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ON THE FORCING GEODETIC AND FORCING STEINER NUMBERS OF A GRAPH

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Abstract

For a connected graph $G = (V, E)$, a set $W \subseteq V$ is called a Steiner set of G if every vertex of G is contained in a Steiner W-tree of G. The Steiner number $s(G)$ of G is the minimum cardinality of its Steiner sets and any Steiner set of cardinality $s(G)$ is a minimum Steiner set of G. For a minimum Steiner set W of G, a subset $T \subseteq W$ is called a forcing subset for W if W is the unique minimum Steiner set containing T. A forcing subset for W of minimum cardinality is a minimum forcing subset of W. The forcing Steiner number of W, denoted by $f_s(W)$, is the cardinality of a minimum forcing subset of W. The forcing Steiner number of G, denoted by $f_s(G)$, is $f_s(G) = \min\{f_s(W)\}$, where the minimum is taken over all minimum Steiner sets W in G. The geodetic number $g(G)$ and the forcing geodetic number $f(G)$ of a graph G are defined in [2]. It is proved in [6] that there is no relationship between the geodetic number and the Steiner number of a graph so that there is no relationship between the forcing geodetic number and the forcing Steiner number of a graph. We give realization results for various possibilities of these four parameters.

Keywords: geodetic number, Steiner number, forcing geodetic number, forcing Steiner number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. It is known that the distance is a metric on the vertex set of G. For basic graph theoretic terminology, we refer to [1]. A *geodetic set* of G is a set S of vertices such that every vertex of G is contained in a geodesic joining some pair of vertices of S. The *geodetic number* $g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinality $g(G)$ is a *minimum geodetic set* or simply a *g-set* of G. A vertex v is said to be a *geodetic vertex* if v belongs to every g-set of G. The geodetic number of a graph was introduced in [5] and further studied in [3]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem. A subset $T \subseteq S$ is called a *forcing subset for S* if S is the unique minimum geodetic set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing geodetic number of S*, denoted by $f(S)$, is the cardinality of a minimum forcing subset of S. The *forcing geodetic number of G*, denoted by $f(G)$, is $f(G) = \min\{f(S)\}$, where the minimum is taken over all minimum geodetic sets S in G. The forcing geodetic number of a graph was introduced and studied in [2]. Santhakumaran *et al.* studied the connected geodetic number of a graph in [7] and also the upper connected geodetic number and the forcing connected geodetic number of a graph in [8].

For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a *Steiner tree* with respect to W or a *Steiner W-tree*. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $S(W) = V$, then W is called a *Steiner set* for G. A Steiner set of minimum cardinality is a *minimum*

Steiner set or simply a *s-set* of G and this cardinality is the *Steiner number* $s(G)$ of G. A vertex v is said to be a *Steiner vertex* if v belongs to every s-set of G. The Steiner number of a graph was introduced and studied in [4]. Chartrand *et al.* proved in [4] that every Steiner set in a connected graph is a geodetic set. However, this particular result was proved to be wrong by Pelayo in [6]. The *forcing Steiner number* $f_s(G)$ of G is defined similar to the forcing geodetic number $f(G)$ of G.

For the graph G given in Figure 1.1(a), $W_1 = \{v_1, v_4, v_5, \}, W_2 = \{v_2, v_4, v_7\}$ and $W_3 = \{v_3, v_5, v_7\}$ are the only three s-sets of G so that $s(G) = 3$ and $f_s(G) = 1$. Also $S = \{v_1, v_2, v_3, v_6\}$ is the unique g-set of G so that $g(G) = 4$ and $f(G) = 0$. For the graph G given in Figure 1.1(b), $W = \{v_1, v_2, v_5, v_6\}$ is the unique s-set of G so that $s(G) = 4$ and $f_s(G) = 0$. Also $S_1 = \{v_1, v_5, v_6\}$ and $S_2 = \{v_2, v_5, v_6\}$ are the only two g-sets of G so that $g(G) = 3$ and $f(G) = 1$. For the graph G given in Figure 1.1(c), $W = \{v_1, v_5\}$ is the unique g-set as well as the unique s-set of G so that $g(G) = s(G) = 2$ and $f(G) = f_s(G) = 0$.

