Discussiones Mathematicae
Graph Theory 31 (2011) 397–409

DISTANCE INDEPENDENCE IN GRAPHS

J. Louis Sewell

Department of Mathematical Sciences
University of Alabama in Huntsville
Huntsville, AL 35899 USA

e-mail: louis.sewell@gmail.com

and

Peter J. Slater

Department of Mathematical Sciences
and Computer Sciences Department
University of Alabama in Huntsville
Huntsville, AL 35899 USA

Abstract

For a set D of positive integers, we define a vertex set $S \subseteq V(G)$ to be D-independent if $u, v \in S$ implies the distance $d(u, v) \notin D$. The D-independence number $\beta_D(G)$ is the maximum cardinality of a D-independent set. In particular, the independence number $\beta(G) = \beta_{\{1\}}(G)$. Along with general results we consider, in particular, the odd-independence number $\beta_{ODD}(G)$ where $ODD = \{1, 3, 5, \ldots\}$.

Keywords: independence number, distance set.

2010 Mathematics Subject Classification: 05C12, 05C38, 05C69, 05C70, 05C76.

1. Introduction

A vertex subset S of a graph $G = (V, E)$ is independent if no two vertices in S are adjacent. Alternatively, one can say that $S \subseteq V(G)$ is independent if for each edge $e = \{u, v\}$ in $E(G)$ we have either (1) $|S \cap e| \leq 1$ or, equivalently, (2) $|S \cap e| < |e| = 2$. The difference in viewpoint between (1) and (2) for general set systems (hypergraphs) led to different generalized graphical independence, covering, domination, enclaveless, ... parameters as discussed in Sinko and Slater [6, 7].

Likewise, defining independence (and other parameters) in terms of distance leads to the generalizations presented here. In particular, vertex subset $S \subseteq V(G)$ is independent if for any two vertices x and y in S the distance between x and y satisfies $d(x, y) > 1$, that is, $d(x, y) \neq 1$ or, equivalently, $d(x, y) \notin \{1\}$. More generally, $S \subseteq V(G)$ is a k-packing if for any distinct x and y in S we have distance the $d(x, y) > k$, that is, $d(x, y) \notin [k] = \{1, 2, \ldots, k\}$. In general, for any set $D \subseteq \mathbb{Z}^+$ of positive integers we say $S \subseteq V(G)$ is *D-independent* if for any two vertices x and y in S we have $d(x, y) \notin D$. The *D-independence number* $\beta_D(G)$ is the maximum cardinality of a D-independent set. Thus, the normal independence number $\beta(G)$ satisfies $\beta(G) = \beta_{\{1\}}(G)$; the packing number $\rho(G) = \beta_{\{1,2\}}(G)$; and the k-packing number $\rho_k(G) = \beta_{[k]}(G)$.

For a new example, consider $D = \{1, 4, 5\}$ and the path $P_n = v_1, v_2, \ldots, v_n$, shown in Figure 1.1. Let vertex set $S \subseteq V(P_n)$ be a $\{1, 4, 5\}$-independent set and $k = \min\{i | v_i \in S\}$. Then, $S^* = \left\{v_{i-(k-1)} | v_i \in S\right\}$ is a $\{1, 4, 5\}$-independent set with the same cardinality as S. So, without loss of generality, suppose $v_1 \in S$. In this case, the vertices labeled above by $*_1$ in Figure 1.1(a) (namely, v_2, v_5, and v_6) cannot be in S since the distance from one of these vertices to v_1 is in $\{1, 4, 5\}$. More generally, in Figure 1.1 a $*_i$ above a vertex indicates that it is at a distance in $\{1, 4, 5\}$ from v_i, and v_i is in S. If we successively, greedily place the next possible vertex to the right of v_1 in S, then the result is the pattern shown in Figure 1.1(a). Notice that here $|S| = \left\lceil \frac{1}{4} n \right\rceil$, showing that $\beta_{\{1,4,5\}}(P_n) \geq \left\lceil \frac{1}{4} n \right\rceil$.

Now suppose $v_1 \in S$, but we do not take a greedy approach to adding vertices to S. In particular, we can use every third vertex as in Figure 1.1(b). Note that $|S| = \left\lceil \frac{1}{3} n \right\rceil$. To show that $\beta(P_n)$ is essentially $\frac{1}{3} n$, we can associate with each $v \in S$ two vertices from $V(P_n) \setminus S$. Consider vertex $v_i \in S$ with $i \leq n - 5$. Then we cannot have v_{i+1}, v_{i+4} nor v_{i+5} in S. If $v_{i+2} \notin S$, then associate v_{i+1} and v_{i+2} with v_i. Otherwise, associate v_{i+1} and v_{i+5} with v_i. Note that here v_{i+3} and v_{i+4} are associated with v_{i+2}. It follows that $\beta_{\{1,4,5\}}(P_n) = \left\lceil \frac{1}{3} n \right\rceil$ for $n \geq 4$.

