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COLOR-BOUNDED HYPERGRAPHS, V: HOST GRAPHS AND SUBDIVISIONS*

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Abstract

A color-bounded hypergraph is a hypergraph (set system) with vertex set X and edge set $\mathcal{E} = \{E_1, \ldots, E_m\}$, together with integers s_i and t_i satisfying $1 \le s_i \le t_i \le |E_i|$ for each $i = 1, \ldots, m$. A vertex coloring φ is proper if for every i, the number of colors occurring in edge E_i satisfies $s_i \le |\varphi(E_i)| \le t_i$. The hypergraph $\mathcal{H}$ is colorable if it admits at least one proper coloring.

We consider hypergraphs $\mathcal{H}$ over a "host graph", that means a graph G on the same vertex set X as $\mathcal{H}$, such that each E_i induces a *connected* subgraph in G. In the current setting we fix a graph or multigraph G_0, and assume that the host graph G is obtained by some sequence of edge subdivisions, starting from G_0.

The colorability problem is known to be NP-complete in general, and also when restricted to 3-uniform "mixed hypergraphs", i.e., color-bounded hypergraphs in which $|E_i| = 3$ and $1 \le s_i \le 2 \le t_i \le 3$ holds for all $i \le m$. We prove that for every fixed graph G_0 and natural number r, colorability is decidable in polynomial time over the class of r-uniform hypergraphs (and more generally of hypergraphs with

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$|E_i| \le r$ for all $1 \le i \le m$) having a host graph G obtained from G_0 by edge subdivisions. Stronger bounds are derived for hypergraphs for which G_0 is a tree.

Keywords: mixed hypergraph, color-bounded hypergraph, vertex coloring, arboreal hypergraph, hypertree, feasible set, host graph, edge subdivision.

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1. Introduction

In this paper we study a direction in hypergraph coloring theory that was initiated recently in [3] and, with an equivalent but different terminology, in [1]. Our main goal is to prove that two NP-hard problems — to decide whether there exists a proper coloring, and to find a coloring with minimum number of colors if there is at least one — become solvable in polynomial (linear) time under a certain type of structural and quantitative conditions. Moreover, we also obtain that the minimum number of colors in a proper coloring has an absolute upper bound over a structure class defined via topological condition.

Our note belongs to the systematic study carried out in a series of papers [3]–[7] concerning "color-bounded hypergraphs" (formal definitions will be given in the next subsection). This structure class was introduced to give a common generalization of the important class of *mixed hypergraphs* introduced in [14, 15], and a further interesting recent model introduced in [8]. For a detailed discussion on problems and results concernig mixed hypergraphs, we refer to the research monograph [16] and the recent survey [13]. In the other direction, an extension of the model of color-bounded hypergraphs can be found in [5, 6, 7]. All these models include the concept of proper hypergraph coloring as a particular case, hence our results presented here have direct consequences on this classical part of combinatorics, too.

1.1. Terminology

The following structure generalizes various earlier ones in the theory of graph coloring. A *color-bounded hypergraph* is a four-tuple

$$\mathcal{H} = \{X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}\}$$

where $(X, \mathcal{E})$ is a hypergraph (set system) with vertex set $X = \{x_1, \dots, x_n\}$ and edge set $\mathcal{E} = \{E_1, \dots, E_m\}$, and $\boldsymbol{s} : \mathcal{E} \to \mathbb{N}$ and $\boldsymbol{t} : \mathcal{E} \to \mathbb{N}$ are integer-valued functions satisfying for all $1 \le i \le m$ the following chain of inequalities:

$$1 \le \boldsymbol{s}(E_i) \le \boldsymbol{t}(E_i) \le |E_i|.$$

We shall assume throughout that X is finite and nonempty. The following notation will be used:

$$s_i := \boldsymbol{s}(E_i), \quad t_i := \boldsymbol{t}(E_i), \quad s = s(\mathcal{H}) := \max_{1 \le i \le m} s_i.$$

A *proper vertex coloring* of $\mathcal{H}$ is a mapping $\varphi : X \to \mathbb{N}$ such that the number $|\varphi(E_i)|$ of colors occurring in E_i satisfies

$$s_i \le |\varphi(E_i)| \le t_i \qquad \text{for all } \ 1 \le i \le m.$$

A *mixed hypergraph* is obtained if each edge is of the type

$$(s_i, t_i) \in \{(1, |E_i| - 1), (2, |E_i|), (2, |E_i| - 1)\}.$$

Moreover, proper coloring of a hypergraph in the classical sense exactly means the conditions $s_i = 2$ and $t_i = |E_i|$ for all i. The recent model of [8] keeps the condition $t_i = |E_i|$ and allows s_i to be given arbitrarily.