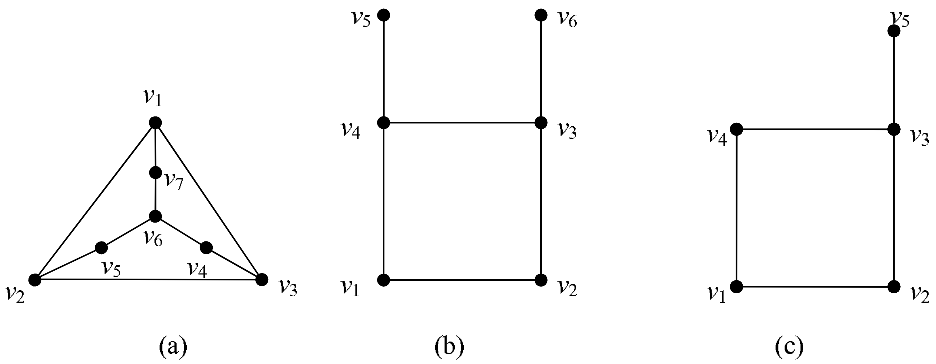

Figure 1.1. G

A vertex v is a *simplicial vertex* of a graph G if the subgraph induced by its neighbors is complete. The following theorems are used in the sequel.

Theorem 1 [3]. *Each simplicial vertex of a connected graph G belongs to every geodetic set of G.*

Theorem 2 [1]. *Let G be a connected graph. Then*

(i) *no cut-vertex of G belongs to any g-set of G.*

(ii) $g(G) = p$ *if and only if* $G = K_p$.

Theorem 3 [2]**.** *Let G be a connected graph. Then*

(a) $f(G) = 0$ *if and only if G has a unique minimum geodetic set.*

(b) $f(G) \leq g(G) - |W|$, *where W is the set of all geodetic vertices of G.*

Theorem 4 [4]**.** *Let G be a connected graph. Then*

(i) *each simplicial vertex belongs to every Steiner set of G.*

(ii) $s(G) = p$ *if and only if $G = K_p$.*

The following theorem is an easy consequence of the corresponding definitions.

Theorem 5. *Let G be a connected graph. Then*

(a) $f_s(G) = 0$ *if and only if G has a unique minimum Steiner set.*

(b) $f_s(G) \leq s(G) - |W|$, *where W is the set of all Steiner vertices of G.*

(c) *For the complete graph $G = K_p$ $(p \geq 2)$, $f_s(G) = 0$.*

Throughout the following G denotes a connected graph with at least two vertices.

2. SPECIAL GRAPHS

In this section, we present some graphs from which various graphs arising in theorems of different sections are generated using identification.

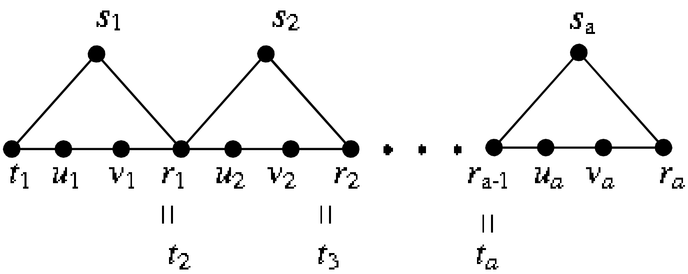

Figure 2.1. G_a

The graph G_a is obtained from the F_i's by identifying the vertices r_{i-1} of F_{i-1} and t_i of F_i $(2 \leq i \leq a)$, where $F_i : s_i, t_i, u_i, v_i, r_i, s_i$ $(1 \leq i \leq a)$ is a copy of the cycle C_5.

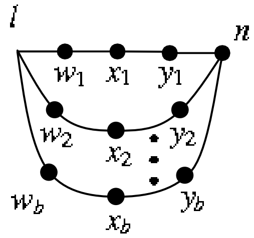

Figure 2.2. H_b

The graph H_b is obtained from P_i $(1 \leq i \leq b)$ by adding two new vertices l and n, and joining the edges lw_i and ny_i $(1 \leq i \leq b)$, where $P_i : w_i, x_i, y_i$ $(1 \leq i \leq b)$ is a copy of the path on three vertices.