The minimum cardinality of a maximally independent vertex set $S \subseteq V(G)$ is the lower-independence number $i(G)$. More generally, for each $D \subseteq \mathbb{Z}^+$ a vertex set $S \subseteq V(G)$ is maximally D-independent if S is D-independent and for each $v \in V(G) \setminus S$ there is a vertex $w \in S$ such that $d(v, w) \in D$.

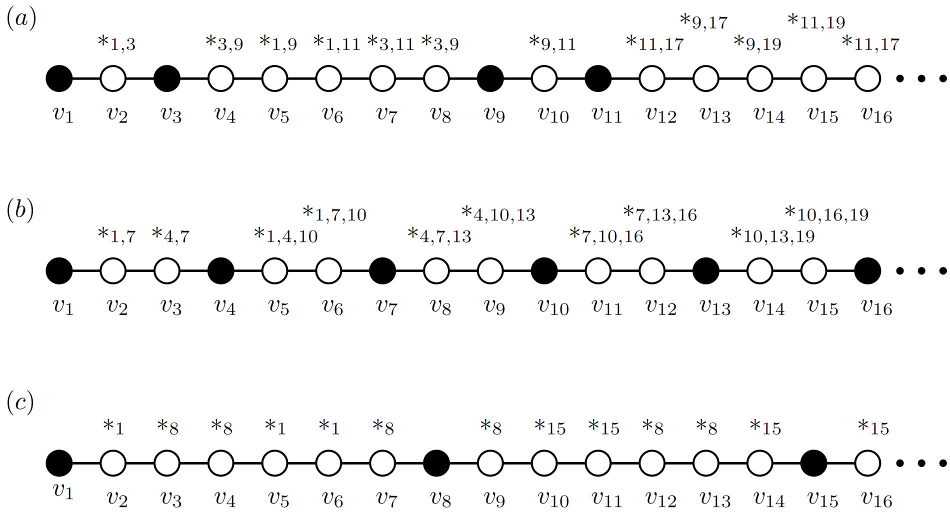

Figure 1.1. $\beta_{\{1,4,5\}}(P_n)$ and $i_{\{1,4,5\}}(P_n)$.

We define the *lower-D-independence number* of G, denoted $i_D(G)$, to be the minimum cardinality of a maximally D-independent set. For example, for the tree $T_{1,k}$ in Figure 1.2, $\{v, w, x\}$ is a maximally $\{3,5\}$-independent set. In fact, $i_{\{3,5\}}(T_{1,k}) = 3$, while $\beta_{\{3,5\}}(T_{1,k}) = k + 2$. Clearly, $i_D(G) \le \beta_D(G)$ for all G and $D \subseteq \mathbb{Z}^+$.

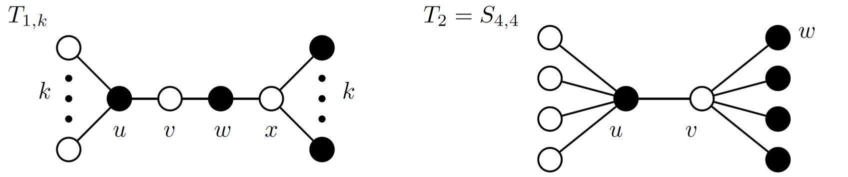

Figure 1.2. Illustrating $i_D(T)$ and $\beta_D(T)$.

For T_2 in Figure 1.2, the set of all endpoints forms a $\beta(T_2)$-set, while the set containing an endpoint, say w, and all vertices at distance two from w form an $i(T_2)$-set. Thus, $\beta_{\{1\}}(T_2) = \beta(T_2) = 8$ and $i_{\{1\}}(T_2) = i(T_2) = 5$. Also, notice that a set formed by any pair of adjacent vertices of T_2 or a set formed by endpoints at distance three are the only maximal $\{2\}$-independent sets of T_2. Thus, $\beta_{\{2\}}(T_2) \equiv i_{\{2\}}(T_2) = 2$. (The symbol $\equiv$ denotes strong equality as introduced in Haynes and Slater [3]. See also [10, 11]. Here, for a graph G, $\beta_{\{2\}}(G) \equiv i_{\{2\}}(G)$ is equivalent to S is a $\beta_{\{2\}}(G)$-set $\Leftrightarrow$ S is an $i_{\{2\}}(G)$-set.) Finally, note that $N[u]$ and $N[v]$ are the only two maximal $\{3\}$-independent sets of T_2. This shows that $\beta_{\{3\}}(T_2) \equiv i_{\{3\}}(T_2) = 6$.

