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NOTE

FORBIDDEN-MINOR CHARACTERIZATION FOR THE CLASS OF COGRAPHIC ELEMENT SPLITTING MATROIDS

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Abstract

In this paper, we prove that an element splitting operation by every pair of elements on a cographic matroid yields a cographic matroid if and only if it has no minor isomorphic to $M(K_4)$.

Keywords: binary matroid, graphic matroid, cographic matroid, minor.

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1. INTRODUCTION

The element splitting operation for binary matroid is defined in [3] as follows: Let A be a matrix over $GF(2)$ that represents the matroid M. Suppose that x and y are distinct elements of M. Let $A'_{x,y}$ be the matrix that is obtained

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by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to x and y where it takes the value 1 and then adjoining an extra column (corresponding to a) with this column being zero everywhere except in the last row where it takes the value 1. Suppose $M'_{x,y}$ is the matroid represented by the matrix $A'_{x,y}$. Then $M'_{x,y}$ is said to be obtained from M by *element splitting* the pair of elements x and y. The transition from M to $M'_{x,y}$ is called an element splitting operation. The matroid $M'_{x,y}$ is called the *element splitting matroid*.

If M is the cycle matroid of a graph G of Figure 1, $M'_{x,y}$ is the cycle matroid of the graph $G'_{x,y}$ of Figure 1.

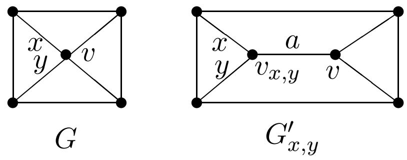

Figure 1

Alternatively, the element splitting operation can be defined in terms of circuits of binary matroids as follows: Let $M = (S, \mathcal{C})$ be a binary matroid, $\{x, y\} \subseteq S$, and $a \notin S$. Let $\mathcal{C}_0 = \{C \in \mathcal{C} : x, y \in C$ or $x, y \notin C\}$; $\mathcal{C}_1$ = set of minimal members of $\{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \phi$ and $x \in C_1, y \in C_2$ such that $C_1 \cup C_2$ contains no member of $\mathcal{C}_0\}$; and $\mathcal{C}_2 = \{C \cup \{a\} : C \in \mathcal{C}$ and C contains exactly one of x and $y\}$. Let $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$. Then $M'_{x,y} = (S \cup \{a\}, \mathcal{C}')$ is the element splitting matroid.

The element splitting operation arises in the following way also [1]: Consider the unique binary extension of M by the element a so that $\{x, y, a\}$ is a triangle. Perform a Delta-Y exchange on the triangle $\{x, y, a\}$. The resulting matroid is produced by performing an element splitting on the pair x, y.

The splitting operation for binary matroid is defined as follows [6]: Let A be a matrix over $GF(2)$ that represents the matroid M. Consider distinct elements x and y of M. Let $A_{x,y}$ be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to x and y where it takes the value 1. Suppose $M_{x,y}$ is the matroid represented by the matrix $A_{x,y}$. Then $M_{x,y}$ is said to be obtained from M by *splitting* away the pair x, y. The relation between the splitting operation and the element splitting operation is that $M'_{x,y} \setminus \{a\} = M_{x,y}$.

Dalvi, Borse and Shikare [3] characterized graphic matroids whose element splitting matroids are also graphic as follows.

Theorem 1.1. *The element splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if it has no minor isomorphic to* $M(K_4)$*, where* K_4 *is the complete graph on* 4 *vertices.* □

The element splitting operation on a cographic matroid may not yield a cographic matroid. In this paper, we characterize those cographic matroids M for which the matroid $M'_{x,y}$ is cographic for every pair of elements $\{x, y\}$ of M. The main result in this paper is the following theorem.

Theorem 1.2. *The element splitting operation, by any pair of elements, on a cographic matroid yields a cographic matroid if and only if it has no minor isomorphic to* $M(K_4)$*, where* K_4 *is the complete graph on* 4 *vertices.*

2. Proof of the Main Theorem

In this section, firstly we provide necessary lemmas.