Host graphs. Suppose that $\mathcal{H} = (X, \mathcal{E})$ is a hypergraph, and $G = (X, F)$ is a graph over the same vertex set X as $\mathcal{H}$, and with edge set F. We say that G is a *host graph* of $\mathcal{H}$ if each $E \in \mathcal{E}$ induces a *connected* subgraph in G. In some cases, if G belongs to an important graph class, particular terminology is applied for $\mathcal{H}$ as follows:

- G is a path $\to$ $\mathcal{H}$ is an *interval hypergraph*;
- G is a tree $\to$ $\mathcal{H}$ is a *hypertree*, also called *arboreal hypergraph*;
- G is a cycle $\to$ $\mathcal{H}$ is a *circular hypergraph*;
- G is a cactus $\to$ $\mathcal{H}$ is a *hypercactus.*

Those classes of hypergraphs have a different behavior with respect to colorability, despite that their host graphs may look not very much different.

It should be noted that the host graph of $\mathcal{H}$ is not unique, and we usually prefer one with as simple structure as possible. For example, every interval hypergraph also is a hypertree and a circular hypergraph at the same time.

Feasible sets, chromatic numbers. We say that $\mathcal{H}$ is *colorable* if it admits at least one proper coloring, and is *uncolorable* otherwise.

If we assume that $\mathcal{H}$ is colorable, several further interesting notions arise. The *feasible set* $\Phi(\mathcal{H})$ of $\mathcal{H}$ is the set of integers $k \in \mathbb{N}$ such that $\mathcal{H}$ admits at least one *strict k-coloring*, that is a coloring with *precisely* k colors. The *lower* and *upper chromatic number* of $\mathcal{H}$ are defined as the *smallest* and *largest* number of colors in $\Phi(\mathcal{H})$, respectively. In notation,

$$\chi(\mathcal{H}) = \min \Phi(\mathcal{H}), \qquad \bar{\chi}(\mathcal{H}) = \max \Phi(\mathcal{H}).$$

A *gap* of the feasible set is a "missing" integer $k \notin \Phi(\mathcal{H})$ such that $\chi(\mathcal{H}) < k < \bar{\chi}(\mathcal{H})$. If at least one gap occurs, the feasible set is said to be *broken*, otherwise it is called *gap-free* or *continuous*. Some problems related to gaps will be mentioned in the concluding section.

Subdivisions. Let G_0 be a fixed graph, *allowing loops and multiple edges*, i.e., not necessarily a simple graph. A *subdivision* of an edge $e = uv \in E(G_0)$ is the replacing of e by a path whose endpoints are u and v; evidently, internal vertices of such path are not in $V(G_0)$. A subdivision of G_0 is obtained by a sequence (possibly empty) of subdivisions of some edges of G_0. For short, G_0-*subdivision* means any graph that is a subdivision of G_0, including G_0.

1.2. Results of this paper

The main issue in our present work is to prove that G_0-subdivisions as host graphs yield a universal upper bound $f(G_0, r, s)$ on the lower chromatic numbers of all color-bounded hypergraphs having rank (the size of a largest hyperedge) at most r and satisfying $\max s_i \le s$, for any fixed graph G_0 and natural numbers r, s. This will be proved in Section 2. There we also consider the algorithmic complexity of deciding whether a given hypergraph is colorable; and if it is, then to determine its lower chromatic number.

In Section 3 we show that the bound $f(G_0, r, s)$ can be made independent of rank r whenever G_0 is a tree. Remarks on the estimates concerning f are given in Section 4, and further problems are raised in Section 5.

Since the classical concept of proper hypergraph coloring exactly means $s_i = 2$ for all $1 \le i \le m$, our theorems immediately imply the analogous results for the chromatic number of hypergraphs in the usual sense.

2. Results for Unrestricted G_0

Here we prove the following results.

