

## GRAPHS WITH EQUAL DOMINATION AND 2-DISTANCE DOMINATION NUMBERS

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### Abstract

Let  $G = (V, E)$  be a graph. The distance between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of the shortest  $(u - v)$  path in  $G$ . A set  $D \subseteq V(G)$  is a dominating set if every vertex of  $G$  is at distance at most 1 from an element of  $D$ . The domination number of  $G$  is the minimum cardinality of a dominating set of  $G$ . A set  $D \subseteq V(G)$  is a 2-distance dominating set if every vertex of  $G$  is at distance at most 2 from an element of  $D$ . The 2-distance domination number of  $G$  is the minimum cardinality of a 2-distance dominating set of  $G$ . We characterize all trees and all unicyclic graphs with equal domination and 2-distance domination numbers.

**Keywords:** domination number, trees, unicyclic graphs.

**2010 Mathematics Subject Classification:** 05C05, 05C69.

### 1. DEFINITIONS

Here we consider simple undirected graphs  $G = (V, E)$  with  $|V| = n(G)$ . The *distance*  $d_G(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $(u - v)$  path in  $G$ . If  $D$  is a set and  $u \in V(G)$ , then  $d_G(u, D) = \min\{d_G(u, v) : v \in D\}$ . The *k-neighbourhood*  $N_G^k[v]$  of a vertex  $v \in V(G)$  is the set of all vertices at distance at most  $k$  from  $v$ . For a set  $D \subseteq V$ , the *k-neighbourhood*  $N_G^k[D]$  is defined to be  $\bigcup_{v \in D} N_G^k[v]$ . A

subset  $D$  of  $V$  is  $k$ -distance dominating in  $G$  if every vertex of  $V(G) - D$  is at distance at most  $k$  from at least one vertex of  $D$ . Let  $\gamma^k(G)$  be the minimum cardinality of a  $k$ -distance dominating set of  $G$ . This kind of domination was defined by Borowiecki and Kuzak [1]. Note that the 1-distance domination number is the *domination number*, denoted  $\gamma(G)$ .

The degree of a vertex  $v$  is  $d_G(v) = |N_G^1(v)|$  and a vertex of degree 1 is called a *leaf*. A vertex which is a neighbour of a leaf is called a *support vertex*. Denote by  $S(G)$  the set of all support vertices of  $G$ . If a support vertex is adjacent to more than one leaf, then we call it a *strong support vertex*. We denote a path on  $n$  vertices by  $P_n = (v_0, \dots, v_{n-1})$  and the cycle on  $n$  vertices by  $C_n$ . For example,  $P_2$  contains two leaves and two support vertices. For any unexplained terms and symbols see [2].

In this paper we study trees and unicyclic graphs for which the domination number and the 2-distance domination number are the same.

## 2. GENERAL RESULTS

First we give some general results for graphs with equal domination and 2-distance domination numbers. Obviously, for any graph  $G$  if  $\gamma(G) = 1$ , then  $\gamma^2(G) = 1$  and thus  $\gamma(G) = \gamma^2(G)$ . We start with a necessary condition for a graph  $G$  with  $1 < \gamma(G) = \gamma^2(G)$ . A set  $D \subseteq V(G)$  is a *2-packing* in  $G$  if  $d_G(u, v) \geq 3$  for every  $u, v \in D$ .

**Proposition 1.** *If  $G$  is a connected graph with  $\gamma(G) = \gamma^2(G)$  and  $\gamma(G) > 1$ , then every minimum dominating set of  $G$  is a 2-packing of  $G$ .*

**Proof.** Suppose  $D$  is a minimum dominating set of  $G$  such that  $|D| \geq 2$  and  $D$  is not a 2-packing. Then there exist  $u, v \in D$  in  $G$  such that  $d_G(u, v) \leq 2$ . Denote by  $x$  a vertex which belongs to  $N_G[u] \cap N_G[v]$  (if  $u$  and  $v$  are adjacent, then possibly  $x = u$  or  $x = v$ ) and let  $D' = (D - \{u, v\}) \cup \{x\}$ . Then  $N_G[u] \subseteq N_G^2[x]$  and  $N_G[v] \subseteq N_G^2[x]$ . Hence  $D'$  is a 2-distance dominating set of  $G$  of smaller cardinality than  $\gamma(G)$ , a contradiction. ■

The condition in Proposition 1 is not sufficient. Consider, for example the cycle  $C_9$ . Next result gives a sufficient condition for a graph  $G$  to have equal domination and 2-distance domination numbers.

