delta \geq 2$, then $\gamma_o^{2,c}(G) = n - 1$ if and only if G is a cycle or G is isomorphic to the complete graph K_4.*

(ii) *If $\delta \geq k \geq 3$, then $\gamma_o^{k,c}(G) = n - 1$ if and only if G is isomorphic to the complete graph K_{k+1} or K_{k+2}.*

Proof. Obviously, if G is a cycle or G is isomorphic to K_4, then $\gamma_o^{2,c}(G) = n - 1$, and if G is isomorphic to the complete graphs K_{k+1} or K_{k+2}, then $\gamma_o^{k,c}(G) = n - 1$.

Conversely, assume that $\gamma_o^{k,c}(G) = n - 1$, and let $P = u_1u_2 \dots u_t$ be the longest path in G. The condition $\delta \geq k \geq 2$ implies that $u_1 \neq u_t$ and $G - \{u_1, u_t\}$ is a connected subgraph of G. If u_1 and u_t are not adjacent in G, then we arrive at the contradiction that $V(G) - \{u_1, u_t\}$ is a connected global offensive k-alliance of G. In the remaining case that u_1 and u_t are adjacent in G, we observe that $C = u_1u_2 \dots u_tu_1$ is a Hamiltonian cycle of G, because P is the longest path in G. This yields $t = n$.

(i) Assume that $k = 2$. Suppose that the Hamiltonian cycle $C = u_1u_2 \dots u_nu_1$ has a chord. If, without loss of generality, u_1u_s with $3 \leq s \leq n-1$ is a chord of C, then we obtain the contradiction that $V(G) - \{u_2, u_n\}$ is a connected global offensive 2-alliance of G or u_2 and u_n are adjacent. Therefore assume in the following that u_2 and u_n are adjacent. If $n = 4$, then $G = K_4$. If $n \geq 5$, then we distinguish the cases $s = 3$ and $s \geq 4$.

Assume first that $s = 3$. Then we obtain the contradiction that $V(G) - \{u_2, u_4\}$ is a connected global offensive 2-alliance of G or u_2 and u_4 are adjacent. If u_2 and u_4 are adjacent, then we have the contradiction that $V(G) - \{u_3, u_n\}$ is a connected global offensive 2-alliance of G or u_3 and u_n are adjacent. However, if u_3 and u_n are adjacent, then $d_G(u_2), d_G(u_n) \geq 4$, and thus we arrive at the contradiction that $V(G) - \{u_2, u_n\}$ is a connected global offensive 2-alliance of G.

Assume now that $s \geq 4$. Then we obtain the contradiction that $V(G) - \{u_1, u_3\}$ is a connected global offensive 2-alliance of G or u_1 and u_3 are adjacent. If u_1 and u_3 are adjacent, then we have the contradiction that $V(G) - \{u_3, u_n\}$ is a connected global offensive 2-alliance of G or u_3 and u_n are adjacent. However, if u_3 and u_n are adjacent, then $d_G(u_1), d_G(u_n) \geq 4$, and thus we arrive at the contradiction that $V(G) - \{u_1, u_n\}$ is a connected global offensive 2-alliance of G.

(ii) Assume that $k \geq 3$. In the following all indices are taken modulo n. If the vertices u_i and u_{i+2} are not adjacent for any index i with $1 \leq i \leq n$, then the hypothesis $\delta \geq k \geq 3$ leads to the contradiction that $V(G) - \{u_i, u_{i+2}\}$ is a connected global offensive k-alliance of G. Hence assume that u_i and u_{i+2} are adjacent for each index $i \in \{1, 2, \ldots, n\}$. Now let s be an arbitrary integer with $3 \leq s \leq n-3$. If u_i and u_{i+s} are not adjacent, then $V(G) - \{u_i, u_{i+s}\}$ is a connected global offensive k-alliance of G, since there exists the edge $u_{i-1}u_{i+1}$ in G. Therefore it remains the case that G is a complete graph. If G is isomorphic to K_{k+1} or K_{k+2}, then $\gamma_o^{k,c}(G) = n-1$. However, if G is isomorphic to K_q for any integer $q \geq k+3$, then $V(G) - \{u_1, u_2\}$ is a connected global offensive k-alliance of G. This contradiction completes the proof of Theorem 5. ■