Theorem 1. *For every fixed graph G_0 and natural numbers r and s, there exists a constant $f(G_0, r, s)$ with the following property: If $\mathcal{H} = \{X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}\}$ is a colorable color-bounded hypergraph with $|E_i| \le r$ and $s_i \le s$ for all $1 \le i \le m$, and $\mathcal{H}$ admits a host graph G that is a G_0-subdivision, then $\chi(\mathcal{H}) \le f(G_0, r, s)$.*

Theorem 2. *For every fixed graph G_0 and natural numbers r and s, there exists a linear-time algorithm that solves the following decision and search problems: Given an input hypergraph $\mathcal{H} = \{X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}\}$ with $|E_i| \le r$ and $s_i \le s$ for all $1 \le i \le m$, and its host graph G that is a G_0-subdivision, decide whether $\mathcal{H}$ is colorable, and if it is, then determine the lower chromatic number.*

Comments on these theorems and related problems will be discussed in Sections 4 and 5. The rest of this section is devoted to the proof of Theorems 1 and 2.

Proof. Let G be any subdivision of G_0, and $\mathcal{H} = \{X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}\}$ a colorable color-bounded hypergraph over the host graph G. The graph G can be represented with an edge-weight function over G_0,

$$w : E(G_0) \to \mathbb{N}$$

where the weight $w(e)$ of an edge e in G_0 is the length of the path in G into which e has been subdivided. Moreover, we assume that

$$\varphi : X \to \mathbb{N}$$

is a proper coloring of $\mathcal{H}$. Our goal is to modify φ to another proper coloring, until the number of colors becomes so small that a constant upper bound $f(G_0, r, s)$ can be guaranteed for it.

The main idea is that the colors of φ will be kept on the vertices originated from G_0 and on vertices in their appropriately fixed surroundings; and then, using a result from [4], we shall prove that all the remaining vertices of $\mathcal{H}$ can be properly recolored using at most s further colors.

(1) Reducing the number of colors

We will specify a subset X_r of X. First, let the set of vertices of G originated from the vertices of G_0 be denoted by X_0, and for every $x \in X_0$ let x^* denote the corresponding vertex in G_0. Moreover, we introduce the notation $d = \lceil r/2 \rceil - 1$ and define X_r to be the set of vertices in G that are at distance at most d from some $x \in X_0$. This means that if $w(e) \le 2d + 1$ for some $e = x^*y^*$ in G_0, then the corresponding x–y path entirely belongs to the subgraph induced by X_r in G; and each connected component of $G - X_r$ is some path P, resulting from an edge $e_P \in E(G_0)$ whose weight is $w(e_P) = |V(P)| + 2d + 1$. Note further that the distance in G between any two components of $G - X_r$ is at least $2d + 2 \ge r$ (because all connections between them pass through at least one vertex of X_0), and consequently every edge of $\mathcal{H}$ can meet at most one component of $G - X_r$.

Recall that $\varphi : X \to \mathbb{N}$ is a proper vertex coloring of $\mathcal{H}$. We call a color *rigid* if it appears on some vertex of X_r, and call it *flexible* if it is in $\varphi(X) \setminus \varphi(X_r)$. The set of vertices $x \in X$ having rigid color in φ will be denoted by X_R. Clearly, $X_r \subseteq X_R$ holds but X_R may also contain some vertices from $X \setminus X_r$. By a renumbering of colors we can ensure that all rigid colors are larger than s.

By what has been said, if two vertices of flexible color(s) are contained in a common edge of $\mathcal{H}$, then they are in the same component of $G - X_r$. For each component P of $G - X_r$ we construct an *interval hypergraph* $\mathcal{H}_P = (X_P, \mathcal{E}_P)$ as follows:

$$X_P = V(P) \setminus X_R, \quad \mathcal{E}_P = \{E_i \cap X_P : E_i \in \mathcal{E} \ \text{ and } \ |E_i \cap X_P| \ge 2\}.$$