**Proposition 2.** *Let  $G$  be the graph obtained from a graph  $H$  and  $n(H)$  copies of  $P_2$ , where the  $i$ th vertex of  $H$  is adjacent to exactly one vertex of the  $i$ th copy of  $P_2$ . Then  $\gamma(G) = \gamma^2(G)$ .*

**Proof.** Let  $G$  be the graph obtained from a graph  $H$  and  $n(H)$  copies of  $P_2$ , where the  $i$ th vertex of  $H$  is adjacent to exactly one vertex of the  $i$ th copy of  $P_2$ . Denote by  $D$  a  $\gamma^2(G)$ -set. Observe that the distance between any two leaves adjacent to two different support vertices in  $G$  is greater than or equal to 5. For this reason, if  $u$  and  $v$  are two leaves adjacent to two different support vertices, then  $u$  and  $v$  cannot be 2-dominated by the same element of  $D$ . This implies that  $\gamma^2(G) \geq |S(G)|$ . Since  $\gamma^2(G) \leq \gamma(G)$ , it follows that  $\gamma(G) = \gamma^2(G)$ . ■

### 3. TREES

In what follows, we constructively characterize all trees  $T$  for which  $\gamma(T) = \gamma^2(T)$ .

Let  $\mathcal{T}$  be the family of all trees  $T$  that can be obtained from sequence  $T_1, \dots, T_j$  ( $j \geq 1$ ) of trees such that  $T_1$  is the path  $P_2$  and  $T = T_j$ , and, if  $j > 1$ , then  $T_{i+1}$  can be obtained recursively from  $T_i$  by the operation  $\mathcal{J}_1$ ,  $\mathcal{J}_2$  or  $\mathcal{J}_3$ :

- **Operation  $\mathcal{J}_1$ .** The tree  $T_{i+1}$  is obtained from  $T_i$  by adding a vertex  $x_1$  and the edge  $x_1y$  where  $y \in V(T_i)$  is a support vertex of  $T_i$ .
- **Operation  $\mathcal{J}_2$ .** The tree  $T_{i+1}$  is obtained from  $T_i$  by adding a path  $(x_1, x_2, x_3)$  and the edge  $x_1y$  where  $y \in V(T_i)$  is neither a leaf nor a support vertex in  $T_i$ .
- **Operation  $\mathcal{J}_3$ .** The tree  $T_{i+1}$  is obtained from  $T_i$  by adding a path  $(x_1, x_2, x_3, x_4)$  and the edge  $x_1y$  where  $y \in V(T_i)$  is a support vertex in  $T_i$ .

Additionally, let  $P_1$  belong to  $\mathcal{T}$ .

The following observation follows immediately from the way in which each tree in the family  $\mathcal{T}$  is constructed.

**Observation 3.** *If a tree  $T$  belonging to the family  $\mathcal{T}$  has at least 2 vertices, then:*

1. *If  $u, v \in S(T)$ , then  $d_T(u, v) \geq 3$ , that is, if  $u, v \in S(T)$ , then  $S(T)$  is a 2-packing in  $T$ ;*
2. *If  $u \in V(T)$ , then  $|N_T[u] \cap S(T)| = 1$ ;*
3.  *$S(T)$  is a minimum dominating set of  $T$ .* ■

We show first that each tree  $T$  belonging to the family  $\mathcal{T}$  is a tree with  $\gamma(T) = \gamma^2(T)$ . To this aim we prove the following lemma.

**Lemma 4.** *If a tree  $T$  of order at least 2 belongs to the family  $\mathcal{T}$ , then  $\gamma^2(T) = |S(T)|$ .*

**Proof.** Let  $T$  be a tree belonging to the family  $\mathcal{T}$  and let  $D$  be a  $\gamma^2(T)$ -set. Since  $S(T)$  is a 2-packing in  $T$ , the distance between any two leaves adjacent to different support vertices is greater than or equal to 5. For this reason, if  $u$  and  $v$  are two leaves adjacent to different support vertices in  $T$ , then  $u$  and  $v$  cannot be 2-distance dominated by the same element of  $D$ . This implies that  $|D| \geq |S|$ . On the other hand, since  $S(T)$  is a dominating set of  $T$ , it is also a 2-distance dominating set of  $T$ . We conclude that  $\gamma^2(T) = |S(T)|$ . ■

By Lemma 4 and Observation 3 we obtain immediately.

