SIMPLICIAL AND NONSIMPLICIAL COMPLETE SUBGRAPHS

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Abstract

Define a complete subgraph \(Q\) to be simplicial in a graph \(G\) when \(Q\) is contained in exactly one maximal complete subgraph (‘maxclique’) of \(G\); otherwise, \(Q\) is nonsimplicial. Several graph classes—including strong \(p\)-Helly graphs and strongly chordal graphs—are shown to have pairs of peculiarly related new characterizations: (i) for every \(k \geq 2\), a certain property holds for the complete subgraphs that are in \(k\) or more maxcliques of \(G\), and (ii) in every induced subgraph \(H\) of \(G\), that same property holds for the nonsimplicial complete subgraphs of \(H\).

One example: \(G\) is shown to be hereditary clique-Helly if and only if, for every \(k \geq 2\), every triangle whose edges are each in \(k\) or more maxcliques is itself in \(k\) or more maxcliques; equivalently, in every induced subgraph \(H\) of \(G\), if each edge of a triangle is nonsimplicial in \(H\), then the triangle itself is nonsimplicial in \(H\).

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A maxclique of a graph is an inclusion-maximal complete subgraph. For each complete subgraph \(Q\) of a graph \(G\), define \(\text{str}_G(Q)\) to be the number of maxcliques of \(G\) that contain \(Q\). Notice that if \(H\) is an induced subgraph of \(G\) and \(Q\) is a complete subgraph of \(H\), then \(\text{str}_H(Q) \leq \text{str}_G(Q)\). As in [5], define \(Q\) to be strength-\(k\) in \(G\) if \(\text{str}_G(Q) \geq k\).
Define $Q$ to be a simplicial clique of $G$ if $\text{str}_G(Q) = 1$ and to be a nonsimplicial clique of $G$ if $Q$ is strength-2 in $G$. A $k$-clique is a complete subgraph of order $k$. When convenient, a complete subgraph $Q$ will be identified with its vertex set $V(Q)$.

The distinguishing feature of each ‘Theorem $n$’ or ‘Corollary $n$’ below can be loosely described as the equivalence of two statements involving a parameterized graph property $\mathcal{P}(k)$ (defined in terms of the strengths of complete subgraphs):

$n.1)$ $G$ satisfies $\mathcal{P}(k)$ for all $k \geq 2$.

$n.2)$ Every induced subgraph of $G$ satisfies $\mathcal{P}(2)$.

Typically, there will also be equivalent statements ($n.0$), asserting $G$ to be in a known graph class, and ($n.3$), expressed in terms of (non)simplicial cliques.

1. Clique Strength and Strong $p$-Helly Graphs

A graph is strong $p$-Helly if every family $Q$ of maxcliques contains a subfamily $Q'$ with $|Q'| \leq p$ such that $\cap Q = \cap Q'$. Reference [2] proves that these are also precisely the graphs that are hereditary $p$-clique-Helly (meaning that, for every family $Q$ of maxcliques, if every $p$ members of $Q$ have a vertex in common, then all the members of $Q$ have a vertex in common). Theorem 1 will contain additional characterizations.

**Theorem 1.** The following are equivalent for every graph $G$ and $p \geq 2$:

(1.0) $G$ is strong $p$-Helly.

(1.1) For every $k \geq 2$ and every $p$-clique $Q$ of $G$, if each $(p - 1)$-clique that is contained in $Q$ is strength-$k$ in $G$, then $Q$ is also strength-$k$ in $G$.

(1.2) For every $p$-clique $Q$ of an induced subgraph $H$ of $G$, if each $(p - 1)$-clique that is contained in $Q$ is strength-2 in $H$, then $Q$ is also strength-2 in $H$.

(1.3) If a $p$-clique $Q$ is simplicial in an induced subgraph $H$ of $G$, then at least one $(p - 1)$-clique that is contained in $Q$ is simplicial in $H$.

**Proof.** (1.1) $\Rightarrow$ (1.2): Suppose $p \geq 2$ and $G$ satisfies condition (1.1). Suppose $H$ is any proper induced subgraph of $G$ and $Q$ is a $p$-clique of $H$ such that, if $Q^-$ is a $(p - 1)$-clique with $Q^- \subset Q$, then $Q^-$ is strength-2 in $H$. 