For an edge $E = X_P \cap E_i$ of $\mathcal{H}_P$ we define

$$s(E) = \max\{s(E_i) - \rho(E_i), 1\}, \qquad t(E) = t(E_i) - \rho(E_i)$$

where $\rho(E_i)$ denotes the number of rigid colors in E_i. Since $|E| > 1$ holds by definition, E contains at least one flexible color, and so E_i has strictly fewer than $t(E_i)$ rigid colors. By assumption, φ is a proper coloring of $\mathcal{H}$, therefore the inequalities

$$1 \le s(E) \le |\varphi(E)| \le t(E) \le |E|$$

are valid and each subhypergraph $\mathcal{H}_P$ is colorable. Some edge $E \in \mathcal{E}_P$ can belong to more than one edge E_i of $\mathcal{E}$ and so, the values $s(E)$ and $t(E)$ can be multiply defined. In this case we keep the strongest constraints; i.e., the largest value of $s(E)$ and the smallest value of $t(E)$.

By the result of [4], for every interval hypergraph $\mathcal{H}_P$ the lower chromatic number equals the maximum value of its color-bound function $\boldsymbol{s}$, that is

$$\chi(\mathcal{H}_P) = \max_{E \in \mathcal{E}_P} s(E) \le s.$$

Thus, due to the large distance between the components of $G - X_r$, there exists a coloring

$$\varphi^+ : X \setminus X_R \to \{1, \ldots, s\}$$

that properly colors each component $\mathcal{H}_P$. Now we define

$$\varphi^*(x) = \begin{cases} \varphi(x) & \text{if } x \in X_R, \\ \varphi^+(x) & \text{if } x \in X \setminus X_R. \end{cases}$$

This φ^* properly colors all edges of $\mathcal{H}$, and so the number of colors in $\varphi^*(X)$ is an upper bound on the lower chromatic number:

$$\chi(\mathcal{H}) \le s + |\varphi(X_r)| \le s + |X_r| \le s + |V(G_0)| + (r-1) \cdot |E(G_0)|,$$

where the number of rigid colors is estimated from above by the largest possible number of vertices in X_r.

(2) Algorithm for deciding colorability

Let us denote $k^* = s + |V(G_0)| + (r-1) \cdot |E(G_0)|$, the universal upper bound on the lower chromatic number. If $\mathcal{H}$ is colorable, then it admits at least one proper coloring

$$\varphi : X \to \{1, 2, \ldots, k^*\}.$$

Having fixed G_0, r and s, this means a bounded number of colors. Moreover, $|X_r|$ is bounded, therefore the number of proper colorings of the subhypergraph induced by X_r with colors taken from $\{s+1, \ldots, k^*\}$ is bounded, too. Also, the number of components P in $G - X_r$ is bounded above by $|E(G_0)|$. To complete the proof, it will suffice to show that it can be decided in linear time for each component P whether an arbitrarily specified coloring of X_r can be extended to a proper coloring of the subhypergraph $\mathcal{H}_P^+$ induced by $X_r \cup V(P)$ in $\mathcal{H}$.

Assume a given coloring φ_r on X_r, and consider a component P in $G - X_r$. Let $V(P) = \{x_1, \ldots, x_q\}$. If $q \leq r$, we simply take all the possible q-tuples $(a_1, \ldots, a_q)$ over the color set $\{1, 2, \ldots, k^*\}$ and for each case check whether or not the extension of φ_r by the assignments $\varphi(x_j) = a_j$ $(1 \leq j \leq q)$ properly colors every edge $E_i \in \mathcal{E}$ intersecting P.

If $q > r$, we construct an auxiliary digraph F that has a linearly arranged structure and whose vertices represent r-tuples over the color set $\{1, 2, \ldots, k^*\}$. The vertex set of F is $V_1 \cup \cdots \cup V_{q-r+1}$, where each V_i consists of certain color sequences $(a_1, \ldots, a_r) \in \{1, 2, \ldots, k^*\}^r$ of length r. Any two elements $u \in V_i$ and $v \in V_j$ $(i \neq j)$ are considered to be different, even if they correspond to the same color sequence. The condition for a sequence $(a_1, \ldots, a_r)$ to be included in V_i is as follows.

- Assigning color a_j to x_{i+j-1} for all $j = 1, \ldots, r$, each $E_\ell \in \mathcal{E}$ contained wholly in the interval $\{x_i, x_{i+1}, \ldots, x_{i+r-1}\}$ is properly colored.
- If $i = 1$ or $i = q - r + 1$, a further constraint is that the corresponding extension of φ_r properly colors all edges intersecting both X_r and P.