**Corollary 5.** *If a tree  $T$  belongs to the family  $\mathcal{T}$ , then  $\gamma(T) = \gamma^2(T)$ .*

Before we prove our next Lemma, observe that for any tree  $T$  with at least 3 vertices,  $\gamma(T) \geq |S(T)|$ .

**Lemma 6.** *If  $T$  is a tree with  $\gamma^2(T) = \gamma(T)$ , then  $T$  belongs to the family  $\mathcal{T}$ .*

**Proof.** Let  $T$  be a tree with  $\gamma^2(T) = \gamma(T)$ . Let  $(v_0, v_1, \dots, v_k)$  be a longest path in  $T$ . If  $k \in \{1, 2\}$ , then  $T$  is  $P_1$  or a star  $K_{1,p}$ , for a positive integer  $p$ , and clearly  $T$  is in  $\mathcal{T}$ .

If  $k \in \{3, 4\}$ , then  $\gamma^2(T) = 1$ , but  $\gamma(T) > 1$ . For this reason now we assume  $k \geq 5$ . We proceed by induction on the number  $n(T)$  of vertices of a tree  $T$  with  $\gamma^2(T) = \gamma(T)$ . If  $n(T) = 6$ , then  $T = P_6$  and  $T$  belongs to the family  $\mathcal{T}$ . (Observe that  $P_6$  may be obtained from  $P_2$  by operation  $\mathcal{T}_3$ ). Now let  $T$  be a tree with  $\gamma^2(T) = \gamma(T)$  and  $n(T) \geq 7$ , and assume that each tree  $T'$  with  $n(T') < n(T)$ ,  $k \geq 5$  and  $\gamma^2(T') = \gamma(T')$  belongs to the family  $\mathcal{T}$ .

If there exists  $v \in S(T)$  such that  $v$  is adjacent to at least two leaves, say  $x_1$  and  $x_2$ , then clearly  $\gamma(T') = \gamma(T)$  and  $\gamma^2(T') = \gamma^2(T)$ , where  $T' = T - x_1$ . Thus,  $\gamma^2(T') = \gamma(T')$  and by the induction,  $T'$  belongs to the family  $\mathcal{T}$ . Moreover,  $T$  may be obtained from  $T'$  by operation  $\mathcal{T}_1$  and we conclude that  $T$  also belongs to the family  $\mathcal{T}$ .

Now assume that each support vertex of  $T$  is adjacent to exactly one leaf. For this reason  $d_T(v_1) = 2$ . If  $d_T(v_2) > 2$ , then  $v_2$  is adjacent to a leaf or  $|N_T(v_2) \cap S(T)| \geq 2$ . In both cases  $v_2$  2-distance dominates all support vertices and leaves at distance at most 2 from  $v_2$ , while  $\gamma(T) \geq |S(T)|$ . Hence  $\gamma(T) > \gamma^2(T)$ , which is impossible. Thus,  $d_T(v_2) = 2$ .

Observe that either  $v_0$  or  $v_1$  is in every minimum dominating set of  $T$ . Assume  $d_T(v_3) > 2$ . If  $v_3$  belongs to some minimum dominating set of  $T$ , say  $D$ , then  $(D \cup \{v_2\}) - \{v_0, v_1, v_3\}$  is a 2-distance dominating set of  $T$  of cardinality smaller than  $\gamma(T)$ , which is impossible. Hence  $v_3$  does not belong to any minimum dominating set of  $T$  and this reason together with  $n(T) \geq 7$  imply that  $v_3$  is not a support vertex of  $T$ . Denote  $T' = T - \{v_0, v_1, v_2\}$ . Since  $d_T(v_3) > 2$ ,  $v_3$  is not a leaf in  $T'$  and since  $k \geq 5$ ,  $v_3$  is not a support vertex in  $T'$ . Moreover, it is no problem to verify that  $\gamma(T') = \gamma(T) - 1$  and  $\gamma^2(T') \geq \gamma^2(T) - 1$ . Hence

$$\gamma^2(T) - 1 \leq \gamma^2(T') \leq \gamma(T') = \gamma(T) - 1 = \gamma^2(T) - 1.$$

Thus,  $\gamma^2(T') = \gamma(T')$  and by the induction,  $T'$  belongs to the family  $\mathcal{T}$ . Moreover,  $T$  may be obtained from  $T'$  by operation  $\mathcal{T}_2$  and we conclude that  $T$  also belongs to the family  $\mathcal{T}$ .