For path P_n we have $\beta(P_n) = \lceil \frac{1}{2}n \rceil$, $i(P_n) = \lceil \frac{1}{3}n \rceil$ and $\beta_{\{1,4,5\}}(P_n) = \lceil \frac{1}{3}n \rceil$. We can see that $i_{\{1,4,5\}}(P_n)$ is approximately $\frac{1}{7}n$. Note that if $S \subseteq V(P_n)$ with $|S| = t$, then at most $6t$ vertices in $V(P_n) \setminus S$ are at a distance in $\{1, 4, 5\}$ from S. Thus $|S| < \frac{1}{7}n$ implies S is not maximally $\{1, 4, 5\}$-independent, and so $i_{\{1,4,5\}}(P_n) \geq \frac{1}{7}n$. As seen in Figure 1.1(c), if S contains any two vertices $v_i, v_{i+7} \in V(P_n)$ at distance 7, then the vertices v_{i+1} through v_{i+6} cannot be in S. This shows that $i_{\{1,4,5\}}(P_n)$ is upper bounded by essentially $\frac{1}{7}n$.

For one more example, the Petersen graph P, we have $i(P) = 3$, $\beta(P) = 4$ and $i_{\{2\}}(P) \equiv \beta_{\{2\}}(P) = 2$.

In Section 2 we focus on the odd-independence case where $D = \{1, 3, 5, 7, \ldots\}$, and in Section 3 we introduce D-covering, D-enclaveless, D-dominating, and D-irredundant sets.

2. Odd-Independence

Observing that the set D can be infinite, an intriguing example is to consider the set $D = \{1, 3, 5, 7, \ldots\}$ of odd positive integers. We call a set $S \subseteq V(G)$ an *odd-independent set* if $u, v \in S$ implies $d(u, v)$ is not odd. Also, we define the odd-independence number, denoted $\beta_{ODD}(G)$, to be the maximum cardinality of an odd-independent set $S \subseteq V(G)$ and the lower-odd-independence number, denoted $i_{ODD}(G)$, to be the minimum cardinality of a maximal odd-independent set $S \subseteq V(G)$.

Consider the path $P_n = v_1, v_2, \ldots, v_n$, and let $S \subseteq V(P_n)$ be a maximal odd-independent set. Then for $v_i, v_j \in S$ the distance $d(v_i, v_j)$ is even; that is, $i - j \equiv 0\,(mod\ 2)$. This shows that $v_i \in S$ implies $S \subseteq \{v_j \in V(P_n) | i - j \equiv 0\,(mod\ 2)\}$. Since S is maximal, $v_i \in S$ and $i - j \equiv 0\,(mod\ 2)$ implies $v_j \in S$. Hence, there are exactly two maximal odd-independent subsets of $V(P_n)$, $S_1 = \{v_i \in V(P_n) | i = 1, 3, 5, \ldots\}$ and $S_2 = \{v_i \in V(P_n) | i = 2, 4, 6, \ldots\} = V(P_n) \setminus S_1$. Since for all n, $|S_1| = \lceil \frac{n}{2} \rceil \geq |S_2| = \lfloor \frac{n}{2} \rfloor$, we have that $\beta_{ODD}(P_n) = \lceil \frac{n}{2} \rceil$ and $i_{ODD}(P_n) = \lfloor \frac{n}{2} \rfloor$.

More generally, let G be any connected bipartite graph with partite sets S and $V(G) \setminus S$. As with P_n, there are exactly two maximal odd-independent subsets of G. To see that these are precisely the partite sets S and $V(G) \setminus S$, notice that the distance between any pair of vertices in S, or any pair of vertices in $V(G) \setminus S$, is even and the distance from any vertex in S to any vertex in $V(G) \setminus S$ is odd. This gives us the following theorem.

Theorem 2.1. *For any connected bipartite graph G with partite sets S and $V(G) \setminus S$, we have $\beta_{ODD}(G) = max\{|S|, |V(G) \setminus S|\}$ and $i_{ODD}(G) = \min\{|S|, |V(G) \setminus S|\}$.*

Proposition 2.2. $\beta_{ODD}(P_n) = \lceil \frac{n}{2} \rceil$, $i_{ODD}(P_n) = \lfloor \frac{n}{2} \rfloor$, $\beta_{ODD}(C_{2k}) \equiv i_{ODD}(C_{2k}) = k$ *and* $\beta_{ODD}(C_{2k+1}) \equiv i_{ODD}(C_{2k+1}) = \lceil \frac{k+1}{2} \rceil$.

