

BOUNDS FOR THE RAINBOW CONNECTION NUMBER OF GRAPHS

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Abstract

An edge-coloured graph G is *rainbow-connected* if any two vertices are connected by a path whose edges have distinct colours. The *rainbow connection number* of a connected graph G , denoted $rc(G)$, is the smallest number of colours that are needed in order to make G rainbow-connected. In this paper we show some new bounds for the rainbow connection number of graphs depending on the minimum degree and other graph parameters. Moreover, we discuss sharpness of some of these bounds.

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1. INTRODUCTION

We use [2] for terminology and notation not defined here and consider finite and simple graphs only.

An edge-coloured graph G is called *rainbow-connected* if any two vertices are connected by a path whose edges have different colours. This concept of rainbow connection in graphs was recently introduced by Chartrand *et al.* in [5]. The rainbow connection number of a connected graph G , denoted $rc(G)$, is the smallest number of colours that are needed in order to make

G rainbow connected. An easy observation is that if G has n vertices then $rc(G) \leq n - 1$, since one may colour the edges of a given spanning tree of G with different colours, and colour the remaining edges with one of the already used colours. Chartrand *et al.* computed the precise rainbow connection number of several graph classes including complete multipartite graphs [5]. The rainbow connection number has been studied for further graph classes in [4] and for graphs with fixed minimum degree in [4, 9, 11].

Rainbow connection has an interesting application for the secure transfer of classified information between agencies (cf. [7]). While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?

The computational complexity of rainbow connectivity has been studied in [3, 10]. It is proved that the computation of $rc(G)$ is NP-hard ([3, 10]). In fact it is already NP-complete to decide if $rc(G) = 2$. More generally it has been shown in [10], that for any fixed $k \geq 2$, deciding if $rc(G) = k$ is NP-complete. Moreover, it is NP-complete to decide whether a given edge-coloured (with an unbounded number of colours) graph is rainbow-connected [3].

2. LOWER BOUNDS

It is an easy observation that $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of a graph G . However, the difference $rc(G) - diam(G)$ can be arbitrarily large. One example is given by the star $K_{1,n-1}$ on $n \geq 3$ vertices, which has $diam(K_{1,n-1}) = 2$, but $rc(K_{1,n-1}) = n - 1$.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition 1. *Let G be a connected graph of order n . Then*

1. $1 \leq rc(G) \leq n - 1$,

- 2. $rc(G) \geq diam(G)$,
- 3. $rc(G) = 1 \Leftrightarrow G$ is complete,
- 4. $rc(G) = n - 1 \Leftrightarrow G$ is a tree.

We will now show an improved lower bound for the rainbow connection number. For a given graph G of order n let $n_i(G)$ denote the number of vertices of G which have degree i for $1 \leq i \leq n - 1$.

Theorem 1. *Let G be a connected graph on $n \geq 3$ vertices. Then*

$$rc(G) \geq \max\{diam(G), n_1(G)\}.$$

Proof. If $diam(G) = 1$, then G is complete. Hence $rc(G) = 1 = diam(G) > 0 = n_1(G)$. So we may assume $n_1(G) > diam(G) \geq 2$, since $rc(G) \geq diam(G)$ holds (see Proposition 1.2). Then any two vertices of degree one in G cannot be adjacent, since G is connected and $n \geq 3$. Thus $d(u, v) \geq 2$ for any two vertices $u, v \in V(G)$ with $d(u) = d(v) = 1$. This implies that the n_1 edges which are incident with the n_1 vertices of G of degree one are all coloured distinct. Therefore, $rc(G) \geq \max\{diam(G), n_1(G)\}$. ■

3. UPPER BOUNDS

We first prove the following useful lemma.

Lemma 1 (contraction lemma). *Let G be a connected graph of order n and H be a connected subgraph of G of order k . Let G' denote the graph which is obtained from G by contracting H to a single vertex v . Then*

$$rc(G) \leq rc(G') + rc(H).$$

Proof. We consider an edge-colouring of $G'(H)$ with $rc(G)$ ($rc(H)$) colours which makes $G'(H)$ rainbow-connected. The two colour sets are chosen to be disjoint.