But assume $Q$ itself is not strength-2 in $H$ [arguing by contradiction]; so $\text{str}_H(Q) = 1$. (Since each $Q^+$ is also strength-2 in $G$, the $k = 2$ case of (1.1) implies $Q$ is strength-2 in $G$.)

Let $g = \text{str}_G(Q)$. Then $Q$ will be in $g - 1$ more maxcliques in $G$ than in $H$. Therefore, each of the $(p-1)$-cliques contained in $Q$ will be strength-$(2 + [g-1])$ in $G$, and so strength-$(g + 1)$ in $G$. But then (1.1) implies that $Q$ is strength-$(g + 1)$ in $G$ [contradicting that $\text{str}_G(Q) = g$].

(1.1) $\iff$ (1.2): Suppose $p \geq 2$ and $G$ satisfies condition (1.2). Suppose $Q$ is a $p$-clique and $Q_1, \ldots, Q_p$ are the $(p-1)$-cliques contained in $Q$. Suppose $k \geq 2$ and each $Q_i$ is strength-$k$ in $G$, but $Q$ itself is not strength-$k$ in $G$ [arguing by contradiction].

Suppose $Q$ is contained in the pairwise-distinct maxcliques $Q^1, \ldots, Q^g$ of $G$ where $\text{str}_G(Q) = g < k$, and suppose each $Q_i$ is contained in the pairwise-distinct maxcliques $Q^1, \ldots, Q^g, Q^1_i, \ldots, Q^{k-g}_i$ of $G$ where each $Q^i_i \cap Q = Q_i$. Let $H$ be the subgraph of $G$ induced by

$$Q \cup \bigcup_{i=1}^{p} \bigcup_{j=1}^{k-g} Q^i_j - \bigcup_{j=1}^{g} (Q^j - Q).$$

Then each $Q_i$ is strength-2 in $H$, but $\text{str}_H(Q) = 1$ [contradicting (1.2)].

(1.2) $\iff$ (1.3): Condition (1.3) simply restates (1.2) using that $Q$ is strength-2 in $H$ if and only if $Q$ is nonsimplicial in $H$.

(1.0) $\iff$ (1.3): This follows from [2, Theorem 4].

Notice that the proof of (1.1) $\iff$ (1.2) in Theorem 1 did not use the characterization of strong-$p$ Helly graphs from [2]. This enables the $p = 2$ and $p = 3$ cases of Theorem 1 to be presented separately as Corollaries 2 and 3.

Let $C_k$ and $P_k$ denote, respectively, a cycle or path on $k$ vertices. For any graphs $G, H_1, \ldots, H_s$, say that $G$ is $\{H_1, \ldots, H_s\}$-free (or simply $H_1$-free if $s = 1$) if $G$ contains no induced subgraph isomorphic to any of the graphs $H_1, \ldots, H_s$. A graph is trivially perfect if it is $\{C_4, P_4\}$-free; see [1, 7] for additional characterizations (and additional names).

**Corollary 2.** The following are equivalent for every graph $G$:

(2.0) $G$ is trivially perfect.

(2.1) For every $k \geq 2$ and every edge $xy$ of $G$, if both $x$ and $y$ are strength-$k$ in $G$, then edge $xy$ is strength-$k$ in $G$. 


(2.2) For every edge $xy$ of an induced subgraph $H$ of $G$, if both $x$ and $y$ are strength-2 in $H$, then edge $xy$ is strength-2 in $H$.

(2.3) If an edge $e$ is simplicial in an induced subgraph $H$ of $G$, then at least one endpoint of $e$ is simplicial in $H$.

**Proof.** The $k = 2$ case of condition (2.1) implies that $G$ is $\{C_4, P_4\}$-free—and so implies (2.0)—by letting $xy$ be an edge of an induced $C_4$ or $P_4$ subgraph. Conversely, if (2.1) fails, suppose $xy \in E(G)$ where $x$ is in a maxclique that does not contain $y$ and $y$ is in a maxclique that does not contain $x$. Then those maxcliques contain edges $xx'$ and $yy'$ where $\{x', x, y, y'\}$ induces either a $P_4$ or a $C_4$ subgraph, making (2.0) fail.

The equivalence of (2.1) and (2.2) is the $p = 2$ case of Theorem 1. Condition (2.3) simply restates (2.2).