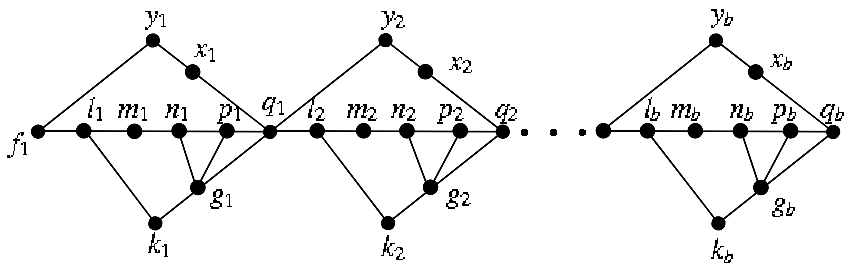

Figure 2.3. L_b

Let $J_i : f_i, l_i, m_i, n_i, p_i, q_i, x_i, y_i, f_i$ $(1 \leq i \leq b)$ be a copy of the cycle C_8. Let R_i be the graph obtained from J_i by adding two new vertices k_i, g_i and the edges $l_ik_i, k_ig_i, g_in_i, g_ip_i, g_iq_i$ $(1 \leq i \leq b)$. The graph L_b is obtained from R_i's by identifying the vertices q_{i-1} of R_{i-1} and f_i of R_i $(2 \leq i \leq b)$.

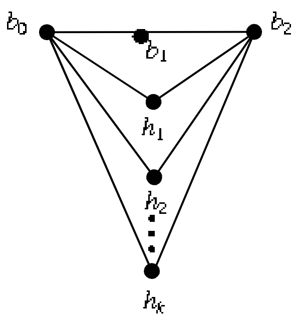

Figure 2.4. T_k

Let $P : b_0, b_1, b_2$ be a path on three vertices. The graph T_k is obtained from P by adding the new vertices $h_1, h_2, \ldots, h_k$ and joining each h_i $(1 \leq i \leq k)$ with b_0 and b_2 in P.

Since there is no relationship between the geodetic number and the Steiner number of a graph, there is no relationship between the corresponding forcing geodetic and forcing Steiner numbers also. In the rest of the section, we give realization results for various possibilities of these four parameters.

3. Realization with Respect to Two Integers

In this section, we give realization results for the four parameters $g(G), s(G)$, $f(G)$ and $f_s(G)$ of a graph G when $0 \leq a < b$, $b \geq 2$ and $b - a - 1 > 0$.

Theorem 6. *For every pair a, b of integers with $0 \leq a < b$, $b \geq 2$ and $b - a - 1 > 0$, there exists a connected graph G such that $f_s(G) = a$ and $s(G) = b$.*

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 1.5(c) and Theorem 1.4(ii), $f_s(G) = 0$ and $s(G) = b$. Now, assume that $a \geq 1$. Let G be the graph obtained from G_a in Figure 2.1 by adding $b - a$ new vertices $z_1, z_2, \ldots, z_{b-a-1}, u$ and joining the $b - a$ edges $t_1 z_i$ $(1 \leq i \leq b - a - 1)$ and $r_a u$. Let $Z = \{z_1, z_2, \ldots, z_{b-a-1}, u\}$ be the set of simplicial vertices of G. By Theorem 1.4(i), every s-set of G contains Z. Let $M_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$. First, we show that $s(G) = b$. Since the vertices u_i, v_i do not lie on the unique Steiner Z-tree of G, it is clear that Z is not a Steiner set of G. We observe that every s-set of G must contain exactly one vertex from each M_i $(1 \leq i \leq a)$ and so $s(G) \geq b - a + a = b$. On the other hand, since the set $W = Z \cup \{v_1, v_2, \ldots, v_a\}$ is a Steiner set of G, it follows that $s(G) \leq |W| = b$. Thus, $s(G) = b$. Next, we show that $f_s(G) = a$. By Theorem 1.4(i), every Steiner set of G contains Z and so it follows from Theorem 1.5(b) that $f_s(G) \leq s(G) - |Z| = a$. Now, it is easily seen that every s-set S is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in M_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap M_j = \emptyset$, which shows that $f_s(G) = a$. ■

Theorem 7. *For every pair a, b of integers with $0 \leq a < b$, $b \geq 2$ and $b - a - 1 > 0$, there exists a connected graph G such that $f_s(G) = f(G) = a$ and $s(G) = g(G) = b$.*

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 1.2(ii) and Theorem 1.3(a), $g(G) = b$ and $f(G) = 0$. For $a \geq 1$, let G be the graph given in Theorem 3.1 for the case $a \geq 1$. Then, as in the proof of Theorem 3.1, it can be proved that $f(G) = a$ and $g(G) = b$. The rest now follows from Theorem 3.1. ■

4. Realization with Respect to Three Integers

In this section, we give realization results for the above said four parameters when $0 \leq a \leq b < c$ and $c - b - 1 > 0$.