Proof. The result for paths follows from the discussion above. An immediate consequence of Theorem 2.1 is the result for even cycles. Now consider the odd cycle C_{2k+1} with $V(C_{2k+1}) = v_1, v_2, \ldots, v_{2k+1}$, and let $S \subseteq V(C_{2k+1})$ be a maximal odd-independent set. We show that $S \subset \{v_t, v_{t+1}, \ldots, v_{t+k}\}$ for some $t = 1, 2, \ldots, 2k+1$ where subscripts are taken modulo $2k+1$. Assume $v_i, v_j \in S$ with $i < j$. Taking $t = i$ if $j - i \leq k$ and $t = j$ otherwise will show the result. Let vertex v_h also be in S with $1 \leq h < i < j \leq 2k+1$. Since $2k+1$ is odd, one of $i-h$, $j-i$, or $(2k+1+h) - j$ is odd. Without loss of generality, assume $i - h$ is odd and let $t = i$. Since $d(v_h, v_i)$ is even, we must have that $i - h > k+1$; otherwise, $d(v_h, v_i) = i - h$. This shows that $(2k+1+h) - i \leq k$ and $\{v_i, v_{i+1}, \ldots, v_{2k+1+h} = v_h\} \subseteq \{v_t, v_{t+1}, \ldots, v_{t+k}\}$. Since $i < j < (2k+1)+h$, the result holds. Without loss of generality, assume $v_t \in S$. Then vertices in $\{v_t, v_{t+1}, \ldots, v_{t+k}\} \cap \{v_{t+2}, v_{t+4}, v_{t+6}, \ldots\}$ are at an even distance from v_t and each other; and each vertex in $\{v_t, v_{t+1}, \ldots, v_{t+k}\} \cap \{v_{t+1}, v_{t+3}, v_{t+5}, \ldots\}$ is at an odd distance from v_t. Since S is maximal, this shows that $S = \{v_t, v_{t+1}, \ldots, v_{t+k}\} \cap \{v_t, v_{t+2}, v_{t+4}, \ldots\}$. Since there are exactly $2k+1$ such maximal odd-independent sets, one for each $t = 1, 2, \ldots, 2k+1$, and each has the same cardinality, we have that $\beta_{ODD}(C_{2k+1}) \equiv i_{ODD}(C_{2k+1}) = \lceil \frac{k+1}{2} \rceil$. ∎

Extending the discussion of odd-independent sets of paths and cycles, we now look at the Cartesian products, namely, grids $P_s \square P_t$, cylinders $P_s \square C_t$ and tori $C_s \square C_t$.

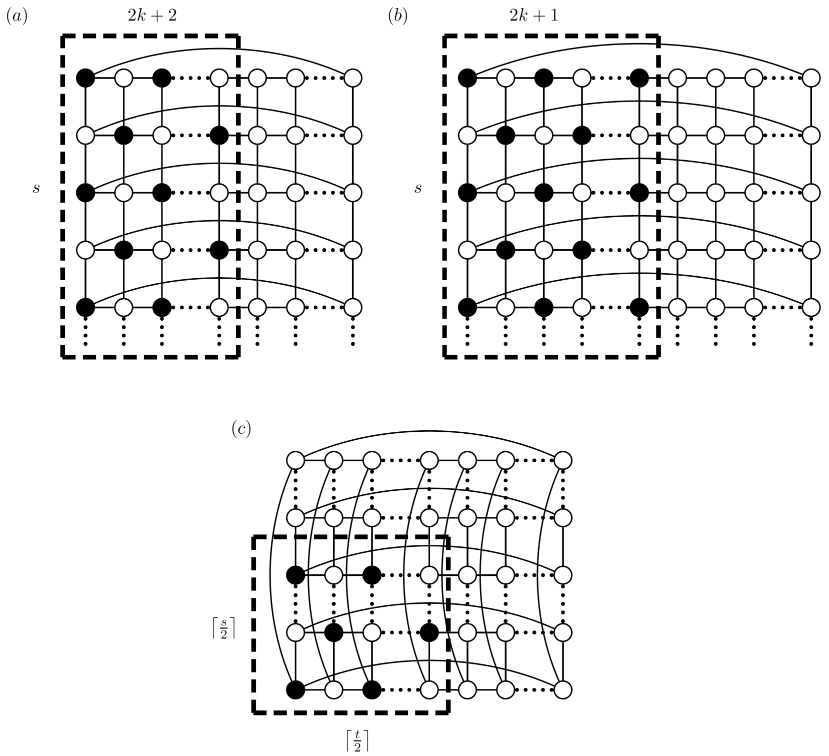

Figure 2.1. (a) $P_s \Box C_{4k+3}$; (b) $P_s \Box C_{4k+1}$; (c) $C_s \Box C_t$, s and t odd.