A graph is clique-Helly if, for every family $\mathcal{F}$ of maxcliques, if every two members of $\mathcal{F}$ have a vertex in common, then all the members of $\mathcal{F}$ have a vertex in common. A graph is hereditary clique-Helly if every induced subgraph is clique-Helly. See [1, 5, 8, 9] for details. Reference [9] also proves that $G$ is hereditary clique-Helly if and only if, for every maxclique $Q$ of an induced subgraph $H$ of $G$, at least one edge of $Q$ is simplicial in $H$. The hereditary clique-Helly graphs are, of course, precisely the hereditary 2-clique-Helly graphs (and so are precisely the strong 2-Helly graphs).

**Corollary 3.** The following are equivalent for every graph $G$:

(3.0) $G$ is hereditary clique-Helly.

(3.1) For every $k \geq 2$ and every triangle $xyz$ of $G$, if each edge $xy$, $xz$, and $yz$ is strength-$k$ in $G$, then triangle $xyz$ is strength-$k$ in $G$.

(3.2) For every triangle $xyz$ of an induced subgraph $H$ of $G$, if each edge $xy$, $xz$, and $yz$ is strength-2 in $H$, then triangle $xyz$ is strength-2 in $H$.

(3.3) If a triangle $\Delta$ is simplicial in an induced subgraph $H$ of $G$, then at least one edge of $\Delta$ is simplicial in $H$.

**Proof.** The equivalence of (3.0) and (3.1) restates [5, Theorem 2]. The equivalence of (3.1) and (3.2) is the $p = 3$ case of Theorem 1. Condition (3.3) simply restates (3.2).

Sections 2 and 3 go in a different direction, generalizing Corollary 3 by replacing triangles with arbitrary cycles.
2. Edge Strength and Chordal Graphs

A chord of a cycle is an edge that joins two nonconsecutive vertices of the cycle (only cycles of length four or more can have chords). A graph is chordal if and only if every cycle of length four or more has a chord; see [1, 7] for thorough discussions. Define a graph to be strength-\(k\) chordal if every cycle of strength-\(k\) edges either has a strength-\(k\) chord or is a strength-\(k\) triangle. Being strength-1 chordal is equivalent to being chordal, and Corollary 6 will characterize being strength-\(k\) chordal for all \(k \geq 1\).

The graph \(G_1\) in Figure 1 is the smallest chordal graph that is not strength-2 chordal—the three edges between the vertices 2, 3, and 5 are each strength-2, but the triangle they form is simplicial in \(G\). The graph \(G_2\) is strength-2 chordal—the nine edges incident to vertices 3 or 4 are each strength-2 (indeed, the edge 34 is strength-4), as are the four triangles that contain edge 34—yet \(G_2\) is not chordal because of the chordless cycle 1, 2, 6, 5, 1. (The graph \(G_2\) is vacuously strength-\(k\) chordal for all \(k > 2\).)

\[
\begin{align*}
G_1: & \quad 1 \quad 2 \quad 3 \\
& \quad 4 \quad 5 \quad 6 \\
G_2: & \quad 1 \quad 3 \quad 4 \\
& \quad 2 \quad 5 \quad 6
\end{align*}
\]

Figure 1. Graph \(G_1\) is chordal, but not strength-2 chordal; \(G_2\) is strength-\(k\) chordal for all \(k \geq 2\), but not chordal.

As is common when working with cycle spaces, a sum of cycles will mean the symmetric difference of the edge sets of those cycles—in other words, an edge \(e\) is in the sum (denoted) \(C_1 \oplus \cdots \oplus C_k\) if and only if \(e\) is in an odd number of the cycles \(C_1, \ldots, C_k\). The notation \(|C|\) will be used to denote the length of a cycle \(C\), and \(C\) is a \(k\)-cycle if \(|C| = k\). Lemma 4 will generalize the following simple fact from [4, Lemma 3.2] (also see [6, Corollary 1]): A graph is chordal if and only if every cycle \(C\) is the sum of \(|C| - 2\) triangles.

**Lemma 4.** A graph is strength-\(k\) chordal if and only if every cycle \(C\) of strength-\(k\) edges is the sum of \(|C| - 2\) strength-\(k\) triangles.