Thus assume  $d_T(v_1) = d_T(v_2) = d_T(v_3) = 2$ . Without loss of generality, denote by  $D$  a minimum dominating set of  $T$  containing  $v_1$ . In this situation  $v_2, v_3$  or  $v_4$  belong to  $D$  to dominate  $v_3$ . If  $v_2$  or  $v_3$  is in  $D$ , then  $D' = (D \cup \{v_2\}) - \{v_1, v_3\}$  is a 2-distance dominating set of  $T$  of cardinality smaller than  $\gamma(T)$ , which is impossible. Hence  $v_4 \in D$ . Observe that  $D'$ , defined as above, 2-distance dominates  $v_4$ . Moreover, if  $w$  is a neighbour of  $v_4$  and  $d_T(w, D - \{v_4\}) \leq 2$ , then  $w$  is 2-distance dominated by  $D'$  and again  $\gamma^2(T') < \gamma(T)$ . Thus  $v_4$  has a neighbour, say  $u$ , such that  $d_T(u, D - \{v_4\}) \geq 3$ . Since  $T$  is a tree and each neighbour of  $u$  is dominated by  $D$ , we conclude that  $u$  is a leaf and for this reason  $v_4$  is a support vertex. Denote  $T' = T - \{v_0, v_1, v_2, v_4\}$ . Since  $u$  is a leaf in  $T'$ ,  $v_4$  is a support vertex in  $T'$ . Moreover, it is no problem to verify that  $\gamma(T') + 1 = \gamma(T)$ . Further, since  $d_T(u, v_0) = 5$ ,  $\gamma^2(T') + 1 = \gamma^2(T)$ . Thus,  $\gamma^2(T') = \gamma(T')$  and by the induction,  $T'$  belongs to the family  $\mathcal{T}$ . Moreover,  $T$  may be obtained from  $T'$  by operation  $\mathcal{T}_3$  and we conclude that  $T$  also belongs to the family  $\mathcal{T}$ . ■

The following Theorem is an immediate consequence of Lemma 6 and Corollary 5.

**Theorem 7.** *Let  $T$  be a tree. Then  $\gamma(T) = \gamma^2(T)$  if and only if  $T$  belongs to the family  $\mathcal{T}$ .* ■

## 4. UNICYCLIC GRAPHS

A unicyclic graph is a graph that contains precisely one cycle. Our next results consider graphs with cycles.

**Lemma 8.** *Let  $G$  be a connected graph with  $\gamma(G) = \gamma^2(G)$ . If  $u, v$  are two leaves of  $G$  adjacent to the same support vertex, then  $\gamma(G+uv) = \gamma^2(G+uv)$ .*

**Proof.** Let  $G$  be a connected graph with  $\gamma(G) = \gamma^2(G)$  and let  $u, v$  be two leaves of  $G$  such that  $d_G(u, v) = 2$  and let  $w$  be the neighbour of  $u$  and  $v$ . By our assumptions and some immediate properties of the domination number of a graph,

$$\gamma^2(G + uv) \leq \gamma(G + uv) \leq \gamma(G) = \gamma^2(G).$$

Hence it suffices to justify that  $\gamma^2(G + uv) \geq \gamma^2(G)$ . Clearly,  $N_{G+uv}^2[x] = N_G^2[x]$  for each  $x \in V(G)$ . Thus, every minimum 2-distance dominating set of  $G + uv$  is also a minimum 2-distance dominating set of  $G$ . Therefore,  $\gamma^2(G + uv) \geq \gamma^2(G)$  and hence  $\gamma(G + uv) = \gamma^2(G + uv)$ . ■

By Theorem 7 and recursively using Lemma 8 we may obtain graphs  $G$  with  $\gamma(G) = \gamma^2(G)$  and containing any number of induced cycles  $C_3$ .

Now we characterize all connected unicyclic graphs  $G$  with  $\gamma(G) = \gamma^2(G)$ . To this aim we introduce some additional notations. Let  $T$  be a tree belonging to the family  $\mathcal{T}$ . We call  $v \in V(T)$  an *active vertex*, if  $v$  is a leaf adjacent to a strong support vertex or  $v \in V(T) - (S(T) \cup \Omega(T))$ . Further, let  $\mathcal{C}_6^+$  be the family of all unicyclic graphs that may be obtained from a tree  $T$  belonging to the family  $\mathcal{T}$  and the cycle  $C_6$  by identifying one vertex of  $C_6$  with a support vertex of  $T$ . In addition, let  $C_6$  belong to  $\mathcal{C}_6^+$ .

