

**THE CONNECTED FORCING CONNECTED VERTEX
DETOUR NUMBER OF A GRAPH**

A.P. SANTHAKUMARAN

Research Department of Mathematics
St. Xavier's College (Autonomous)
Palayamkottai – 627 002, India
e-mail: apskumar1953@yahoo.co.in

AND

P. TITUS

Department of Mathematics
Anna University Tirunelveli
Tirunelveli – 627 007, India
e-mail: titusvino@yahoo.com

Abstract

For any vertex x in a connected graph G of order $p \geq 2$, a set S of vertices of V is an x -detour set of G if each vertex v in G lies on an x - y detour for some element y in S . A *connected x -detour set* of G is an x -detour set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x -detour set of G is the *connected x -detour number* of G and is denoted by $cd_x(G)$. For a minimum connected x -detour set S_x of G , a subset $T \subseteq S_x$ is called a *connected x -forcing subset* for S_x if the induced subgraph $G[T]$ is connected and S_x is the unique minimum connected x -detour set containing T . A connected x -forcing subset for S_x of minimum cardinality is a *minimum connected x -forcing subset* of S_x . The *connected forcing connected x -detour number* of S_x , denoted by $cf_{cdx}(S_x)$, is the cardinality of a minimum connected x -forcing subset for S_x . The *connected forcing connected x -detour number* of G is $cf_{cdx}(G) = \min cf_{cdx}(S_x)$, where the minimum is taken over all minimum connected x -detour sets S_x in G . Certain general properties satisfied by connected x -forcing sets are studied. The connected forcing connected vertex detour numbers

of some standard graphs are determined. It is shown that for positive integers a, b, c and d with $2 \leq a < b \leq c \leq d$, there exists a connected graph G such that the forcing connected x -detour number is a , connected forcing connected x -detour number is b , connected x -detour number is c and upper connected x -detour number is d , where x is a vertex of G .

Keywords: vertex detour number, connected vertex detour number, upper connected vertex detour number, forcing connected vertex detour number, connected forcing connected vertex detour number.

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1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices x and y in a connected graph G , the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G . An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G , while for $S \subseteq V$, $I[S] = \bigcup_{x, y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g -set*. The geodetic number of a graph was introduced in [1, 7] and further studied in [3].

The vertex geodomination number was introduced in [9] and further studied in [10]. For any vertex x in a connected graph G , a set S of vertices of G is an *x -geodominating set* of G if each vertex v of G lies on an $x - y$ geodesic in G for some element y in S . The minimum cardinality of an x -geodominating set of G is defined as the *x -geodomination number* of G and is denoted by $g_x(G)$. An x -geodominating set of cardinality $g_x(G)$ is called a *g_x -set*. The connected vertex geodomination number was introduced and studied in [12]. A *connected x -geodominating set* of G is an x -geodominating set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x -geodominating set of G is the *connected x -geodomination number* of G and is denoted by $cg_x(G)$. A connected x -geodominating set of cardinality $cg_x(G)$ is called a *cg_x -set* of G .

For vertices x and y in a connected graph G , the *detour distance* $D(x, y)$ is the length of a longest $x - y$ path in G . An $x - y$ path of length $D(x, y)$ is called an $x - y$ *detour*. The closed interval $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of G , while for $S \subseteq V$, $I_D[S] = \bigcup_{x, y \in S} I_D[x, y]$. A set S of vertices is a *detour set* if $I_D[S] = V$, and the minimum cardinality of a detour set is the *detour number* $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set*. The detour number of a graph was introduced in [4] and further studied in [5].

Graph theory, as such, has tremendous applications in theoretical Chemistry. Distance concepts in graphs, in particular, the graph invariants arising from distance concepts are widely applied in Chemical Graph Theory. In [8], the *distance matrix* and the *detour matrix* of a connected graph are used to discuss the applications of the graph parameters *Wiener index*, the *detour index*, the *hyper-Wiener index* and the *hyper-detour index* to a class of graphs viz. bridge and chain graphs, which in turn, capture different aspects of certain molecular graphs associated to the molecules arising in special situations of molecular problems in theoretical Chemistry. For more applications of these parameters, one may refer to [8] and the references therein. It is in this context, the vertex geodomination number and the vertex detour number of a graph introduced and studied in [9, 10, 11, 12, 13, 14] are used in determining certain aspects in Chemical graphs.