Proposition 6. *Let G be a graph of order n, and let k, p be two integers such that $k \geq 1$ and $-1 \leq p \leq \alpha(G) - 2$. If $\delta(G) \geq k$ and $\kappa(G) \geq \alpha(G) - p$, then*

$$\gamma_o^{k,c}(G) \leq n(G) - \alpha(G) + p + 1.$$

Proof. Let $I \subset V(G)$ be an independent set of cardinality $\alpha(G)-p-1$. The hypothesis $\kappa(G) \geq \alpha(G) - p$ implies that $G[V(G) - I]$ is connected. Since I is an independent set, the condition $\delta(G) \geq k$ shows that each vertex in I has at least k neighbors in $V(G) - I$. Thus $V(G) - I$ is a connected global offensive k-alliance of G such that $|V(G) - I| \leq n - (\alpha(G) - p - 1)$, and the proof is complete. ■

If H is the complete bipartite graph $K_{k,k}$, then $\delta(H) = \alpha(H) = \kappa(H) = k$ and $\gamma_o^{k,c}(H) = k + 1 = n(H) - \alpha(H) + 1$. This example demonstrates that Proposition 6 is the best possible, at least for $p = 0$.

Theorem 7. *Let G be a connected graph and k an integer with $1 \leq k \leq \Delta(G)$. Then*

$$\gamma_o^{k,c}(G) \leq 2\gamma_o^k(G) - k + 1.$$

Proof. Let S be a $\gamma_o^k(G)$-set. Since $k \leq \Delta(G)$, we observe that $|S| = \gamma_o^k(G) \leq n(G) - 1$. Now let $x \in V(G) - S$ be an arbitrary vertex.

If $G[S \cup \{x\}]$ is connected, then the inequality $k \leq \gamma_o^k(G)$ implies that $\gamma_o^{k,c}(G) \leq \gamma_o^k(G) + 1 \leq 2\gamma_o^k(G) - k + 1$, and we are done.

Thus assume next that $G[S \cup \{x\}]$ is not connected. We will add successively vertices from $V(G) - (S \cup \{x\})$ to $S \cup \{x\}$ in order to decrease the number of components, at least one in each step, until we obtain a set of vertices whose induced subgraph is connected. Note that if we partition $S \cup \{x\}$ into two parts A and B such that there is no edge between A and B, and we take vertices $a \in A$ and $b \in B$ such that the distance between a and b is minimum in G, then the property of S of being dominating implies that $d_G(a,b) \leq 3$. It follows that in each step of increasing $S \cup \{x\}$ we need to add at most 2 vertices from $V(G) - (S \cup \{x\})$. Let r_1 and r_2 be the number of steps where we include one vertex and two vertices from $V(G) - (S \cup \{x\})$, respectively, and define $r = r_1 + r_2$. Let $S_0 \subset S \cup \{x\}$ be the set of vertices of the component of $G[S \cup \{x\}]$ to which x belongs, and let $S_i \subset S$ be the set of vertices connected to $\bigcup_{j=0}^{i-1} S_j$ in step $i \geq 1$. Clearly, $|S_0| \geq k + 1$ and $|S_i| \geq 1$ for $1 \leq i \leq r$. Furthermore, since S is a global offensive k-alliance, in the steps where two vertices from $V(G) - (S \cup \{x\})$ are added, we observe that $|S_i| \geq k + 1$. This leads to

$$\gamma_o^k(G) = |S| = |S_0 - \{x\}| + \sum_{i=1}^{r} |S_i| \geq k + r_2(k+1) + r_1$$

and therefore $r_1 \leq \gamma_o^k(G) - k - r_2(k+1)$. As a further consequence, we see that $S \cup \{x\}$ together with all vertices from $V(G) - (S \cup \{x\})$ added in steps 1 to r form a connected global offensive k-alliance of G. Altogether, we deduce that