Theorem 2.3. (1) *For positive integers s and t,*

$$\beta_{ODD}(P_s \Box P_t) = \left\lceil \frac{st}{2} \right\rceil .$$

(2) (i) *For positive integer s and positive even integer t,*

$$\beta_{ODD}(P_s \Box C_t) = \frac{st}{2}.$$

(ii) *For positive integer s and positive odd integer t,*

$$\beta_{ODD}(P_s \Box C_t) = \begin{cases} \left\lceil \frac{s}{2} \right\rceil \cdot \frac{t+1}{2} & \textit{if} \;\; t = 4k+1\,, \\ \frac{s(t+1)}{4} & \textit{if} \;\; t = 4k+3\,. \end{cases}$$

(3) (i) *For positive even integers s and t,*

$$\beta_{ODD}(C_s \square C_t) = \frac{st}{2}.$$

(ii) *For positive even integer s and positive odd integer t,*

$$\beta_{ODD}(C_s \square C_t) = \frac{s(t+1)}{4}.$$

(iii) *For positive odd integers s and t,*

$$\beta_{ODD}(C_s \square C_t) \geq \left\lceil \frac{\lceil \frac{s}{2} \rceil \cdot \lceil \frac{t}{2} \rceil}{2} \right\rceil.$$

Proof. (1) By Theorem 2.1, the s by t grid $P_s \square P_t$ satisfies $\beta_{ODD}(P_s \square P_t) = \lceil \frac{st}{2} \rceil$ and $i_{ODD}(P_s \square P_t) = \lfloor \frac{st}{2} \rfloor$.

(2) (i) The s by t cylinder $P_s \square C_t$ is bipartite when t is even, yielding the same values as for $P_s \square P_t$.

(ii) For odd t, let $S \subseteq V(P_s \square C_t) = \{v_{i,j} | 1 \leq i \leq s, 1 \leq j \leq t\}$ be a maximal odd-independent set. Notice that for each i no more than $\lceil \frac{t+1}{4} \rceil$ vertices from $X_i = \{v_{i,j} | 1 \leq j \leq t\}$ can be in S, per the above result for odd-independent sets of odd cycles. If $t = 4k + 3$ for some k, then this bound is achieved with the pattern shown in Figure 2.1(a), or any shift of this pattern, yielding $|S| = s \cdot \lceil \frac{t+1}{4} \rceil = \frac{s(t+1)}{4}$. For $t = 4k + 1$ we first show that for each i the intersection of S with $X_i \cup X_{i+1}$ can contain no more than $2k + 1$ vertices. As already noted, no more than $\lceil \frac{t+1}{4} \rceil = k + 1$ vertices can be in $S \cap X_i$ or $S \cap X_{i+1}$. Without loss of generality, assume the $k + 1$ vertices $v_{i,1}, v_{i,3}, \ldots, v_{i,2k+1}$ are in S. Then the vertices $v_{i+1,1}, v_{i+1,3}, \ldots, v_{i+1,2k+1}$ and $v_{i+1,2k+2}, v_{i+1,2k+3}, \ldots, v_{i+1,4k+1}$ are at an odd distance from at least one vertex in S. The remaining k vertices in X_{i+1} are at an even distance from each other and from the vertices in $S \cap X_i$. This gives the upper bound of $\beta_{ODD}(P_s \square C_t) \leq \lceil \frac{s}{2} \rceil \cdot \frac{t+1}{2}$. This bound is achieved with the pattern shown in Figure 2.1(b), or any shift of this pattern. Combining the above results, for positive s and odd positive t we have that $\beta_{ODD}(P_s \square C_t) = \begin{cases} \lceil \frac{s}{2} \rceil \cdot \frac{t+1}{2} & \text{if } t = 4k+1, \\ \frac{s(t+1)}{4} & \text{if } t = 4k+3. \end{cases}$

(3) Given the torus $C_s \square C_t$, we consider three cases: s and t are even; s is even and t is odd; and s and t are both odd.

(i) When s and t are even, $C_s \square C_t$ is bipartite and Theorem 2.1 implies $\beta_{ODD}(C_s \square C_t) \equiv i_{ODD}(C_s \square C_t) = \lceil \frac{st}{2} \rceil = \frac{st}{2}$.
(ii) For even s and odd t, the same reasoning used to determine $\beta_{ODD}(P_s \square C_t)$ under this restriction shows that $\beta_{ODD}(C_s \square C_t) = \beta_{ODD}(P_s \square C_t)$.
(iii) Finally, when s and t are both odd $\beta_{ODD}(C_s \square C_t) \geq \left\lceil \frac{\lceil \frac{s}{2} \rceil \cdot \lceil \frac{t}{2} \rceil}{2} \right\rceil$ as evidenced by the pattern in Figure 2.1(c). (We believe, in fact, that for odd s and t the value of $\beta_{ODD}(C_s \square C_t)$ is exactly $\left\lceil \frac{\lceil \frac{s}{2} \rceil \cdot \lceil \frac{t}{2} \rceil}{2} \right\rceil$.) ■

The results for β_{ODD} of grids, cylinders and tori are summarized in Table 2.1 above with approximate values for ease of comparison.