The vertex detour number was introduced in [11]. For any vertex x in a connected graph G , a set S of vertices of G is an x -*detour set* if each vertex v of G lies on an $x - y$ detour in G for some element y in S . The minimum cardinality of an x -*detour set* of G is defined as the x -detour number of G and is denoted by $d_x(G)$. An x -detour set of cardinality $d_x(G)$ is called a d_x -*set* of G . An elaborate study of results on vertex detour number with several interesting applications is given in [11].

The connected x -detour number was introduced and studied in [13] and further studied in [14]. A *connected x -detour set* of G is an x -*detour set* S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x -detour set of G is the *connected x -detour number* of G and is denoted by $cd_x(G)$. A connected x -detour set of cardinality $cd_x(G)$ is called a cd_x -*set* of G . A connected x -detour set S_x is called a *minimal connected x -detour set* if no proper subset of S_x is a connected x -detour set. The upper connected x -detour number, denoted by $cd_x^+(G)$, is defined as the maximum cardinality of a minimal connected x -detour set of G . For the graph G given in Figure 1.1, the minimum vertex detour sets,

the vertex detour numbers, the minimum connected vertex detour sets, the connected vertex detour numbers, the minimal connected vertex detour sets and the upper connected vertex detour numbers are given in Table 1.1.

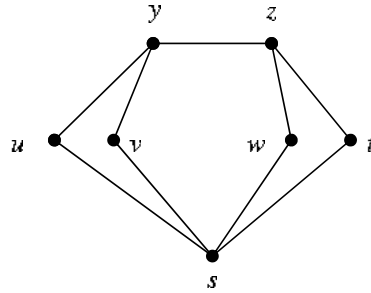


Figure 1.1

Vertex x	d_x -sets	$d_x(G)$	cd_x -sets	$cd_x(G)$	Minimal connected x -detour sets	$cd_x^+(G)$
s	$\{u,v\}, \{u,z\}, \{u,w\}, \{u,t\}, \{v,z\}, \{v,w\}, \{v,t\}, \{y,z\}, \{y,w\}, \{y,t\}, \{t,w\}$	2	$\{y, z\}$	2	$\{y,z\}, \{u,v,y\}, \{u,v,s\}, \{u,w, s\}, \{u,t,s\}, \{v,w, s\}, \{v,t,s\}, \{w,t,s\}, \{w,t,z\}$	3
y	$\{z\}, \{t\}, \{w\}$	1	$\{z\}, \{t\}, \{w\}$	1	$\{z\}, \{t\}, \{w\}, \{u,v,s\}, \{u,v,y\}$	3
z	$\{y\}, \{u\}, \{v\}$	1	$\{y\}, \{u\}, \{v\}$	1	$\{y\}, \{u\}, \{v\}, \{w,t,z\}, \{w,t,s\}$	3
u	$\{z\}, \{w\}, \{v\}, \{t\}$	1	$\{z\}, \{w\}, \{v\}, \{t\}$	1	$\{z\}, \{w\}, \{v\}, \{t\}$	1
v	$\{z\}, \{w\}, \{u\}, \{t\}$	1	$\{z\}, \{w\}, \{u\}, \{t\}$	1	$\{z\}, \{w\}, \{u\}, \{t\}$	1
w	$\{y\}, \{u\}, \{v\}, \{t\}$	1	$\{y\}, \{u\}, \{v\}, \{t\}$	1	$\{y\}, \{u\}, \{v\}, \{t\}$	1
t	$\{y\}, \{u\}, \{v\}, \{w\}$	1	$\{y\}, \{u\}, \{v\}, \{w\}$	1	$\{y\}, \{u\}, \{v\}, \{w\}$	1

Table 1.1

A vertex of degree one is called an *end vertex* of G . The following theorems will be used in the sequel.