Define  $\mathcal{C}$  to be the family of all unicyclic graphs that belong to  $\mathcal{C}_6^+$  or may be obtained from a tree  $T$  belonging to the family  $\mathcal{T}$  by adding an edge between two active vertices of  $T$ .

The following two lemmas prove that  $\gamma(G) = \gamma^2(G)$  for every graph  $G$  belonging to the family  $\mathcal{C}$ .

**Lemma 9.** *Each graph belonging to the family  $\mathcal{C}_6^+$  has equal domination and 2-distance domination numbers.*

**Proof.** Let  $G \in \mathcal{C}_6^+$ . Obviously  $\gamma(C_6) = \gamma^2(C_6)$ . Thus let  $G$  be obtained from a tree  $T$  belonging to the family  $\mathcal{T}$  and the cycle  $C_6 = (v_1, \dots, v_6, v_1)$  by identifying the vertex  $v_1$  with a support vertex of  $T$ .

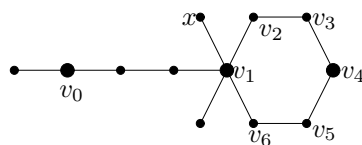


Figure 1. Graph  $G \in \mathcal{C}_6^+$ .  $\{v_0, v_1, v_4\}$  is the  $\gamma(G)$ -set.

Since  $G$  is unicyclic and connected,  $G - v_5v_6$  is a tree. It is no problem to observe, that  $G - v_5v_6$  may be obtained from  $T$  by adding to  $T$  first the path  $P_4 = (v_2, v_3, v_4, v_5)$  and the edge  $v_1v_2$ , and then  $v_6$  and the edge  $v_1v_6$ . Since  $T \in \mathcal{T}$  and  $G - v_5v_6$  may be obtained from  $T$  by operations  $\mathcal{T}_3$  and  $\mathcal{T}_1$ , we conclude that  $G - v_5v_6 \in \mathcal{T}$ . Thus by Lemma 4,  $\gamma^2(G - v_5v_6) = |S(G - v_5v_6)|$  and by Lemma 5,  $\gamma(G - v_5v_6) = \gamma^2(G - v_5v_6)$ .

Let  $D$  be a  $\gamma^2(G)$ -set. Since  $G$  is obtained from  $T$  and  $C_6$  by identifying  $v_1$  with a support vertex of  $T$  and  $\gamma^2(T) = |S(T)|$ ,  $|D| \geq |S(T)|$ . Denote by  $x$  a leaf adjacent to  $v_1$  in  $G$ . Then there exists a vertex  $y$  such that  $y \in N_G^2[x] \cap D$ . In any choice of  $y$ , at least one vertex belonging to  $\{v_1, \dots, v_6\} - \{y\}$  belongs also to  $D$  (because  $D$  is 2-distance dominating). Thus  $|D| \geq |S(T)| + 1$ . On the other hand,  $S(G) \cup \{v_4\}$  is a 2-distance dominating set of  $G$  of cardinality  $|S(G)| + 1$ . Thus

$$\begin{aligned} |S(G)| + 1 = \gamma^2(G) &\leq \gamma(G) \leq \gamma(G - v_5v_6) \\ &= \gamma^2(G - v_5v_6) = |S(G - v_5v_6)|. \end{aligned} \tag{1}$$

Since  $|S(G)| = |S(G - v_5v_6)| - 1$ , we have equalities throughout the inequality chain (1). In particular,  $\gamma^2(G) = \gamma(G)$ . ■

**Lemma 10.** *If  $G$  is a graph obtained from a tree  $T$  belonging to the family  $\mathcal{T}$  by adding an edge between two active vertices of  $T$ , then  $\gamma(G) = \gamma^2(G)$ .*

**Proof.** Let  $T$  be a tree belonging to the family  $\mathcal{T}$ . Denote by  $u$  and  $v$  two active vertices of  $T$  and let  $D$  be a  $\gamma^2(G)$ -set, where  $G = T + uv$ . If  $u$  and  $v$  are leaves adjacent to the same support vertex, then the result follows from Lemma 8.