Theorem 8. *For integers a, b, c with $0 \leq a \leq b < c$ and $c - b - 1 > 0$, there exists a connected graph G such that $f_s(G) = a$, $f(G) = b$ and $s(G) = g(G) = c$.*

Proof. We consider three cases.

Case 1. $a = 0$. Let G be the graph obtained from H_b in Figure 2.2 by adding the new vertices $u, m, z_1, z_2, \ldots, z_{c-b-1}$ and joining the edges $nu, ml, mn, lz_1, lz_2, \ldots, lz_{c-b-1}$. Let $Z = \{z_1, z_2, \ldots, z_{c-b-1}, u\}$ be the set of simplicial vertices of G. The vertices w_i, x_i, y_i $(1 \leq i \leq b)$ do not lie on any Steiner Z-tree of G. It easily follows from Theorem 1.4(i) that $W = Z \cup \{x_1, x_2, \ldots, x_b\}$ is the unique minimum Steiner set of G so that $s(G) = c$. Hence, by Theorem 1.5(a), $f_s(G) = 0 = a$. Now, we show that $g(G) = c$. Let $M_i = \{w_i, x_i, y_i\}$, $(1 \leq i \leq b)$. We observe that every geodetic set of G must contain at least one vertex from each M_i $(1 \leq i \leq b)$ and so by Theorem 1.1, $g(G) \geq c - b + b = c$. On the other hand, since $W = Z \cup \{x_1, x_2, \ldots, x_b\}$ is a geodetic set of G, it follows that $g(G) \leq |W| = c$. Thus $g(G) = c$. Next, we show that $f(G) = b$. Since every g-set contains Z, it follows from Theorem 1.3(b) that $f(G) \leq g(G) - |Z| = c - (c - b) = b$. Also, it is easily seen that every g-set S of G is of the form $Z \cup \{c_1, c_2, \ldots, c_b\}$, where $c_i \in M_i$ $(1 \leq i \leq b)$. Let T be any proper subset of S with $|T| < b$. Then it is clear that there exists some j such that $T \cap M_j = \emptyset$, which shows that $f(G) = b$.

Case 2. $a = b$. This follows from Theorem 3.2 by taking b as c.

Case 3. $1 \leq a < b$. Let G be the graph obtained from G_a and H_{b-a} by identifying the vertex r_a of G_a with the vertex l of H_{b-a} and then adding the new vertices $m, u, z_1, z_2, \ldots, z_{c-b-1}$ and adding the edges

$nu, lm, mn, t_1z_1, t_1z_2, \dots, t_1z_{c-b-1}$. Let $Z = \{z_1, z_2, \dots, z_{c-b-1}, u\}$ be the set of simplicial vertices of G. Let $M_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$ and $Q_i = \{w_i, x_i, y_i\}$ $(1 \leq i \leq b-a)$.

First, we show that $g(G) = c$. We observe that every geodetic set of G must contain at least one vertex from each M_i $(1 \leq i \leq a)$ and at least one vertex from each Q_i $(1 \leq i \leq b-a)$. Hence, by Theorem 1.1, $g(G) \geq c - b + a + b - a = c$. On the other hand, since the set $S_1 = Z \cup \{u_1, u_2, \dots, u_a\} \cup \{x_1, x_2, \dots, x_{b-a}\}$ is a geodetic set of G, it follows that $g(G) \leq |S_1| = c$. Thus $g(G) = c$.

Next, we show that $f(G) = b$. As in Case 1, $f(G) \leq b$ and it is easily seen that every g-set S of G is of the form $Z \cup \{c_1, c_2, \dots, c_a\} \cup \{d_1, d_2, \dots, d_{b-a}\}$, where $c_i \in M_i$ $(1 \leq i \leq a)$ and $d_j \in Q_j (1 \leq j \leq b-a)$. Let T be any proper subset of S with $|T| < b$. Then it is clear that there exists some i such that $T \cap M_i = \emptyset$ or there exists some j such that $T \cap Q_j = \emptyset$, which shows that $f(G) = b$.