Theorem 1.1 [13]. *If G is the complete graph K_n ($n \geq 3$), the n -cube Q_n ($n \geq 2$), the cycle C_n ($n \geq 3$), the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 4$) or the complete bipartite graph $K_{m,n}$ ($m, n \geq 2$), then $cd_x(G) = 1$ for every vertex x in G .*

Theorem 1.2 [13]. *Let x be any vertex of a connected graph G . If $y \neq x$ is an end vertex of G , then y belongs to every x -detour set of G .*

Theorem 1.3 [14]. *For any vertex x in a connected graph G , $1 \leq cd_x(G) \leq cd_x^+(G) \leq p$.*

Theorem 1.4 [14]. *Let G be a connected graph with cut vertices and let S_x be an x -detour set of G . If v is a cut vertex of G , then every component of $G - \{v\}$ contains an element of $S_x \cup \{x\}$.*

Throughout this paper, G denotes a connected graph with at least two vertices.

2. CONNECTED FORCING SUBSETS IN CONNECTED VERTEX DETOUR SETS OF A GRAPH

Let x be a vertex of a connected graph G . Although G contains a minimum connected x -detour set, there are connected graphs which may contain more than one minimum connected x -detour set. For example, the graph G given in Figure 1.1 contains more than one minimum connected t -detour set. For each minimum connected x -detour set S_x in a connected graph G , there is always some subset T of S_x that uniquely determines S_x as the minimum connected x -detour set containing T . Such sets are called "vertex forcing subsets" and we discuss these sets in minimum connected x -detour sets.

Definition 2.1. Let x be a vertex of a connected graph G and S_x a minimum connected x -detour set of G . A subset $T \subseteq S_x$ is called an *x -forcing subset* for S_x if S_x is the unique minimum connected x -detour set containing T . An x -forcing subset for S_x of minimum cardinality is a *minimum x -forcing subset* of S_x . The *forcing connected x -detour number* of S_x , denoted by $f_{cdx}(S_x)$, is the cardinality of a minimum x -forcing subset for S_x .

The *forcing connected x -detour number* of G is $f_{cdx}(G) = \min \{f_{cdx}(S_x)\}$, where the minimum is taken over all minimum connected x -detour sets S_x in G .

A subset $T' \subseteq S_x$ is called a *connected x -forcing subset* for S_x if the induced subgraph $G[T']$ is connected and S_x is the unique minimum connected x -detour set containing T' . A connected x -forcing subset for S_x of minimum cardinality is a *minimum connected x -forcing subset* of S_x . The *connected forcing connected x -detour number* of S_x , denoted by $cf_{cdx}(S_x)$, is the cardinality of a minimum connected x -forcing subset for S_x . The *connected forcing connected x -detour number* of G is $cf_{cdx}(G) = \min\{cf_{cdx}(S_x)\}$, where the minimum is taken over all minimum connected x -detour sets S_x in G .

Example 2.2. For the graph G given in Figure 2.1, the minimum connected x -detour sets, the connected x -detour numbers, the forcing connected x -detour numbers and the connected forcing connected x -detour numbers for every vertex x of G are given in Table 2.1.

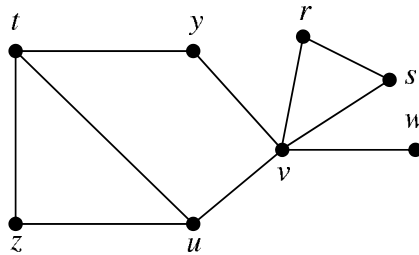


Figure 2.1

Proposition 2.3. For any vertex x in a connected graph G , $0 \leq f_{cdx}(G) \leq cf_{cdx}(G) \leq cd_x(G) \leq p$.