Now going back to G , any edge with both endvertices not in H receives the same colour it had in G' . Any edge with one endvertex in H receives the colour of the edge of G' from v to that other endvertex. Any edge with both endvertices in H receives the colour it has in H . The resulting edge-colouring makes G rainbow-connected and therefore $rc(G) \leq rc(G') + rc(H)$. ■

It is known that a cycle C_k with $k \geq 3$ vertices has rainbow connection number $rc(C_k) = \min\{k - 2, \lceil \frac{k}{2} \rceil\}$ (cf. [5]). Hence, if a connected graph G of order $n \geq k \geq 3$ contains a cycle C_k , then $rc(G) \leq ((n - k + 1) - 1) + \min\{k - 2, \lceil \frac{k}{2} \rceil\} = n - \max\{2, \lfloor \frac{k}{2} \rfloor\}$ by Proposition 1.1 and Lemma 1.

Corollary 1. *Let G be a connected graph with n vertices and circumference $c(G)$. Then,*

$$rc(G) \leq n - \left\lfloor \frac{c(G)}{2} \right\rfloor.$$

It is well known that $c(G) \geq \delta(G) + 1$ for every graph G with minimum degree $\delta(G) \geq 2$.

Corollary 2. *Let G be a connected graph with n vertices and minimum degree $\delta(G) \geq 2$. Then*

$$rc(G) \leq n - \left\lfloor \frac{\delta(G) + 1}{2} \right\rfloor.$$

For 2-connected graphs Dirac [6] has shown the following lower bound for the circumference of a graph.

Theorem 2 (Dirac, 1952, [6]). *Let G be a 2-connected graph of order n and minimum degree $\delta(G)$. Then*

$$c(G) \geq \min\{n, 2\delta(G)\}.$$

Corollary 3. *Let G be a 2-connected graph of order n and minimum degree $\delta(G)$. Then*

$$rc(G) \leq n - \min\left\{\left\lfloor \frac{n}{2} \right\rfloor, \delta(G)\right\}.$$

In fact a stronger upper bound for the rainbow connection number $rc(G)$ of a connected graph G has been shown in [4].

Theorem 3 [4]. *If G is a connected graph with minimum degree $\delta(G)$ then $rc(G) \leq n - \delta(G)$.*

Let us discuss the sharpness of this result. For this purpose we first mention some known results.

Theorem 4 [11]. *If G is a connected graph with n vertices and $\delta(G) \geq 3$, then $rc(G) \leq \frac{3n-1}{4}$.*

Theorem 5 [4]. *Any non-complete graph with $\delta(G) \geq \frac{n}{2} + \log_2 n$ has $rc(G) = 2$.*

Theorem 6 [8]. *Let G be a connected graph of order n and size m . If $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2} - 1$, then $rc(G) = 2$.*

For $\delta = 2$ we show the following theorem.

Theorem 7 . *Let G be a connected graph of order $n \geq 3$ and with minimum degree $\delta(G) = 2$. If $G \notin \{K_3, C_4, K_4 - e, C_5\}$, then $rc(G) \leq n - 3$.*

Proof. We have $c(G) \geq \delta(G) + 1 \geq 3$. If $c(G) \geq 6$, then $rc(G) \leq n - 3$ by Corollary 1. Hence we may assume that $3 \leq c(G) \leq 5$. If G contains two cycles C_{k_1} and C_{k_2} having at most one common vertex, then $rc(G) \leq n + 1 - \max\{2, \lfloor \frac{k_1}{2} \rfloor\} - \max\{2, \lfloor \frac{k_2}{2} \rfloor\} \leq n - 3$ by Lemma 1. Hence we may assume that any two cycles of G have at least two common vertices.