Now, we show that $s(G) = c$. It is clear that Z is not a Steiner set of G. We observe that every minimum Steiner set of G must contain exactly one vertex from each M_i $(1 \leq i \leq a)$ and only the vertex x_i $(1 \leq i \leq b-a)$ from each Q_i $(1 \leq i \leq b-a)$. Hence, by Theorem 1.4(i), $s(G) \geq c$. On the other hand, $S = Z \cup \{u_1, u_2, \dots, u_a\} \cup \{x_1, x_2, \dots, x_{b-a}\}$ is a Steiner set of G and so $s(G) \leq c$. Hence $s(G) = c$.

Next, we show that $f_s(G) = a$. Since every s-set of G contains $W = Z \cup \{x_1, x_2, \dots, x_{b-a}\}$,it follows from Theorem 1.5(b) that $f_s(G) \leq s(G) - |W| = c - (c - b + b - a) = a$. Now, it is easily seen that every s-set S of G is of the form $W \cup \{c_1, c_2, \dots, c_a\}$, where $c_i \in M_i$ $(1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap M_j = \emptyset$, which shows that $f_s(G) = a$. ■

Theorem 9. *For integers a, b, c with $0 \leq a \leq b < c$ and $c - b - 1 > 0$, there exists a connected graph G such that $f(G) = a$, $f_s(G) = b$ and $s(G) = g(G) = c$.*

Proof. We consider three cases.

Case 1. $a = 0$. Let G be the graph obtained from L_b in Figure 2.3 by adding the new vertices $u, z_1, z_2, \dots, z_{c-b-1}$ and adding the $c - b$ edges f_1z_i $(1 \leq i \leq c-b-1)$ and q_bu. First, we show that $s(G) = c$. Let $Z = \{z_1, z_2, \dots, z_{c-b-1}, u\}$ be the set of simplicial vertices of G. It is clear that there is only one Steiner Z-tree of G and it is given in Figure 4.1. Hence Z is not a Steiner set of G. For $1 \leq i \leq b$, let $M_i = \{m_i, n_i\}$. We observe that every s-set of G must contain exactly one vertex from each M_i so that by Theorem 1.4(i), $s(G) \geq c - b + b = c$. Now, $W = Z \cup \{m_1, m_2, \dots, m_b\}$ is a Steiner set of G so that $s(G) \leq c - b + b = c$. Thus $s(G) = c$.

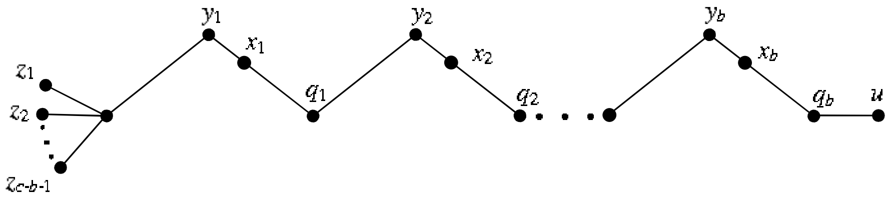

Figure 4.1. The unique Steiner Z-tree of G in Case 1 of Theorem 4.2

Next, we show that $f_s(G) = b$. Since every s-set contains Z, it follows from Theorem 1.5(b) that $f_s(G) \leq s(G) - |Z| = c - (c - b) = b$. Now, it is easily seen that every s-set S is of the form $Z \cup \{c_1, c_2, \dots, c_b\}$, where $c_i \in M_i$ $(1 \leq i \leq b)$. Let T be any proper subset of S with $|T| < b$. Then it is clear that there exists some j such that $T \cap M_j = \emptyset$, which shows that $f_s(G) = b$.

Now, we show that $g(G) = c$. It is clear that Z is not a geodetic set of G. It is easily seen that every geodetic set must contain only the vertices p_i $(1 \leq i \leq b)$ and $W = Z \cup \{p_1, p_2, \dots, p_b\}$ is the unique g-set of G so that $g(G) = c$ and by Theorem 1.3(a), $f(G) = 0$.

Case 2. $a = b$. This follows from Theorem 3.2 by taking b as c.