Proof. Let x be a vertex of G . It is clear that $f_{cdx}(G) \geq 0$. Let S_x be any minimum connected x -detour set of G . Since every connected x -forcing subset of S_x is x -forcing subset of S_x , it follows that $f_{cdx}(G) \leq cf_{cdx}(G)$. Also, since $cf_{cdx}(S_x) \leq cd_x(G)$ and $cf_{cdx}(G) = \min\{cf_{cdx}(S_x) : S_x \text{ is a minimum connected } x\text{-detour set in } G\}$, it follows that $cf_{cdx}(G) \leq cd_x(G)$. Thus $0 \leq f_{cdx}(G) \leq cf_{cdx}(G) \leq cd_x(G) \leq p$. ■

The proof of the following proposition is straight forward and so we omit it.

Vertex x	cd_x -sets	$cd_x(G)$	$f_{cdx}(G)$	$cf_{cdx}(G)$
t	$\{y,v,w,r\}, \{y,v,w,s\},$ $\{u,v,w,r\}, \{u,v,w,s\}$	4	2	3
y	$\{v,w,r\}, \{v,w,s\}$	3	1	1
z	$\{v,w,r\}, \{v,w,s\}$	3	1	1
u	$\{v,w,r\}, \{v,w,s\}$	3	1	1
v	$\{y,v,w,r\}, \{y,v,w,s\}$ $\{u,v,w,r\}, \{u,v,w,s\}$	4	2	3
w	$\{v,y,r\}, \{v,y,s\}, \{u,v,r\},$ $\{u,v,s\}$	3	2	3
r	$\{v,w,y\}, \{v,w,u\}$	3	1	1
s	$\{v,w,y\}, \{v,w,u\}$	3	1	1

Table 2.1

Proposition 2.4. *Let x be any vertex of a connected graph G . Then*

- (i) $f_{cdx}(G) = cf_{cdx}(G) = 0$ if and only if G has a unique minimum connected x -detour set.
- (ii) $f_{cdx}(G) = cf_{cdx}(G) = 1$ if and only if G has at least two minimum connected x -detour sets, one of which is a unique minimum connected x -detour set containing one of its elements.
- (iii) $f_{cdx}(G) = cf_{cdx}(G) = cd_x(G)$ if and only if no minimum connected x -detour set of G is the unique minimum connected x -detour set containing any of its proper subsets.

Proposition 2.5. *There is no graph G with $cf_{cdx}(G) = p$ for any vertex x in G .*

Proof. If there is a graph G and a vertex x in G with $cf_{cdx}(G) = p$, then it follows from Proposition 2.3 that $cd_x(G) = p$. This implies that $V(G)$ is the unique cd_x -set of G and so by Proposition 2.4(i), $cf_{cdx}(G) = 0$, which is a contradiction. ■

Corollary 2.6. *There is no graph G with $f_{cdx}(G) = p$ for any vertex x in G .*

Proof. This follows from Propositions 2.3 and 2.5. ■

Definition 2.7. Let x be any vertex of a connected graph G . A vertex v in G is a connected x -detour vertex if v belongs to every minimum connected x -detour set of G .

Example 2.8. For the graph G given in Figure 2.1, the vertices v and w are the connected t -detour vertices of G . It is clear that if $y \neq x$ is an end vertex of a graph G , then y is a connected x -detour vertex of G .

Proposition 2.9. *Let x be a vertex of a connected graph G and S_x a minimum connected x -detour set of G . Then*

- (i) *no connected x -detour vertex of G belongs to any minimum x -forcing subset of S_x .*
- (ii) *no end vertex of G belongs to any minimum connected x -forcing subset of S_x .*

Proof. (i) Let v be a connected x -detour vertex of G . Then v belongs to every minimum connected x -detour set of G . Let $T \subseteq S_x$ be arbitrary minimum x -forcing subset for arbitrary minimum connected x -detour set S_x of G . We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S_x is the unique minimum connected x -detour set containing T' so that T' is an x -forcing subset for S_x with $|T'| < |T|$, which is a contradiction to T a minimum x -forcing subset for S_x .