Let C_k by a cycle of G with maximum length, that is, $k = c(G)$. With $\delta(G) \geq 2$ we conclude that $G \cong K_3$ or $k \geq 4$. Hence we may assume $n \geq k \geq 4$. If $n = k = 4$, then $G \cong C_4$ or $G \cong K_4 - e$. In both cases $rc(G) = 2 = n - \delta$. If $n = k = 5$, then $G \cong C_5$ or $C_5 + e \subseteq G$. Then $rc(C_5) = 3 = n - \delta$ and $rc(C_5 + e) = 2 < n - \delta$. Hence we may assume $n > k \geq 4$. Let $H = G[V(G) \setminus V(C_k)]$. If H is not edgeless, then with $\delta \geq 2$ there are two cycles in G having at least two common vertices. Since $4 \leq k \leq 5$ we conclude that there is a cycle of order at least $(k-1)+2 = k+1 > k = c(G)$, a contradiction. Hence we may assume that H is independent. If $k = 4$ then $G \cong K_{2,t}$ for some $t \geq 3$. Then $rc(G) = \min\{\sqrt{t}, 4\} < n - 2$ (cf. [5]). If $k = 5$ then G contains an induced subgraph F on six vertices, say $v_1, v_2, v_3, v_4, v_5, v_6$, such that $E(F) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_1v_6, v_6v_3\}$. Then $rc(F) = 3$ and so $rc(G) \leq n - 3$ by Proposition 1.1 and Lemma 1. ■

Sharpness of Theorem 3

1. $\delta = 1$
 Let G be a tree on $n \geq 3$ vertices. Then $rc(G) = n - 1$.
2. $\delta = 2$
 By Theorem 7 we have $rc(G) = n - 2$ if and only if $G \in \{K_3, C_4, K_4 - e, C_5\}$.

3. $3 \leq \delta < \frac{n}{2} + \log_2 n$
Then $rc(G) \leq \frac{3n-1}{4} < n - \delta$ by Theorem 4. Hence the bound is not sharp for this range of δ when $n \geq 12$.
4. $\frac{n}{2} + \log_2 n \leq \delta \leq n - 3$
Then $rc(G) = 2 < n - \delta$ by Theorem 5. Hence the bound is not sharp for this range of δ when $n \geq 14$.
5. $\delta = n - 2$
For $n \geq 3$ and $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$ let $G \cong K_n - tK_2$. Then $\delta(G) = n - 2$ and $rc(G) = 2 = n - (n - 2)$ by Theorem 6.
6. $\delta = n - 1$
Then G is complete and thus $rc(G) = 1 = n - (n - 1)$.

We will now generalize Theorem 3 by considering pairs of non-adjacent vertices. For a connected non-complete graph of order n let $\sigma_2(G) = \min\{d(u) + d(v) \mid u, v \in V(G) \text{ and } uv \notin E(G)\}$. If G is complete then let $\sigma_2(G) = 2n - 2$.

For 2-connected graphs Bermond [1] has shown the following lower bound for the circumference of a graph.

Theorem 8 (Bermond, 1976, [1]). *Let G be a 2-connected graph of order $n \geq 3$. Then*

$$c(G) \geq \min\{n, \sigma_2(G)\}.$$

Corollary 4. *Let G be a 2-connected graph of order $n \geq 3$. Then*

$$rc(G) \leq n - \min\left\{\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{\sigma_2(G)}{2} \right\rfloor\right\}.$$

In fact we will now improve and generalize this upper bound for connected graphs.

Theorem 9. *Let G be a connected graph of order n . Then $rc(G) \leq n - \lfloor \frac{\sigma_2(G)}{2} \rfloor$.*

Note that $n - \lfloor \frac{\sigma_2(G)}{2} \rfloor \leq n - \delta(G)$ for any connected graph G since $\sigma_2(G) \geq 2\delta(G)$. Hence Theorem 9 improves Theorem 3.

Proof. Fixing δ , our proof is by induction on n where the basis of induction $n = \delta + 1$ is trivial since graphs induced by cliques have rainbow connection number 1. Hence we may assume $n > \delta + 1$.

Let K be a maximal clique of G consisting only of vertices whose degree is δ . Since there is at least one vertex with degree δ and since G is connected we have $1 \leq k = |K| \leq \delta$.