**Proof.** First suppose \(G\) is a chordal graph in which every cycle \(C\) of strength-\(k\) edges with \(|C| = l \geq 3\) is the sum of \(l - 2\) strength-\(k\) triangles...
\( \Delta_1, \ldots, \Delta_{l-2} \). If \( l = 3 \), then \( C \) itself is a strength-\( k \) triangle \( \Delta_1 \). Suppose \( l \geq 4 \) [toward showing that \( C \) has a strength-\( k \) chord]. Because \( G \) is chordal, each edge of \( C \) must be in some triangle \( \Delta_i \). The pigeon-hole principle implies that some \( \Delta_i \) must contain two (necessarily consecutive) edges of \( C \). Then the third side of \( \Delta_i \) is a chord of \( C \). Since \( \Delta_i \) is strength-\( k \), that third side is a strength-\( k \) chord of \( C \).

Conversely, suppose \( G \) is a strength-\( k \) chordal graph and \( C \) is a cycle of strength-\( k \) edges. Argue by induction on \(|C| = l \geq 3\). If \( l = 3 \), then \( C \) is a strength-\( k \) triangle and so \( C \) is trivially the sum of \( l - 2 = 1 \) strength-\( k \) triangles. Now suppose \( l \geq 4 \). Since \( G \) is strength-\( k \) chordal, cycle \( C \) has a strength-\( k \) chord \( e \). Then \( C = C_a \oplus C_b \) where \( C_a \) and \( C_b \) are cycles of strength-\( k \) edges from \( E(C_a) \cup E(C_b) \cup \{e\} \), with \( \{e\} = C_a \cap C_b \), \(|C_a| = a\), \(|C_b| = b\), and \( a + b = l + 2 \). The induction hypothesis implies that \( C_a \) [respectively, \( C_b \)] is the sum of \( a - 2 \) [or \( b - 2 \)] strength-\( k \) triangles. This makes \( C \) the sum of \((a - 2) + (b - 2) = l - 2 \) strength-\( k \) triangles.

**Theorem 5.** The following are equivalent for every graph \( G \):

(5.1) \( G \) is strength-\( k \) chordal for all \( k \geq 2 \).

(5.2) Every induced subgraph of \( G \) is strength-2 chordal.

(5.3) Every cycle of nonsimplicial edges in an induced subgraph \( H \) of \( G \) either has a chord that is nonsimplicial in \( H \) or is a nonsimplicial triangle of \( H \).

(5.4) Every cycle \( C \) of nonsimplicial edges in an induced subgraph \( H \) of \( G \) is the sum of \(|C| - 2 \) nonsimplicial triangles of \( H \).

**Proof.** (5.1) \( \Rightarrow \) (5.2): Suppose \( G \) satisfies condition (5.1). Suppose \( H \) is any induced subgraph of \( G \) and \( C \) is a cycle of edges that are strength-2 in \( H \), but \( C \) is not the sum of \(|C| - 2 \) triangles that are strength-2 in \( H \) [arguing by contradiction, using Lemma 4]; further suppose \(|C| \) is minimum with respect to all that. By the minimality of \(|C| \), every chord of \( C \) is simplicial in \( H \). This implies that every triangle \( \Delta \) with \( V(\Delta) \subseteq V(C) \) is simplicial in \( H \). Thus, for every edge \( e \) and triangle \( \Delta \), if \( e \in E(C) \cap E(\Delta) \) and \( V(\Delta) \subseteq V(C) \), then \( \text{str}_H(e) > \text{str}_H(\Delta) \). But since every maxclique of \( G \) that contains such a \( \Delta \) also contains such edges \( e \), the same inequality holds with \( H \) replaced by \( G \) [contradicting (5.1), using Lemma 4 with \( k = \min \{\text{str}_G(e) : e \in E(C)\} \).