Case 3. $1 \leq a < b$. Let G be the graph obtained from G_a and L_{b-a} by identifying the vertex r_a of G_a and the vertex f_1 of L_{b-a} and then adding the new vertices $u, z_1, z_2, \dots, z_{c-b-1}$ and joining the edges $uq_{b-a}, t_1z_1, t_1z_2, \dots, t_1z_{c-b-1}$. Let $Z = \{z_1, z_2, \dots, z_{c-b-1}, u\}$ be the set of simplicial vertices of G. Let $M_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$ and $Q_i = \{m_i, n_i\}$ $(1 \leq i \leq b - a)$.

First, we show that $g(G) = c$. It is clear that Z is not a geodetic set of G. We observe that every g-set of G must contain exactly one vertex from each M_i $(1 \leq i \leq a)$ and only the vertex p_i $(1 \leq i \leq b-a)$ so that by Theorem 1.1, $g(G) \geq c$. On the other hand, $S_1 = Z \cup \{v_1, v_2, \dots, v_a\} \cup \{p_1, p_2, \dots, p_{b-a}\}$ is a geodetic set of G and so $g(G) \leq |S_1| = c$. Thus $g(G) = c$.

Now, we show that $f(G) = a$. Since every g-set of G contains $W = Z \cup \{p_1, p_2, \dots, p_{b-a}\}$, it follows from Theorem 1.3(b) that $f(G) \leq g(G) - |W| = c - (c - a) = a$. Now, it is easily seen that every g-set S is of the form $W \cup \{c_1, c_2, \dots, c_a\}$, where $c_i \in M_i$ $(1 \leq i \leq a)$. Then it is easily seen that $f(G) = a$.

Next, we show that $s(G) = c$. It is clear that Z is not a Steiner set of G. We observe that every s-set of G must contain exactly one vertex from each M_i $(1 \leq i \leq a)$ and exactly one vertex from each Q_i $(1 \leq i \leq b - a)$. Thus, by Theorem 1.4(i), $s(G) \geq c - b + a + b - a = c$. Since the set $S' = Z \cup \{v_1, v_2, \ldots, v_a\} \cup \{n_1, n_2, \ldots, n_{b-a}\}$ is a Steiner set of G, we have $s(G) \leq |S'| = c$. Hence $s(G) = c$.

Now, we show that $f_s(G) = b$. Since every s-set of G contains Z, it follows from Theorem 1.5(b) that $f_s(G) \leq s(G) - |Z| = c - (c - b) = b$. Now, it is easily seen that every s-set S is of the form $Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{d_1, d_2, \ldots, d_{b-a}\}$, where $c_i \in M_i$ $(1 \leq i \leq a)$ and $d_j \in Q_j$ $(1 \leq j \leq b - a)$. Then it is easily seen that $f_s(G) = b$. Thus the proof is complete. ∎

5. Realization with Respect to Four Integers

In this section, we give realization results for the above said four parameters when $0 \leq a \leq b < c \leq d$ and $c - b - 2 > 0$.

Theorem 10. *For integers a, b, c and d with $0 \leq a \leq b < c \leq d$ and $c - b - 2 > 0$, there exists a connected graph G such that $f(G) = a$, $f_s(G) = b$, $g(G) = c$ and $s(G) = d$.*

Proof. We consider two cases.

Case 1. $c = d$.

Subcase 1a. $a = b$. Then the graph G constructed in Theorem 3.2 satisfies the requirements of this theorem.

Subcase 1b. $0 \leq a < b$. Then the graph G constructed in Theorem 4.2 satisfies the requirements of this theorem.

Case 2. $c < d$.