Lemma 2.1 [5]**.** *A binary matroid is cographic if and only if it has no minor isomorphic to* F_7, F_7^*, $M(K_5)$, *or* $M(K_{3,3})$. □

Lemma 2.2 [5]**.** *A binary matroid is graphic if and only if it has no minor isomorphic to* F_7, F_7^*, $M^*(K_5)$, *or* $M^*(K_{3,3})$. □

Lemma 2.3 [5]**.** *Every* 3*-connected binary matroid having at least four elements has a minor isomorphic to* $M(K_4)$. □

Lemma 2.4. *Every binary matroid having no* $M(K_4)$ *minor is graphic and cographic.*

Proof. Suppose that M be a binary matroid without $M(K_4)$ as a minor. If M is not graphic or cographic, then by Lemmas 2.1 and 2.2, M contains F_7, F_7^*, $M(K_5)$, $M(K_{3,3})$, $M^*(K_5)$ or $M^*(K_{3,3})$ as a minor. Since all the six matroids are binary and 3-connected, by Lemma 2.3, each of these have $M(K_4)$ as a minor and hence M has $M(K_4)$ as a minor, a contradiction. ■

Lemma 2.5. *Let* M *be a graphic matroid having no* $M(K_4)$ *minor and let* $x, y \in E(M)$ *be such that* $M'_{x,y}$ *is not cographic. Then there is a minor* N

of M such that no two elements of N are in series and $N'_{x,y}\setminus\{a\}/\{x\}\cong F$ or $N'_{x,y}\setminus\{a\}/\{x,y\}\cong F$ or $N'_{x,y}\cong F$ or $N'_{x,y}/\{x\}\cong F$ or $N'_{x,y}/\{y\}\cong F$ or $N'_{x,y}/\{x,y\}\cong F$ for some $F\in\{M(K_5),\ M(K_{3,3})\}$.

Proof. The proof is similar to the proof of Lemma 2.3 of [3]. ■

***Proof of Theorem* 1.2.** Let M be a cographic matroid. Suppose that M has a minor N isomorphic to $M(K_4)$. Then $N'_{x,y}\cong F_7^*$ for x,y corresponding to any pair of non-adjacent edges of K_4. So $N'_{x,y}$ and hence $M'_{x,y}$ is not cographic.

Suppose M has no minor isomorphic to $M(K_4)$. Then, by Lemma 2.4, M is graphic. We claim that $M'_{x,y}$ is cographic. Suppose that $M'_{x,y}$ is not cographic for some $x,y\in E(M)$. By, Theorem 1.1, $M'_{x,y}$ is graphic. Hence $M'_{x,y}$ does not contain F_7 and F_7^* as minors. By, Lemmas 2.1 and 2.5, it is enough to prove that M does not have a minor N such that no two elements of N are in series and $N'_{x,y}\setminus\{a\}/\{x\}\cong F$ or $N'_{x,y}\setminus\{a\}/\{x,y\}\cong F$ or $N'_{x,y}\cong F$ or $N'_{x,y}/\{x\}\cong F$ or $N'_{x,y}/\{y\}\cong F$ or $N'_{x,y}/\{x,y\}\cong F$ for some $F\in\{M(K_5),\ M(K_{3,3})\}$. Since M is graphic, N is graphic. Let G be a graph corresponding to N. Then G is planar and has minimum degree at least three. Considering circuits of $M'_{x,y}$, we note that every 1-cycle or 2-cycle of G must contain exactly one of x and y. This implies that G is loopless.

Case (i). Suppose that $F=M(K_{3,3})$.
Note that $N'_{x,y}\setminus\{a\}=N_{x,y}$. If $N'_{x,y}\setminus\{a\}/\{x\}\cong M(K_{3,3})$, then $N_{x,y}/\{x\}\cong M(K_{3,3})$. Hence, by Case (i) of Lemma 3.3 of [2], N is isomorphic to the cycle matroid of graphs (i) or (ii) of Figure 2. As K_4 is a minor of each of these graphs, we obtain a contradiction. If $N'_{x,y}\setminus\{a\}/\{x,y\}\cong M(K_{3,3})$, then $N_{x,y}/\{x,y\}\cong M(K_{3,3})$. So, by Case (ii) of Lemma 3.3 of [2], N is isomorphic to the cycle matroid of graph (iii) of Figure 2 and thus has K_4 as a minor, a contradiction.