(ii) Let u be an end vertex of G . If $u = x$, then u does not belong to any minimum connected x -detour set of G and so u does not belong to any minimum connected x -forcing subset of S_x . If $u \neq x$, then u is a connected x -detour vertex of G . Let T be a minimum connected x -forcing subset for any minimum connected x -detour set S_x . By an argument similar to result (i) of this theorem, $T' = T - \{u\}$ is an x -forcing subset for S_x . Also, since u is an end vertex, it is clear that $G[T']$ is connected and so T' is a connected x -forcing subset for S_x with $|T'| < |T|$, which is a contradiction to T a minimum connected x -forcing subset for S_x . ■

Remark 2.10. A connected x -detour vertex may belong to a minimum connected x -forcing subset of S_x . For the vertex x in the graph G given in Figure 2.2, it is clear that y is a connected x -detour vertex and $S_x = \{u_1, x, y, v_1\}$ is a minimum connected x -forcing subset of a minimum connected x -detour set S_x .

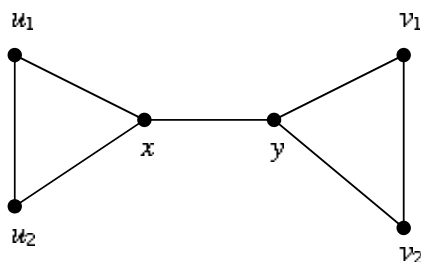


Figure 2.2

Corollary 2.11. Let x be a vertex of a connected graph G . If G contains k end-vertices, then $cf_{cdx}(G) \leq cd_x(G) - k + 1$.

Proof. This follows from Theorem 1.2 and Proposition 2.9(ii). ■

Corollary 2.12.

- (i) If T is a non-trivial tree, then $cf_{cdx}(T) = 0$ for every vertex x in T .
- (ii) If G is the complete graph K_n ($n \geq 3$), the n -cube Q_n ($n \geq 2$), the cycle C_n ($n \geq 3$), the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 4$) or the complete bipartite graph $K_{m,n}$ ($m, n \geq 2$), then $cf_{cdx}(G) = cd_x(G) = 1$ for every vertex x in G .

Proof. (i) For any vertex x in T , cd_x -set of T is unique and so the result follows from Proposition 2.4(i).

(ii) For each of the graphs in (ii), it is easily seen that there is more than one minimum connected x -detour set for any vertex x . Hence it follows from Proposition 2.4(i) that $cf_{cdx}(G) \neq 0$ for each of these graphs. Now it follows from Theorem 1.1 and Proposition 2.3 that $cf_{cdx}(G) = cd_x(G) = 1$ for each of the graphs. ■

Since $cd_x(G) \leq cd_x^+(G)$, the next result follows from Proposition 2.3.

Corollary 2.13. For any vertex x in a connected graph G , $0 \leq f_{cdx}(G) \leq cf_{cdx}(G) \leq cd_x(G) \leq cd_x^+(G) \leq p$.

In view of Corollary 2.13, we have the following realization result.

Theorem 2.14. *If a, b, c and d are positive integers with $2 \leq a < b \leq c \leq d$, then there exists a connected graph G such that $f_{cdx}(G) = a$, $cf_{cdx}(G) = b$, $cd_x(G) = c$ and $cd_x^+(G) = d$ for some vertex x in G .*

Proof. For each integer i with $1 \leq i \leq a - 1$, let F_i be a copy of K_2 , where v_i and v'_i are the vertices of F_i . Let $K_{1,c-b}$ be the star with center x and $U = \{u_1, u_2, \dots, u_{c-b}\}$ the set of end vertices of $K_{1,c-b}$. Let H be the graph obtained by joining the vertex x to the vertices of F_i ($1 \leq i \leq a - 1$). Let $K = (K_2 \cup (d - c + 1)K_1) + \overline{K_2}$, where $Z = V(K_2) = \{z_1, z_2\}$, $Y = V((d - c + 1)K_1) = \{y_1, y_2, \dots, y_{d-c+1}\}$ and $X = V(\overline{K_2}) = \{x_1, x_2\}$. Let $P : w_1, w_2, \dots, w_{b-a}$ be a path of order $b - a$, where $W = V(P) = \{w_1, w_2, \dots, w_{b-a}\}$. Let G be the graph obtained from H, K and P by identifying x in H and w_1 in P , and identifying w_{b-a} in P and x_1 in K . The graph G is shown in Figure 2.3.