Subcase 2a. $a = b = 0$. Let G be the graph obtained from T_k in Figure 2.4 with $k = d - c + 2$ by adding the new vertices $z_1, z_2, \ldots, z_{c-2}$ and joining each z_i $(1 \leq i \leq c - 2)$ with b_1. We show that $g(G) = c$. Let $Z = \{z_1, z_2, \ldots, z_{c-2}\}$. It is clear that Z is not a geodetic set of G. Also it is easily verified that $Z \cup \{v\}$, where $v \in V(G) - Z$ is not a geodetic set of G. It is clear that $S = Z \cup \{b_0, b_2\}$ is a geodetic set of G and so by Theorem 1.1, $g(G) = c$. Now, we show that S is the only g-set of G. Suppose that there exists a g-set $S_1 \neq S$. By Theorem 1.2(i), $b_1 \notin S_1$. Hence there exists at least one vertex of type h_i $(1 \leq i \leq d-c+2)$ such that $h_i \in S_1$ and either $b_0 \notin S_1$ or $b_2 \notin S_1$. Then it is clear that S_1 is not a geodetic set of G, which is a contradiction. Hence S is the unique g-set of G and so it follows from Theorem 1.3(a) that $f(G) = 0$. Next, we show that $s(G) = d$. By Theorem 1.4(i), every Steiner set contains Z and it is easily seen that every Steiner set also contains each h_i $(1 \leq i \leq d-c+2)$ and so $s(G) \geq c-2+d-c+2 = d$. Since $W = Z \cup \{h_1, h_2, \ldots, h_{d-c+2}\}$ is a Steiner set of G, we have $s(G) = d$. Since every Steiner set contains W, W is the unique s-set of G and so it follows from Theorem 1.5(a) that $f_s(G) = 0 = a$.

Subcase 2b. $a = 0$ and $b \geq 1$. Let G be the graph obtained from L_b and T_{d-c} by identifying the vertex q_b of L_b and the vertex b_0 of T_{d-c} and then adding the new vertices $x, y, z_1, z_2, \ldots, z_{c-b-2}$ and joining the edges $xf_1, yb_2, z_1b_1, z_2b_1, \ldots, z_{c-b-2}b_1$. Let $Z = \{x, y, z_1, z_2, \ldots, z_{c-b-2}\}$. It is clear that $S = Z \cup \{p_1, p_2, \ldots, p_b\}$ is the unique g-set of G. Then as in Case 1 of Theorem 4.2 and Subcase 2a of this theorem, $g(G) = c$ and $f(G) = 0$. Also it is clear that any s-set is of the form $W = Z \cup \{c_1, c_2, \ldots, c_b\} \cup \{h_1, h_2, \ldots, h_{d-c}\}$, where $c_i \in \{m_i, n_i\}$ $(1 \leq i \leq b)$. Hence $s(G) = d$ and as in earlier theorems, it can be seen that $f_s(G) = b$.

Subcase 2c. $0 < a = b$. Let G be the graph obtained from G_a and T_{d-c} by identifying the vertex r_a of G_a and the vertex b_0 of T_{d-c} and then adding the new vertices $x, y, z_1, z_2, \ldots, z_{c-a-2}$ and joining the edges $xt_1, yb_2, z_1b_1, z_2b_1, \ldots, z_{c-a-2}b_1$. Let $Z = \{x, y, z_1, z_2, \ldots, z_{c-a-2}\}$ be the set of simplicial vertices of G. It is clear that any g-set is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in \{u_i, v_i\}$ $(1 \leq i \leq a)$. Then as in earlier theorems, it can be verified that $g(G) = c$ and $f(G) = a$. Also it is clear that any s-set is of the form $W = Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{h_1, h_2, \ldots, h_{d-c}\}$, where $c_i \in \{u_i, v_i\}$ $(1 \leq i \leq a)$. Hence $s(G) = d$ and as in earlier theorems it can be verified that $f_s(G) = a$.

Subcase 2d. $0 < a < b$. Let G_1 be the graph obtained from G_a and L_{b-a} by identifying the vertex r_a of G_a and the vertex f_1 of L_{b-a}. Now, let G be the graph obtained from G_1 and T_{d-c} by identifying the vertex q_{b-a} of G_1 and the vertex b_0 of T_{d-c} and then adding the new vertices $x, y, z_1, z_2, \ldots, z_{c-b-2}$ and joining the edges $xt_1, yb_2, b_1z_1, b_1z_2, \ldots, b_1z_{c-b-2}$. Let $Z = \{x, y, z_1, z_2, \ldots, z_{c-b-2}\}$ be the set of simplicial vertices of G. It is clear that any g-set is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{p_1, p_2, \ldots, p_{b-a}\}$, where $c_i \in \{u_i, v_i\}$ $(1 \leq i \leq a)$. Then as in earlier theorems it can be seen that $g(G) = c$ and $f(G) = a$. Also it is clear that any s-set is of the form $W = Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{d_1, d_2, \ldots, d_{b-a}\} \cup \{h_1, h_2, \ldots, h_{d-c}\}$, where $c_i \in \{u_i, v_i\}$ $(1 \le i \le a)$ and $d_i \in \{m_i, n_i\}$ $(1 \le i \le b-a)$. Hence $s(G) = d$ and as in earlier theorems, it can be seen that $f_s(G) = b$. ∎

Theorem 11. *For integers a, b, c and d with $0 \le a \le b < c \le d$ and $c-b-2 > 0$, there exists a connected graph G such that $f_s(G) = a$, $f(G) = b$, $g(G) = c$ and $s(G) = d$.*

Proof. We consider two cases.