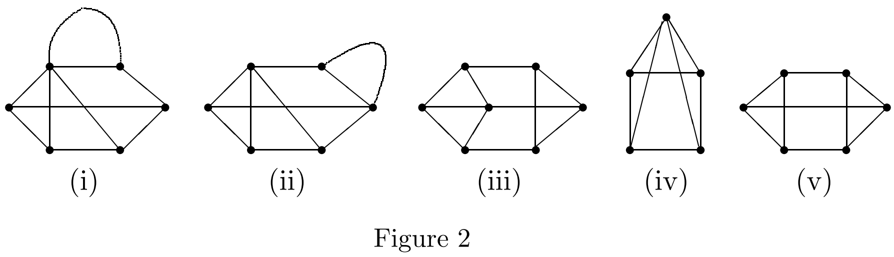

Figure 2

Suppose that $N'_{x,y} \cong M(K_{3,3})$. Then G has 5 vertices, 8 edges and the minimum vertex degree at least three. If G has a 2-cycle then we get a 3-circuit in $M'_{x,y}$ containing a, a contradiction. This implies that G is simple. Therefore, by Appendix 1 of [4], G is isomorphic to the graph (iv) of Figure 2 and has K_4 as a minor, a contradiction. Suppose that $N'_{x,y}/\{x\} \cong M(K_{3,3})$. Then G has 6 vertices, 9 edges. Further, G is simple. Since minimum degree in G is at least 3, G is isomorphic to the graph (v) of Figure 2 (see Appendix 1 of [4]) and hence has K_4 as a minor, a contradiction. Finally, suppose that $N'_{x,y}/\{x,y\} \cong M(K_{3,3})$. Then a graph corresponding to M has 7 vertices and 10 edges. This implies that G has at least one vertex of degree two, which is a contradiction.

Case (ii). Suppose that $F = M(K_5)$.
If $N'_{x,y} \setminus \{a\}/\{x\} \cong M(K_5)$ or $N'_{x,y} \setminus \{a\}/\{x,y\} \cong M(K_5)$, then $N_{x,y}/\{x\} \cong M(K_5)$ or $N_{x,y}/\{x,y\} \cong M(K_5)$. So, by Cases (i) and (ii), respectively of Lemma 3.4 of [2], N is isomorphic to the cycle matroid of one of the graphs of Figure 3. As all each of these graphs has K_4 as a minor, we obtain a contradiction.

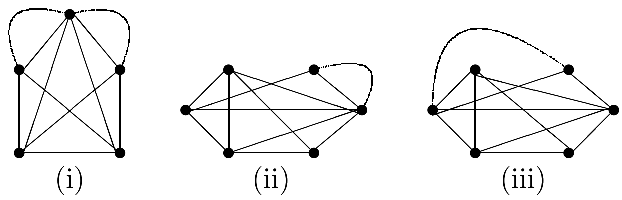

Figure 3

Since $N'_{x,y}$ is not Eulerian, $N'_{x,y} \ncong M(K_5)$. Suppose that $N'_{x,y}/\{x\} \cong M(K_5)$. Then G has 5 vertices, 10 edges. By Appendix 1 of [4], G must be non-simple. If x belongs to a 2-cycle of G, then $N'_{x,y}$ has a 3-circuit containing x and consequently $M(K_5)$ has a 2-circuit, a contradiction. This implies that G has exactly one 2-cycle. Hence G can be obtained from a simple planar graph with 5 vertices and 9 edges by adding an edge in parallel. By Appendix 1 of [4], there is only one graph with 5 vertices and 9 edges which has $M(K_4)$ as a minor, a contradiction. Finally, suppose that $N'_{x,y}/\{x,y\} \cong M(K_5)$. Then G is a planar graph with 6 vertices, 11 edges and has minimum degree at least 3. It follows that G is simple. There are 3 such non-isomorphic graphs (see Appendix 1 of [4]). As each of these graphs has $M(K_4)$ as a minor,

G cannot be isomorphic to any one of them. This completes the proof of the theorem. ■

References

[1] S. Akkari and J. Oxley, *Some local extremal connectivity results for matroids*, Combinatorics, Probability and Computing **2** (1993) 367–384.

[2] Y.M. Borse, K. Dalvi and M.M. Shikare, *Excluded-minor characterization for the class of cographic splitting matroids*, Ars Combin., to appear.

[3] K. Dalvi, Y.M. Borse and M.M. Shikare, *Forbidden-minor characterization for the class of graphic element splitting matroids*, Discuss. Math. Graph Theory **29** (2009) 629–644.

[4] F. Harary, Graph Theory (Addison-Wesley, Reading, 1969).

[5] J.G. Oxley, Matroid Theory (Oxford University Press, Oxford, 1992).

[6] T.T. Raghunathan, M.M. Shikare and B.N. Waphare, *Splitting in a binary matroid*, Discrete Math. **184** (1998) 267–271.

[7] M.M. Shikare and B.N. Waphare, *Excluded-minors for the class of graphic splitting matroids*, Ars Combin. **97** (2010) 111–127.

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