(5.1) \( \Leftarrow \) (5.2): Suppose \( G \) satisfies condition (5.2), toward proving \( G \) is strength-\( k \) chordal by induction on \( k \geq 2 \). The \( k = 2 \) basis step is immediate.
For the inductive step, suppose $G$ is strength-$k$ chordal and $C$ is a cycle of edges that are strength-$(k+1)$ in $G$, but $C$ is not the sum of $|C| - 2$ triangles that are strength-$(k+1)$ in $G$ [arguing by contradiction, using Lemma 4]; further suppose $|C| = l \geq 3$ is minimal with respect to all that. By the minimality of $l$, cycle $C$ has no chords that are strength-$(k+1)$ in $G$. Since $G$ is strength-$k$ chordal, $C$ is the sum of triangles $\Delta_1, \ldots, \Delta_{l-2}$ of $G$ that are strength-$k$ in $G$, where each $\Delta_i$ is made from edges of $C$ that are strength-$(k+1)$ in $G$ together with chords $e$ of $C$ with $\text{str}_G(e) = k$. Therefore if $\Delta_i$ and $\Delta_j$ share a chord of $C$, then $V(\Delta_i) \cup V(\Delta_j)$ must induce a complete subgraph $Q_1$ that is strength-$k$ in $G$. Performing similar consolidations of complete subgraphs $n \leq l - 3$ times partitions $\{\Delta_1, \ldots, \Delta_{l-2}\}$ into $l - 2 - n$ parts that are sets of contiguous triangles that are strength-$k$ in $G$ and whose vertices induce $l - 2 - n$ complete subgraphs that are strength-$k$ in $G$ and that cover $V(C)$. Performing this consolidation $n = l - 3$ times shows that $V(C)$ induces a complete subgraph $Q_n$ that is strength-$k$ in $G$. Since $C$ has no chords that are strength-$(k+1)$ in $G$, it follows that $\text{str}_G(Q_n) = k$. Yet each $e \in E(C)$ is strength-$(k+1)$ in $G$ and so is in a maxclique $Q_e$ of $G$ that has $E(Q_e) \cap E(Q_n) = \{e\}$ (again using that $C$ has no chords that are strength-$(k+1)$ in $G$). But then $V(Q_n)$ together with one vertex from $V(Q_e) - V(Q_n)$ for each $e \in E(C)$ would induce a subgraph $H$ of $G$ such that each edge of $C$ is strength-2 in $H$ while $\text{str}_H(Q_n) = 1$ and each chord $e$ of $C$ has $\text{str}_H(e) = 1$ [contradicting (5.2)].

(5.2) $\Leftrightarrow$ (5.3) $\Leftrightarrow$ (5.4) follows since (5.3) and (5.4) simply restate (5.2) (using Lemma 4).

For $k \geq 3$, a $k$-sun—sometimes called a complete $k$-sun or trampoline, see [1, 3, 5, 7]—is a graph that consists of an even-length cycle $v_1, \ldots, v_{2k}, v_1$, together with all of the $\binom{2k}{2}$ chords between even-subscripted vertices. (The graph $G_1$ in Figure 1 is a 3-sun, and the subgraph $H$ constructed in the (5.1) $\Leftarrow$ (5.2) proof of Theorem 5 is an $l$-sun.) A graph is strongly chordal if it is chordal and no induced subgraph is isomorphic to any $k$-sun; see [1, 3, 5, 7] for other characterizations of this widely-studied concept.

**Corollary 6.** The following are equivalent for every graph $G$:

(6.0) $G$ is strongly chordal.

(6.1) $G$ is strength-$k$ chordal for all $k \geq 1$.

(6.2) $G$ is chordal and every induced subgraph of $G$ is strength-2 chordal.
(6.3) \( G \) is chordal and every cycle of nonsimplicial edges in an induced subgraph \( H \) of \( G \) either has a chord that is nonsimplicial in \( H \) or is a nonsimplicial triangle of \( H \).

(6.4) \( G \) is chordal and every cycle \( C \) of nonsimplicial edges in an induced subgraph \( H \) of \( G \) is the sum of \(|C| - 2\) nonsimplicial triangles of \( H \).

**Proof.** The equivalence of (6.0) and (6.1) restates [5, Theorem 1]. The equivalence of conditions (6.i) and (6.j) when \( 1 \leq i < j \leq 4 \) follows immediately from the equivalence of conditions (5.i) and (5.j).

3. **Vertex Strength and Chordal Graphs**

Recognizing that cycles are determined by their vertices just as well as by their edges, define a graph to be **vertex strength-\( k \) chordal** if every cycle of strength-\( k \) vertices either has a strength-\( k \) chord or is a strength-\( k \) triangle. (Strength-\( k \) chordal graphs could have been called ‘edge strength-\( k \) chordal’ graphs.) Being vertex strength-1 chordal is equivalent to being chordal. Clearly, every cycle of strength-\( k \) edges is a cycle of strength-\( k \) vertices, and so every vertex strength-\( k \) chordal graph is strength-\( k \) chordal. The three graphs in Figure 2 are strength-2 chordal but not vertex strength-2 chordal (because each vertex shown as ‘hollow’ is a strength-2 vertex).