Case 1. $c = d$.

Subcase 1a. $a = b$. Then the graph G constructed in Theorem 3.2 satisfies the requirements of this theorem.

Subcase 1b. $0 \le a < b$. Then the graph G constructed in Theorem 4.1 satisfies the requirements of this theorem.

Case 2. $c < d$.

Subcase 2a. $a = b = 0$. Then the graph G constructed in Subcase 2a of Theorem 5.1 satisfies the requirements of this theorem.

Subcase 2b. $a = 0, b \ge 1$. Let G be the graph obtained from H_b and T_{d-c} by identifying the vertex n of H_b and the vertex b_0 of T_{d-c} and then adding the new vertices $x, y, m, z_1, z_2, \ldots, z_{c-b-2}$ and joining the edges $xl, yb_2, ml, mn, z_1b_1, z_2b_1, \ldots, z_{c-b-2}b_1$. Let $Z = \{x, y, z_1, z_2, \ldots, z_{c-b-2}\}$ be the set of simplicial vertices of G. Then as in Theorems 4.1 and 5.1, $W = Z \cup \{x_1, x_2, \ldots, x_b\} \cup \{h_1, h_2, \ldots, h_{d-c}\}$ is the unique s-set of G so that $s(G) = d$ and $f_s(G) = 0$. Also it is clear that any g-set is of the form $W = Z \cup \{c_1, c_2, \ldots, c_b\}$, where $c_i \in \{w_i, x_i, y_i\}$ $(1 \le i \le b)$. Hence $g(G) = c$ and $f(G) = b$.

Subcase 2c. $0 < a = b$. Then the graph G constructed in Subcase 2c of Theorem 5.1 satisfies the requirements of this theorem.

Subcase 2d. $0 < a < b$. Let G' be the graph obtained from G_a and H_{b-a} by identifying the vertex r_a of G_a and the vertex l of H_{b-a}. Now, let G be the graph obtained from G' and T_{d-c} by identifying the vertex n of G' and the vertex b_0 of T_{d-c} and then adding the new vertices $x, y, m, z_1, z_2, \ldots, z_{c-b-2}$ and joining the edges $xt_1, yb_2, ml, mn, z_1b_1, z_2b_1, \ldots, z_{c-b-2}b_1$. Let $Z = \{x, y, z_1, z_2, \ldots, z_{c-b-2}\}$ be the set of simplicial vertices of G. Then, as in Theorems 4.1 and 5.1, any s-set is of the form $W = Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{x_1, x_2, \ldots, x_{b-a}\} \cup \{h_1, h_2, \ldots, h_{d-c}\}$, where $c_i \in \{u_i, v_i\}$ $(1 \le i \le a)$. Hence

$s(G) = d$ and $f_s(G) = a$. Also it is clear that any g-set is of the form $S = Z \cup \{c_1, c_2, \ldots, c_a\} \cup \{d_1, d_2, \ldots, d_{b-a}\}$,where $c_i \in \{u_i, v_i\}$ $(1 \le i \le a)$ and $d_i \in \{w_i, x_i, y_i\}$ $(1 \le i \le b-a)$. Hence $g(G) = c$ and $f(G) = b$ (as in earlier theorems). ■

We leave the following problems open.

Problem 12. For integers a, b, c and d with $0 \le a \le b < c \le d$ and $c - b - 2 > 0$, there exists a connected graph G such that

(i) $f(G) = a$, $f_s(G) = b$, $s(G) = c$ and $g(G) = d$.

(ii) $f_s(G) = a$, $f(G) = b$, $s(G) = c$ and $g(G) = d$.

6. Conclusion

We also leave open the possible realization results for the case when $b = c$ in Sections 4 and 5.

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