Table 2.1. β_{ODD} for grids, cylinders and tori.

	s even t even	s odd t even	s even t odd	s odd t odd
$P_s \square P_t$	$\frac{st}{2}$	$\frac{st}{2}$	$\frac{st}{2}$	$\frac{st}{2}$
$P_s \square C_t$	$\frac{st}{2}$	$\frac{st}{2}$	$\frac{st}{4}$	$\frac{st}{4}$
$C_s \square C_t$	$\frac{st}{2}$	$\frac{st}{4}$	$\frac{st}{4}$	$\geq \frac{st}{8}$

Theorem 2.4. *For any graph G and distance sets D_1 and D_2, $D_1 \subseteq D_2$ implies $\beta_{D_2}(G) \leq \beta_{D_1}(G)$.*

Proof. Let G be a graph and D_1,D_2 be distance sets such that $D_1 \subseteq D_2$. Let vertex set $S \subseteq V(G)$ be a $\beta_{D_2}(G)$-set. Given $u, v \in S$ we have $d(u,v) \notin D_2$, which implies $d(u,v) \notin D_1$. Hence, $\beta_{D_2}(G) \leq \beta_{D_1}(G)$. ■

This shows that for all graphs G, $\beta_{ODD}(G) \leq \beta(G)$. By definition, for every graph G and distance set D, $i_D(G) \leq \beta_D(G)$. Together, this gives us $i_{ODD}(G) \leq \beta_{ODD}(G) \leq \beta(G)$ and $i(G) \leq \beta(G)$ for every graph G. Given this, it is perhaps surprising that the lower-independence number is incomparable to both the lower-odd-independence number and the odd-independence number. We first note that using Theorem 2.1 we have Theorem 2.5.

Theorem 2.5. *For connected bipartite graph B, $i(B) \leq i_{ODD}(B) \leq \frac{n}{2} \leq \beta_{ODD}(B) \leq \beta(B)$.*

As noted, i is incomparable with i_{ODD} and β_{ODD}. In fact, H_1, H_2 and H_3, with $i(H_1) < i_{ODD}(H_1) < \beta_{ODD}(H_1)$, $i_{ODD}(H_2) < i(H_2) < \beta_{ODD}(H_2)$ and $i_{ODD}(H_3) < \beta_{ODD}(H_3) < i(H_3)$ are illustrated in Figure 2.2.

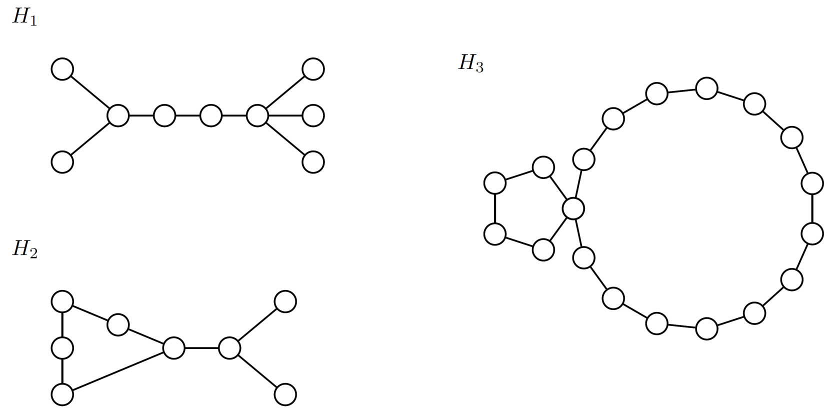

Figure 2.2. Incomparability of i with i_{ODD} and β_{ODD}. In particular, $i(H_1) = 2 < i_{ODD}(H_1) = 4 < \beta_{ODD}(H_1) = 5$, $i_{ODD}(H_2) = 2 < i(H_2) = 3 < \beta_{ODD}(H_2) = 4$ and $i_{ODD}(H_3) = 2 < \beta_{ODD}(H_3) = 5 < i(H_3) = 6$.

3. Other Distance Parameters

A set $R \subseteq V(G)$ is a *cover* if for each edge $\{u, v\} \in E(G)$ we have $\{u, v\} \cap R \neq \emptyset$. The covering number, denoted $\alpha(G)$, is the minimum cardinality of a cover. It is easy to see that R is a cover if and only if $S = V(G) \setminus R$ is independent, and we have the following result of Gallai.

Theorem 3.1 (Gallai [2])**.** *For any graph G of order $n = |V(G)|$, we have $\alpha(G) + \beta(G) = n$.*

The upper-covering number, denoted $\Lambda(G)$, is the maximum cardinality of a minimal cover. Using complementarity of independent sets and covers, we have the following.