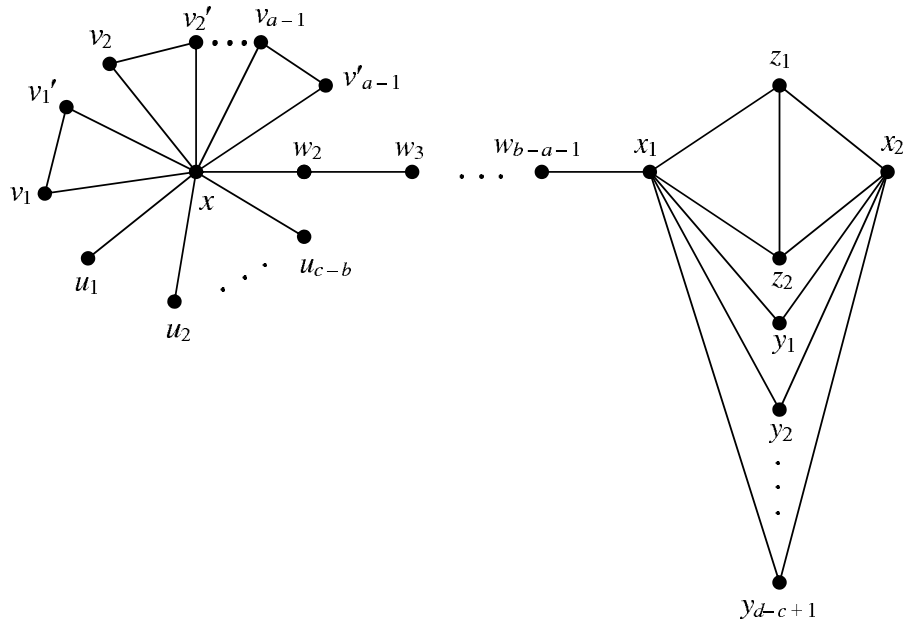


Figure 2.3

Step 1. We show that $cd_x(G) = c$. Let S_x be a connected x -detour set of G . By Theorem 1.4, every component of $G - \{w_i\}$ ($1 \leq i \leq b - a$) contains at least one vertex from $S_x \cup \{x\}$. Also, since $x = w_1$ is a cut vertex of G ,

every component of $G - \{x\}$ contains at least one vertex from S_x . Now, it follows that $U \subset S_x$ and S_x contains at least one vertex from each of F_i ($1 \leq i \leq a-1$) and $K - \{w_{b-a}\}$. Also since $G[S_x]$ is connected, it is clear that $w_i \in S_x$ for $1 \leq i \leq b-a$. Hence $|S_x| \geq c$.

On the other hand, if $d-c+1 > 1$, let $S_x = U \cup W \cup \{v_1, v_2, \dots, v_{a-1}, z_1\}$ be the set formed by taking all the end vertices, all the cut vertices and exactly one vertex from each F_i and Z , and if $d-c+1 = 1$, let $S_x = U \cup W \cup \{v_1, v_2, \dots, v_{a-1}, z_1\}$ be the set formed by taking all the end vertices, all the cut vertices and exactly one vertex from each F_i and $Z \cup \{y_1\}$. Then $D(x, z_1) = b-a+3$ and each vertex of K lies on an $x-z_1$ detour and each vertex of F_i lies on an $x-v_i$ detour. Also, it is clear that the induced subgraph $G[S_x]$ is connected. Hence S_x is a connected x -detour set of G and so $cd_x(G) \leq c$. Therefore, $cd_x(G) = c$.