Thus assume  $u$  and  $v$  are adjacent to different support vertices of  $T$  or at most one of  $u$  and  $v$  is a leaf. In both cases,  $S(T) = S(G)$  and similarly like in  $T$ , the distance between any two leaves adjacent to different support vertices in  $G$  is greater than or equal to 5. For this reason, if  $u$  and  $v$

are two leaves adjacent to different support vertices in  $G$ , then  $u$  and  $v$  cannot be 2-distance dominated by the same element of  $D$ . This implies that  $\gamma^2(G) \geq |S(G)|$ . Hence

$$|S(G)| \leq \gamma^2(G) \leq \gamma(G) \leq \gamma(T) = \gamma^2(T) = |S(T)| = |S(G)|.$$

Therefore  $\gamma(G) = \gamma^2(G)$ . ■

For a cycle  $C_n$  on  $n \geq 3$  vertices it is no problem to see that  $\gamma(C_n) = \lceil \frac{n}{3} \rceil$  and  $\gamma^2(C_n) = \lceil \frac{n}{5} \rceil$ .

**Lemma 11.** *If  $G$  is a connected unicyclic graph with  $\gamma(G) = \gamma^2(G)$ , then  $G$  belongs to the family  $\mathcal{C}$ .*

**Proof.** Let  $G$  be a unicyclic graph, where  $C_k = (v_1, \dots, v_k, v_1)$  is the unique cycle of  $G$ . If  $d_G(v_i) > 2$  for some  $v_i \in V(C_k)$ , then let  $T(v_i)$  be the tree attached to the vertex  $v_i$  and let  $v_i$  be the root of  $T(v_i)$ . Let  $D$  be a minimum dominating set of  $G$  containing all support vertices of  $G$ .

By Proposition 1, at most  $\lfloor \frac{k}{3} \rfloor$  vertices of  $C_k$  belong to  $D$  and the distance between any two elements of  $D$  is at least 3. Thus there exists an edge, without loss of generality say  $v_2v_3$  (where  $v_2, v_3 \in V(C_k)$ ), such that  $v_2 \notin D$  and  $v_3 \notin D$ . Note that neither  $v_2$  nor  $v_3$  is a support vertex. Since  $G$  is unicyclic and connected,  $G - v_2v_3$  is a tree. Moreover, by our assumptions and some immediate properties of the domination number of a graph,

$$\gamma(G) = \gamma^2(G) \leq \gamma^2(G - v_2v_3) \leq \gamma(G - v_2v_3). \quad (2)$$

However, since  $v_2, v_3 \notin D$ ,  $D$  is also a dominating set in  $G - v_2v_3$ . Therefore,  $\gamma(G) = \gamma(G - v_2v_3)$  and thus we have equalities throughout the inequality chain (2). In particular,  $\gamma^2(G - v_2v_3) = \gamma(G - v_2v_3)$  and since  $G - v_2v_3$  is a tree, Theorem 7 implies that  $G - v_2v_3$  belongs to the family  $\mathcal{T}$ . By Observation 3, each vertex of  $G - v_2v_3$  is a support vertex or is a neighbour of exactly one support vertex. Of course  $v_2, v_3 \notin S(G - v_2v_3)$ . Hence denote by  $s_2$  and  $s_3$  the support vertices adjacent in  $G - v_2v_3$  to  $v_2$  and  $v_3$ , respectively. Observe that  $s_2$  and  $s_3$  may not be support vertices in  $G$ .

If  $s_2 = s_3$ , then  $v_1 = s_2$ . If  $v_1$  is a support vertex in  $G$ , then  $G$  may be obtained from the tree  $G - v_2v_3$  by adding an edge between two active vertices adjacent to the same support vertex and thus  $G \in \mathcal{C}$ . If  $v_1 \notin S(G)$ , then at least one of  $v_2, v_3$  is of degree 2 in  $G$ . Assume first  $d_G(v_2) = d_G(v_3) = 2$ . Then  $v_2$  and  $v_3$  are leaves in  $G - v_2v_3$  and for



this reason  $G$  again may be obtained from the tree  $G - v_2v_3$  by adding an edge between two active vertices. Thus assume, without loss of generality,  $d_G(v_2) = 2$  and  $d_G(v_3) \geq 3$ . Observe that since  $v_1 \notin S(G)$ , every element of  $V(G) - \{v_1, v_2\}$  is within distance 2 from a vertex belonging to  $D - \{v_1\}$ . Thus,  $D - \{v_1\}$  2-distance dominates  $V(G) - \{v_1, v_2\}$ . Denote by  $x$  an element of  $D \cap V(T(v_3))$ , which is at distance 3 from  $v_1$  and let  $(x, y, v_3, v_1)$  be the shortest path from  $x$  to  $v_1$ . Define  $D' = (D - \{x, v_1\}) \cup \{y\}$ . Now every element of  $V(G)$  is within distance 2 from an element of  $D'$ , so  $D'$  is a 2-distance dominating set of  $G$  smaller than  $\gamma(G)$ , which contradicts that  $\gamma(G) = \gamma^2(G)$ .