$$\begin{aligned}
\gamma_o^{k,c}(G) &\leq |S| + 1 + r_1 + 2r_2 \\
&\leq \gamma_o^k(G) + 1 + \gamma_o^k(G) - k - r_2(k+1) + 2r_2 \\
&= 2\gamma_o^k(G) - k + 1 - r_2(k+1) + 2r_2 \\
&\leq 2\gamma_o^k(G) - k + 1,
\end{aligned}$$

and the proof is complete. ■

If H is the complete bipartite graph $K_{k,p}$, then $\gamma_o^k(H) = k$ and $\gamma_o^{k,c}(H) = k + 1$. This example shows that the bound given in Theorem 7 is tight.

Theorem 8. *Let G be a connected graph and $k \geq 1$ an integer. If $\delta(G) \geq k+1$, then*

$$\gamma_o^{k+1,c}(G) \leq \frac{\gamma_o^{k,c}(G) + n(G)}{2}.$$

Proof. Let S be a $\gamma_o^{k,c}(G)$-set, and let A be the set of isolated vertices in the subgraph $G - S$. Then the subgraph $G - (S \cup A)$ contains no isolated vertices. If D is a minimum dominating set of $G - (S \cup A)$, then the well-known inequality of Ore [10] implies

$$|D| \leq \frac{|V(G) - (S \cup A)|}{2} \leq \frac{|V(G) - S|}{2} = \frac{n(G) - \gamma_o^{k,c}(G)}{2}.$$

If $S' = S \cup D$, then $G[S']$ is connected. In addition, for each vertex $v \in V(G) - (S' \cup A)$, we have

$$\begin{aligned} |N(v) \cap S'| &= |N(v) \cap S| + |N(v) \cap D| \\ &\geq |N(v) - S| + k + 1 \\ &= |N(v) - S'| + |N(v) \cap D| + k + 1 \\ &\geq |N(v) - S'| + k + 2. \end{aligned}$$

Since $\delta(G) \geq k+1$, every vertex of A has at least $k+1$ neighbors in S, and therefore S' is a connected global offensive $(k+1)$-alliance of G and thus

$$\begin{aligned} \gamma_o^{k+1,c}(G) &\leq |S'| = |S| + |D| = \gamma_o^{k,c}(G) + |D| \\ &\leq \gamma_o^{k,c}(G) + \frac{n(G) - \gamma_o^{k,c}(G)}{2} = \frac{n(G) + \gamma_o^{k,c}(G)}{2}. \end{aligned}$$

■

The inequality $|N(v) \cap S'| \geq |N(v) - S'| + k + 2$ for each vertex $v \in V(G) - (S' \cup A)$ in the proof of Theorem 8 leads immediately to the next result.

Theorem 9. *Let G be a connected graph and $k \geq 1$ an integer. If $\delta(G) \geq k+2$, then*

$$\gamma_o^{k+2,c}(G) \leq \frac{\gamma_o^{k,c}(G) + n(G)}{2}.$$

If $H = K_{k+3}$, then $\gamma_o^{k+2,c}(H) = \gamma_o^{k+1,c}(H) = k+2$ and $\gamma_o^{k,c}(H) = k+1$ and thus

$$\gamma_o^{k+2,c}(H) = \gamma_o^{k+1,c}(H) = k+2 = \frac{\gamma_o^{k,c}(H) + n(H)}{2}.$$

Let $k \geq 2$ be an even integer, and let $F = K_{k+6} - M$, where M is a perfect matching of the complete graph K_{k+6}. Then $\gamma_o^{k+2,c}(F) = \gamma_o^{k+1,c}(F) = k+4$ and $\gamma_o^{k,c}(F) = k+2$, and so

$$\gamma_o^{k+2,c}(F) = \gamma_o^{k+1,c}(F) = k+4 = \frac{\gamma_o^{k,c}(F) + n(F)}{2}.$$

These two graphs H and F demonstrate that Theorem 8 as well as Theorem 9 are the best possible.

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