Consider the graph G' obtained from G by deleting the vertices of K . Suppose the connected components of G' are G_1, G_2, \dots, G_t where G_i has n_i vertices and minimum degree sum $s_i = \sigma_2(G_i)$ for $i = 1, \dots, t$. Let $K_i \subseteq K$ with $|K_i| = k_i$ be the vertices of K with a neighbour in G_i , and assume that $|K_1| \geq |K_i|$ for $i = 2, \dots, t$ (notice that it may be that $t = 1$ and G' is connected).

Consider first the case where $K_1 = K$. By the induction hypothesis, $rc(G_i) \leq n_i - \frac{s_i}{2}$. Clearly, we may give the edges of K and the edges from K to G_1 the same colour. Hence,

$$rc(G) \leq t + \sum_{i=1}^t \left(n_i - \frac{s_i}{2} \right) = t + n - k - \sum_{i=1}^t \frac{s_i}{2}.$$

By the maximality of K , each pair of non adjacent vertices of G_i has degree sum at least $\sigma_2(G) - 2(k - 1)$ in G_i . Hence, $s_i \geq \sigma_2(G) - 2(k - 1)$, and therefore

$$\begin{aligned} rc(G) &\leq t + n - k - \sum_{i=1}^t \frac{s_i}{2} \leq t + n - k - t \frac{\sigma_2(G)}{2} + tk - t \\ &= n - t \frac{\sigma_2(G)}{2} + (t - 1)k \leq n - \frac{\sigma_2(G)}{2}. \end{aligned}$$

Now assume that $|K| > |K_1| = k_1 > 1$. By contracting K_1 to a single vertex v , we obtain a contraction G^* of G with $n - k_1 + 1$ vertices and minimum degree sum $\sigma_2(G^*) \geq \sigma_2(G) - 2(k_1 - 1)$, so by induction

$$rc(G^*) \leq n - (k_1 - 1) - \frac{\sigma_2(G^*)}{2} \leq n - (k_1 - 1) - \frac{\sigma_2(G)}{2} + (k_1 - 1) = n - \frac{\sigma_2(G)}{2}.$$

Now going back to G , any edge with both endvertices not in K_1 receives the same colour it had in G^* . Any edge with one endvertex in K_1 receives the colour of the edge of G^* from v to that other endvertex. Any edge with both endvertices in K_1 receives the colour of an edge of G^* from v to another vertex in $K \setminus K_1$. The resulting colouring makes G rainbow-connected and therefore $rc(G) \leq n - \frac{\sigma_2(G)}{2}$.

Finally, if $k_1 = 1$ (and since $|K| > |K_1| = k_1$ we have $k \geq 2$), contract the graph induced by K into a single vertex v and notice that the contracted graph G^* has minimum degree sum $\sigma_2(G^*) \geq \sigma_2(G)$, and $n - k + 1$ vertices. Hence, by induction hypothesis, $rc(G^*) \leq n - k + 1 - \frac{\sigma_2(G^*)}{2}$. Going back to G and colouring the edges of the graph induced by the clique K with a new colour, we obtain

$$rc(G) \leq n - k - \frac{\sigma_2(G)}{2} + 2 \leq n - \frac{\sigma_2(G)}{2}. \quad \blacksquare$$

Our final result is based on the chromatic number of the complement \overline{G} of a given graph G .

Theorem 10. *Let G be a connected graph with chromatic number $\chi(G)$. Then*

$$rc(G) \leq 2\chi(\overline{G}) - 1.$$

Proof. We consider a k -vertex-colouring of \overline{G} with $k = \chi(\overline{G})$ colours. Let V_i for $1 \leq i \leq k$ be the colour classes of \overline{G} . Then $G[V_i]$ is complete for $1 \leq i \leq k$. Since G is connected there are $k - 1$ edges in $E(G)$ connecting these k complete subgraphs in G , where the endvertices of each of these $k - 1$ edges belong to two different complete subgraphs. Now choosing one colour for all edges of a complete subgraph, $k - 1$ extra colours for the $k - 1$ connecting edges and arbitrary colours for the remaining edges we obtain an edge colouring of G with $2k - 1$ colours which makes G rainbow-connected. \blacksquare

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