Theorem 3.2 (McFall and Nowakowski [4])**.** *For any graph G of order $n = |V(G)|$, we have $\Lambda(G) + i(G) = n$.*

The complementation relation between covering and independence can be generalized. As described in [8], we have the following. Let $\mathcal{F}$ be any family of subsets of some set X. Define $M(X, \mathcal{F})$ and $m(X, \mathcal{F})$ as follows:

$$M(X,\mathcal{F}) = \max\{|S| : S \in \mathcal{F}\}, \quad m(X,\mathcal{F}) = \min\{|S| : S \in \mathcal{F}\}. \tag{3.1}$$

Families $\mathcal{F}_1$ and $\mathcal{F}_2$ of subsets X will be called *complement-related* if $S \in \mathcal{F}_1$ if and only if $X - S \in \mathcal{F}_2$. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are complement-related. Since the complement of any set in $\mathcal{F}_1$ is in $\mathcal{F}_2$, $m(X,\mathcal{F}_2) \leq |X| - M(X,\mathcal{F}_1)$; since the complement of any set in $\mathcal{F}_2$ is in $\mathcal{F}_1$, $M(X,\mathcal{F}_1) \geq |X| - m(X,\mathcal{F}_2)$. Thus $M(X,\mathcal{F}_1) + m(X,\mathcal{F}_2) = |X|$. Note that one could let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the complement-related familes of independent sets and covering sets, respectively. Then $M(V(G),\mathcal{F}_1) = \beta(G)$ and $m(V(G),\mathcal{F}_2) = \alpha(G)$ implies $\beta(G) + \alpha(G) = n$. Recall that $i(G)$, the lower-independence number (or the independent domination number), is the minimum cardinality of a maximal independent set. In general, let $\mathcal{F}^+$ denote the family of those members of $\mathcal{F}$ which are set-theoretically maximal with respect to membership, and $\mathcal{F}^-$ those which are minimal. It is easily seen that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are complement-related, then so are $\mathcal{F}_1^+$ and $\mathcal{F}_2^-$. Hence $m(X,\mathcal{F}_1^+) + M(X,\mathcal{F}_2^-) = |X|$.

Theorem 3.3 (Set Complementation [8])**.** *If families $\mathcal{F}_1$ and $\mathcal{F}_2$ of subsets of X are complement-related, then $M(X,\mathcal{F}_1) + m(X,\mathcal{F}_2) = |X| = m(X,\mathcal{F}_1^+) + M(X,\mathcal{F}_2^-)$.*

Also, see Slater [9] for a general Y-valued Matrix Complementation Theorem for any (complementable) set of reals $Y \subseteq \mathbb{R}$, and Slater [12] discusses complementarity and duality.

If we replace considering edges by considering closed neighborhoods and mimic the definitions of independence and cover, we have the concepts of enclaveless and dominating. A set $S \subseteq V(G)$ is *enclaveless* if it does not entirely contain any closed neighborhood $N[v]$, that is, $|S \cap N[v]| < |N[v]|$ for each $v \in V(G)$; the maximum cardinality of an enclaveless set is the *enclaveless number* $\Psi(G)$, and the *lower-enclaveless number* $\psi(G)$ is the minimum cardinality of a maximally enclaveless set; a set $R \subseteq V(G)$ is dominating if $|R \cap N[v]| \geq 1$ for each $v \in V(G)$; and the minimum cardinality of a dominating set is the domination number $\gamma(G)$, and the upper-domination number $\Gamma(G)$ is the maximum cardinality of a minimal dominating set. Clearly the families of enclaveless sets and dominating sets are complement-related, and the Set Complementation Theorem implies the next result.

Theorem 3.4 (Slater [8])**.** *For any graph G of order n, $\Psi(G)+\gamma(G) = n = \psi(G) + \Gamma(G)$.*

As we did for independence, we can define distance generalizations of these (and other) parameters. For $D \subseteq \mathbb{Z}^+$, vertex set $S \subseteq V(G)$ is D-independent if the distance $d(x, y) \in D$ implies $|S \cap \{x, y\}| \leq 1$. We define $R \subseteq V(G)$ to be a *D-cover* if $d(x, y) \in D$ implies $|R \cap \{x, y\}| \geq 1$, and $\alpha_D(G)$ and $\Lambda_D(G)$ denote the minimum and maximum cardinalities of minimal D-covers and are called the *D-covering number* and *upper-D-covering number*, respectively.

Call vertex set R a *D-dominating set* if, for each $v \in V(G)\setminus R$, there is a vertex $w \in R$ such that $d(v, w) \in D$. The *D-domination number* and *upper-D-domination number*, $\gamma_D(G)$ and $\Gamma_D(G)$, respectively, are the minimum and maximum cardinalities of minimally D-dominating sets. Vertex v will be called a *D-enclave* of $S \subseteq V(G)$ if $v \in S$ and $\{w \in V(G)|d(v, w) \in D\} \subseteq S$, and S is *D-enclaveless* if it has no D-enclaves. That is, S is D-enclaveless if for each $v \in S$ there is a vertex $w \in R = V(G) \setminus S$ with $d(v, w) \in D$. The *D-enclaveless number* and *lower-D-enclaveless number*, $\Psi_D(G)$ and $\psi_D(G)$, respectively, are the maximum and minimum cardinalities of maximal D-enclaveless sets.

In particular, vertex set $S \subseteq V(G)$ is D-independent if and only if $R = V(G)\setminus S$ is a D-cover, and S is D-enclaveless if and only if $R = V(G)\setminus S$ if D-dominating. Hence, generalizing Theorems 3.1, 3.2, and 3.4, by the Set Complementation Theorem, we have the next result.