In what follows we assume  $s_2 \neq s_3$  and we consider three cases.

1. If  $s_2 \in S(G)$  and  $s_3 \in S(G)$ , then  $v_2$  and  $v_3$  are both active vertices in  $G - v_2v_3$ . Therefore  $G$  may be obtained from the tree  $G - v_2v_3$  by adding the edge  $v_2v_3$  and thus  $G$  belongs to the family  $\mathcal{C}$ .

2. Without loss of generality, assume that  $s_2 \notin S(G)$  and  $s_3 \in S(G)$ . Then  $v_2$  is the unique leaf adjacent to  $s_2$  in  $G - v_2v_3$ . Therefore  $d_G(v_2) = 2$  and  $s_2 = v_1$ . Observe, that since  $v_1 \notin S(G)$ , each element of  $V(G) - \{v_1\}$  is within distance 2 from an element of  $D - \{v_1\}$ . Thus,  $D - \{v_1\}$  2-distance dominates  $V(G) - \{v_1\}$ .

If  $d_G(v_1) \geq 3$ , then since  $v_1$  is not a support vertex in  $G$ ,  $D \cap V(T(v_1)) \neq \emptyset$ . Denote by  $x$  an element of  $D \cap V(T(v_1))$ , which is at distance 3 from  $v_1$  and let  $(x, y, z, v_1)$  be the shortest path from  $x$  to  $v_1$ . Define  $D' = (D - \{x, v_1\}) \cup \{y\}$ . It is no problem to see that  $D'$  is a 2-distance dominating set of  $G$ , which contradicts that  $\gamma(G) = \gamma^2(G)$ . We conclude that  $d_G(v_1) = 2$ .

If  $s_3 \neq v_4$ , then  $d_G(v_3) \geq 3$ . Define  $D' = (D - \{s_3\}) \cup \{v_3\}$ . Then, since  $d_G(v_1, v_3) = 2$ ,  $D' - \{v_1\}$  is a 2-distance dominating set of  $G$ , contradicting that  $\gamma(G) = \gamma^2(G)$ . We conclude that  $s_3 = v_4$  and since  $v_4$  is a support vertex,  $d_G(v_4) \geq 3$  and  $v_1 \neq v_4$ . Moreover,  $v_5, v_6 \notin D$  and for this reason  $v_5, v_6 \notin S(G)$ . Denote by  $v_0$  a vertex belonging to  $D$  and at distance 2 from  $v_k$ . If  $v_0 \neq v_k$ , then  $(D - \{v_1, v_4\}) \cup \{v_3\}$  is a 2-distance dominating set of  $G$  of smaller cardinality than  $\gamma(G)$ , a contradiction. Therefore,  $v_0 = v_4$  and since  $d_G(v_4, v_k) = 2$  we obtain  $v_k = v_6$ .

We have already proven, that under our conditions  $d_G(v_1) = d_G(v_2) = 2$  and  $v_4$  is a support vertex. Suppose  $d_G(v_6) \geq 3$ . Then since  $v_6$  is not a support vertex in  $G$ ,  $D \cap V(T(v_6)) \neq \emptyset$ . Denote by  $x$  an element of  $D \cap V(T(v_6))$ , which is at distance 3 from  $v_1$  and let  $(x, y, v_6, v_1)$  be the shortest path from  $x$  to  $v_1$ . Define  $D' = (D - \{x, v_1\}) \cup \{y\}$ . Now  $D'$  is

a 2-distance dominating set of  $G$ , which contradicts that  $\gamma(G) = \gamma^2(G)$ . Therefore  $d_G(v_6) = 2$ .

Suppose  $d_G(v_5) \geq 3$ . Then since  $v_5$  is not a support vertex in  $G$ ,  $D \cap V(T(v_5)) \neq \emptyset$ . Denote by  $x$  an element of  $D \cap V(T(v_5))$ , which is at distance 3 from  $v_4$  and let  $(x, y, v_5, v_4)$  be the shortest path from  $x$  to  $v_4$ . Define  $D' = (D - \{x, v_1, v_4\}) \cup \{y, v_3\}$ . Now  $D'$  is a 2-distance dominating set of  $G$ , which contradicts that  $\gamma(G) = \gamma^2(G)$ . Therefore  $d_G(v_5) = 2$ . Similarly we prove that  $d_G(v_3) = 2$ .