![Figure 2](image)

**Figure 2.** From left to right, the kite, gem, and net graphs.

**Lemma 7.** A graph is vertex strength-\( k \) chordal if and only if every cycle \( C \) of strength-\( k \) vertices is the sum of \(|C| - 2\) strength-\( k \) triangles.

**Proof.** This is proved by a straightforward modification of the proof of Lemma 4 (observing that every strength-\( k \) edge has strength-\( k \) endpoints).

**Theorem 8.** The following are equivalent for every graph \( G \):

(8.0) \( G \) is \{kite, gem, net\}-free strongly chordal.
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(8.1) $G$ is vertex strength-$k$ chordal for all $k \geq 2$.

(8.2) Every induced subgraph of $G$ is vertex strength-2 chordal.

(8.3) Every cycle of nonsimplicial vertices in an induced subgraph $H$ of $G$ either has a chord that is nonsimplicial in $H$ or is a nonsimplicial triangle of $H$.

(8.4) Every cycle $C$ of nonsimplicial vertices in an induced subgraph $H$ of $G$ is the sum of $|C| - 2$ nonsimplicial triangles of $H$.

Proof. (8.0) $\Rightarrow$ (8.1): Suppose $k \geq 2$ and $G$ satisfies condition (8.0), and $C$ is a cycle of strength-$k$ vertices. Since $G$ is chordal, $C$ is the sum of $|C| - 2$ triangles. Suppose any of those triangles—say triangle $v_1v_2v_3$—has $\text{str}_G(v_1v_2v_3) < k$ [arguing by contradiction, using Lemma 7, showing that $G$ would contain an induced kite, gem, net, or 3-sun]. Then for each $i \in \{1, 2, 3\}$, there exists a vertex $w_i$ with each $w_i \sim v_i$ and $w_i \not\in \{v_1, v_2, v_3\}$ and $w_i$ not adjacent to some $v_j$. Thus $|\{w_1, w_2, w_3\}| > 1$. Let $H$ be the subgraph of $G$ that is induced by $\{v_1, v_2, v_3, w_1, w_2, w_3\}$. If $|\{w_1, w_2, w_3\}| = 2$, then $H$ is an induced kite or gem [a contradiction]. If $|\{w_1, w_2, w_3\}| = 3$, then either $H$ is an induced net or 3-sun or $H$ contains an induced kite or gem [a contradiction].

(8.1) $\Rightarrow$ (8.2): Suppose $G$ satisfies condition (8.1). Suppose $H$ is any induced subgraph of $G$ and $C$ is a cycle of vertices that are strength-2 in $H$, but $C$ is not the sum of $|C| - 2$ triangles that are strength-2 in $H$ [arguing by contradiction, using Lemma 7]; further suppose $|C|$ is minimum with respect to all that. By the minimality of $|C|$, the cycle $C$ is chordless and so (since vertex strength-$k$ chordal implies chordal) $C$ is a triangle $\Delta$ where $\text{str}_H(\Delta) = 1$. Thus $\text{str}_H(v) > \text{str}_H(\Delta)$ for every $v \in V(\Delta)$. But since every maxclique of $G$ that contains $\Delta$ also contains every $v \in V(\Delta)$, it follows that $\text{str}_G(v) > \text{str}_G(\Delta)$ [contradicting (8.1) with $k = \min\{\text{str}_G(v) : v \in V(\Delta)\}$].

(8.0) $\Leftarrow$ (8.2): Suppose $G$ satisfies condition (8.2). Then no induced subgraph $H$ of $G$ can be isomorphic to $C_k$ with $k \geq 4$ (a chordless cycle of vertices that are strength-2 in $G$), or to a $k$-sun, kite, gem, or net graph (each containing a triangle $\Delta$ of vertices that are strength-2 in $G$ while $\text{str}_G(\Delta) = 1$). Thus (8.0) holds.

(8.2) $\iff$ (8.3) $\iff$ (8.4) follows since (8.3) and (8.4) simply restate (8.2).

If a graph $G$ is vertex strength-$k$ chordal for all $k \geq 2$, then $G$ is strongly chordal and so is certainly vertex strength-1 chordal. Therefore, the $k \geq 2$
restriction in condition (8.1) could just as well be replaced with \( k \geq 1 \), and no ‘Corollary 9’ is needed to parallel Corollary 6.

References


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