Theorem 3.5. *For any graph G of order n, we have $\alpha_D(G)+\beta_D(G) = n = \Lambda_D(G) + i_D(G)$ and $\Psi_D(G) + \gamma_D(G) = n = \psi_D(G) + \Gamma_D(G)$.*

If $S \subseteq V(G)$, $v \notin S$, and $d(v, w) \in D$, then S is a D-cover implies that $w \in S$, and so S D-dominates v. Hence, any D-cover S will D-dominate any vertex $v \notin S$ if there is some vertex y such that $d(v, y) \in D$ or, equivalently, if the eccentricity $ecc(v) \geq \min\{d|d \in D\}$.

Theorem 3.6. *If $ecc(v) \geq \min\{d|d \in D\}$ for all $v \in V(G)$, then every D-cover of G is D-dominating, so $\gamma_D(G) \leq \alpha_D(G)$ and $\beta_D(G) \leq \Psi_D(G)$.*

Call $S \subseteq V(G)$ a *D-irredundant set* if for each $v \in S$ there is a vertex $w \in V(G) \setminus (S \setminus \{v\}) = (V(G) \setminus S) \cup \{v\}$ such that $d(w, x) \notin D$ for each $x \in S \setminus \{v\}$ and if $w \neq v$ then $d(w, v) \in D$. The *D-irredundance number* and *lower-D-irredundance number*, $IR_D(G)$ and $ir_D(G)$, respectively, are the maximum and minimum cardinalities of maximally D-irredundant sets for G.

Observation 3.7. *A D-independent set S is maximally D-independent if and only if S if minimally D-dominating. A D-dominating set R is minimally D-dominating if and only if R is maximally D-irredundant.*

Hence we have the following generalization from $D = \{1\}$ in [1] for a parametric chain.

Theorem 3.8. *For any graph G, $ir_D(G) \leq \gamma_D(G) \leq i_D(G) \leq \beta_D(G) \leq \Gamma_D(G) \leq IR_D(G)$.*

4. Related Work

Many questions concerning the general distance-set parameters introduced are under study (bounds, extremal values, Nordhaus-Gaddum results, etc.), along with other D-parameters.

We note that such generalizations also apply to edge sets, such as D-cycles, D-paths and D-geodesics. For example, several different interesting definitions of a D-matching are possible. Letting the *D-power* of G be the graph G^D with $V(G^D) = V(G)$ and $uv \in E(G^D)$ if and only if $d_G(u,v) \in D$, one can observe that Theorems 3.5, 3.6 and 3.8 can be proven by considering G^D. In defining a D-matching, one can consider matchings in G^D. Another way to consider D-independence for edges is to consider D-independent (vertex) sets in the line graph $L(G)$.

Many of these results will appear in Sewell [5].

References

[1] E.J. Cockayne, S.T. Hedetniemi, and D.J. Miller, *Properties of hereditary hypergraphs and middle graphs*, Canad. Math. Bull. **21** (1978) 461–468.

[2] T. Gallai, *Über extreme Punkt-und Kantenmengen*, Ann. Univ. Sci. Budapest, Eotvos Sect. Math. **2** (1959) 133–138.

[3] T.W. Haynes and P.J. Slater, *Paired domination in graphs*, Networks **32** (1998) 199–206.

[4] J.D. McFall and R. Nowakowski, *Strong indepedence in graphs*, Congr. Numer. **29** (1980) 639–656.

[5] J.L. Sewell, *Distance Generalizations of Graphical Parameters*, (Univ. Alabama in Huntsville, 2011).

[6] A. Sinko and P.J. Slater, *Generalized graph parametric chains*, submitted for publication.

[7] A. Sinko and P.J. Slater, *$\mathcal{R}$-parametric and $\mathcal{R}$-chromatic problems*, submitted for publication.

[8] P.J. Slater, *Enclaveless sets and MK-systems*, J. Res. Nat. Bur. Stan. **82** (1977) 197–202.

[9] P.J. Slater, *Generalized graph parametric chains*, in: Combinatorics, Graph Theory and Algorithms (New Issues Press, Western Michigan University 1999) 787–797.

[10] T.W. Haynes, M.A. Henning and P.J. Slater, *Strong equality of upper domination and independence in trees*, Util. Math. **59** (2001) 111–124.

[11] T.W. Haynes, M.A. Henning and P.J. Slater, *Strong equality of domination parameters in trees*, Discrete Math. **260** (2003) 77–87.

[12] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *LP-duality, complementarity and generality of graphical subset problems*, in: Domination in Graphs Advanced Topics, T.W. Haynes *et al.* (eds) (Marcel-Dekker, Inc. 1998) 1–30.

Received 4 January 2010
Revised 6 January 2011
Accepted 10 January 2011