Therefore,  $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_5) = d_G(v_6) = 2$  and  $v_4$  is a support vertex. Hence  $G$  may be obtained from a tree  $T$  and the cycle  $C_6$  by identifying one vertex of  $C_6$  with a support vertex of  $T$ . Clearly,  $D - \{v_1\}$  is a dominating set of  $T$ , so

$$\gamma^2(T) \leq \gamma(T) \leq \gamma(G) - 1 = \gamma^2(G) - 1. \quad (3)$$

On the other hand, any 2-distance dominating set of  $T$  may be extended to a dominating set of  $G$  by adding to it  $v_1$ . Thus  $\gamma^2(G) \leq \gamma^2(T) + 1$  and we have equalities through the inequality chain (3). In particular,  $\gamma^2(T) = \gamma(T)$ . By Theorem 7,  $T$  belongs to the family  $\mathcal{T}$ . Hence  $G$  may be obtained from  $T \in \mathcal{T}$  and the cycle  $C_6$  by identifying one vertex of  $C_6$  with a support vertex of  $T$ . Thus  $G \in \mathcal{C}_6^+$ .

3. If  $s_2 \notin S(G)$  and  $s_3 \notin S(G)$ , then  $d_G(v_2) = 2$  and  $d_G(v_3) = 2$ . Moreover,  $v_1 = s_2$  and  $v_4 = s_3$ . Since  $v_1$  is not a support vertex, each element of  $V(G) - \{v_1\}$  is within distance 2 from an element of  $D - \{v_1\}$ . Thus,  $D - \{v_1\}$  2-distance dominates  $V(G) - \{v_1\}$ . By the same reasoning,  $D - \{v_4\}$  2-distance dominates  $V(G) - \{v_4\}$ . Similarly as in previous case, we deduce that  $d_G(v_1) = d_G(v_4) = 2$ . Since  $v_1 \neq v_4$ , the unique cycle contains at least 6 vertices,  $v_5, v_6 \notin D$  and  $v_5, v_6 \notin S(G)$ .

If  $d_G(v_5) \geq 3$ , then since  $v_5$  is not a support vertex,  $D \cap V(T(v_5)) \neq \emptyset$ . Denote by  $x$  an element of  $D \cap V(T(v_5))$ , which is at distance 3 from  $v_4$  and let  $(x, y, v_5, v_4)$  be the shortest path from  $x$  to  $v_4$ . Define  $D' = (D - \{x, v_4\}) \cup \{y\}$ . Now  $D'$  is a 2-distance dominating set of  $G$ , which contradicts that  $\gamma(G) = \gamma^2(G)$ . Therefore  $d_G(v_5) = 2$ .

Since  $D$  is dominating,  $v_6$  has a neighbour in  $D$ . If there exists  $x \in N_G(v_6) \cap D$  such that  $x \neq v_1$ , then  $(D - \{v_1, v_4\}) \cup \{v_3\}$  is a 2-distance dominating set of  $G$ , which contradicts that  $\gamma(G) = \gamma^2(G)$ . Thus we conclude that  $\{v_1\} = N_G(v_6) \cap D$ . Therefore the unique cycle of  $G$  contains exactly 6 vertices. By similar reasoning as for  $v_5$ , we obtain that  $d_G(v_6) = 2$ . Hence

each vertex of the unique cycle is of degree 2 and  $G = C_2$ . Therefore  $G$  belongs to the family  $\mathcal{C}$ . ■

The following results are consequences of Theorem 7 and Lemmas 9 and 11.

**Theorem 12.** *Let  $G$  be a connected unicyclic graph. Then  $\gamma(G) = \gamma^2(G)$  if and only if  $G$  belongs to the family  $\mathcal{C}$ .* ■

**Theorem 13.** *Let  $G$  be a unicyclic graph. Then  $\gamma(G) = \gamma^2(G)$  if and only if exactly one connected component of  $G$  is a unicyclic graph belonging to the family  $\mathcal{C}$  and each other connected component of  $G$  is a tree belonging to the family  $\mathcal{T}$ .* ■

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Received 18 December 2009

Revised 15 June 2010

Accepted 25 August 2010