Step 2. We show that $f_{cdx}(G) = a$. Since every minimum connected x -detour set of G contains U, W , exactly one vertex from each F_i ($1 \leq i \leq a-1$) and one vertex from Z or $Z \cup \{y_1\}$ according to $d > c$ or $d = c$, respectively, let $S = U \cup W \cup \{v_1, v_2, \dots, v_{a-1}, z_1\}$ be a minimum connected x -detour set of G and let $T \subseteq S$ be any minimum x -forcing subset of S . Then by Proposition 2.9(i), $T \subseteq S - (U \cup W)$. Therefore, $|T| \leq a$. If $|T| < a$, then there is a vertex $y \in S - (U \cup W)$ such that $y \notin T$. Now there are two cases.

Case 1. Let $y \in \{v_1, v_2, \dots, v_{a-1}\}$, say $y = v_1$. Let $S' = (S - \{v_1\}) \cup \{v'_1\}$. Then $S' \neq S$ and S' is also a minimum connected x -detour set of G such that it contains T , which is a contraction to T a minimum x -forcing subset of S .

Case 2. Let $y = z_1$. Then, similar to Case 1, we see that $|T| < a$ is not possible. Thus $|T| = a$ and so $f_{cdx}(G) = a$.

Step 3. We show that $cf_{cdx}(G) = b$. As in Step 2, let $S = U \cup W \cup \{v_1, v_2, \dots, v_{a-1}, y_1\}$ be a minimum connected x -detour set of G and $T \subseteq S$ any minimum connected x -forcing subset of S . Then by Proposition 2.9(ii), $T \subseteq S - U$. Therefore, $|T| \leq b$. If $|T| < b$, then there is a vertex $y \in S - U$ such that $y \notin T$. Also, since $G[T]$ is connected, $y \notin W$. Now there are two cases.

Case 1. Let $y \in \{v_1, v_2, \dots, v_{a-1}\}$, say $y = v_1$. Let $S' = (S - \{v_1\}) \cup \{v'_1\}$. Then $S' \neq S$ and S' is also a minimum connected x -detour set of G such that it contains T , which is a contraction to T a minimum connected x -forcing subset of S .

Case 2. Let $y = z_1$. Then, similar to Case 1, we see that $|T| < b$ is not possible. Thus $|T| = b$ and so $cf_{cdx}(G) = b$.

Step 4. We show that $cd_x^+(G) = d$. Let $M = U \cup W \cup Y \cup \{v_1, v_2, \dots, v_{a-1}\}$. It is clear that M is a connected x -detour set of G . We claim that M is a minimal connected x -detour set of G . Assume, to the contrary, that M is not a minimal connected x -detour set. Then there is a proper subset T of M such that T is a connected x -detour set of G . Let $s \in M$ and $s \notin T$. Since $G[T]$ is connected, it follows from Theorem 1.4 that $s = y_i$ for some $i = 1, 2, \dots, d-c+1$. For convenience, let $s = y_1$. Since y_1 does not lie on any $x-y_j$ detour where $j = 2, 3, \dots, d-c+1$, it follows that T is not a connected x -detour set of G , which is a contradiction. Thus M is a minimal connected x -detour set of G and so $cd_x^+(G) \geq |M| = d$. Now suppose $cd_x^+(G) > d$. Let N be a minimal connected x -detour set of G with $|N| > d$. Then there exists at least one vertex $w \in N$ such that $w \notin M$. Since every minimal connected x -detour set contains all the end vertices and the cut vertices, and exactly one vertex from each F_i , it follows that $w \in \{x_2, z_1, z_2\}$. Let $w \in \{z_1, z_2\}$, say $w = z_1$. Since every vertex of K lies on an $x-z_1$ detour, we have $(N - V(K)) \cup \{x_1, z_1\}$ is a connected x -detour set and it is a proper subset of N , which is a contradiction to N a minimal connected x -detour set of G . Let $w = x_2$. Since $z_1, z_2 \notin N$, $Y \subset N$ and so $M \subset N$, which is a contradiction to N a minimal connected x -detour set of G . Thus there is no minimal connected x -detour set N of G with $|N| > d$. Hence $cd_x^+(G) = d$. ■

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