We put a directed edge from $(a_1, \ldots, a_r) \in V_i$ to $(a'_1, \ldots, a'_r) \in V_{i+1}$ if and only if omitting the first element of the former and the last element of the latter, we get the same sequence; that is, $a_2 = a'_1$, $a_3 = a'_2, \ldots, a_r = a'_{r-1}$.

By construction, every proper coloring of $\mathcal{H}$ with at most k^* colors defines a subset of $V(F)$, one vertex from each V_i, corresponding to the color sequence of length r starting at x_i $(i = 1, \ldots, q - r + 1)$. Moreover, this subset is a directed path from V_1 to V_{q-r+1}. Conversely, we see that any directed path from V_1 to V_{q-r+1} in F defines a vertex coloring of P that extends φ_r and properly colors all those edges of $\mathcal{H}$ which have at least one vertex in $\{x_1, \ldots, x_q\}$. Thus, there exists some coloring of $V(P)$ which properly extends φ_r if and only if there is a $V_1 \to V_{q-r+1}$ directed path. Moreover, $\mathcal{H}$ is colorable if and only if for at least one coloring of X_r such a path exists for every component P of $G - X_r$.

One can construct F and test the existence of a $V_1 \to V_{q-r+1}$ path in $O(|V(P)|)$ time for every fixed 3-tuple (φ_r, r, G_0). Indeed, for every $1 \leq i \leq q - r + 1$ the set V_i can be determined by checking each of the possible $(k^*)^r$ color sequences for the bounded number of edges from $\mathcal{H}_P^+$. Moreover, edges in F occur between consecutive sets V_i, V_{i+1} only, and for each such pair of sets we have to investigate adjacency for at most $(k^*)^{2r}$ vertex pairs.

Thus, the overall running time of the algorithm is linear in the sum of the edge weights of G_0 and hence, it is linear in $|X|$.

(3) Algorithm for the lower chromatic number

Running the previous algorithm for $k = s, s+1, \ldots, s + |V(G_0)| + (r-1) \cdot |E(G_0)|$ we can identify the smallest integer k admitting a proper k-coloring of $\mathcal{H}$. This gives the correct value of χ because the lower chromatic number is in the given range whenever $\mathcal{H}$ is colorable. ■

3. Bounds for Hypertrees

If G_0 is a tree, we can give a better upper bound on the lower chromatic number, using only the terms s and $|E(G_0)|$; that is, independently of the sizes of the edges. A major tool is the following:

Recoloring Lemma ([4]). *Let a color-bounded hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ and a proper coloring φ of $\mathcal{H}$ be given. Consider two colors $\alpha, \beta \in \varphi(X)$, a partition of the vertex set X into three parts (A, B, C), and the following set of conditions:*

(1) *$\alpha \notin \varphi(B)$ and $\beta \notin \varphi(B)$.*

(2) *For every edge $E_i \in \mathcal{E}$ intersecting both A and C:*

  (a) *$\alpha \in \varphi(E_i \cap C)$;*

  (b) *If $\alpha \in \varphi(E_i \cap A)$, then $\beta \in \varphi(E_i)$; and*

  (c) *$|\varphi(E_i \cap B)| \geq s_i - 1$.*

If the conditions (1) and (2) hold, then a coloring φ' obtained from φ by transposing colors α and β on the vertex set C is proper.

Here we prove:

Theorem 3. *For every fixed tree G_0 and natural number s, there exists a constant $f(G_0, s)$ with the following property: If $\mathcal{H} = \{X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}\}$ is a colorable color-bounded hypertree with $s_i \leq s$ for all $1 \leq i \leq m$, and $\mathcal{H}$ admits a host tree G that is a G_0-subdivision, then $\chi(\mathcal{H}) \leq f(G_0, s)$. Moreover, a coloring realizing this bound can be determined in polynomial time, provided that a proper vertex coloring of $\mathcal{H}$ is given in the input.*

Proof. We use the notation introduced in the proof of Theorem 1. Moreover an $x-y$ path (together with its endpoints) in G obtained by the subdivision of the edge $x^*y^* \in E(G_0)$ will be called a *subdivision-path.* It can be supposed that the tree G_0 is rooted at a vertex v^* and, correspondingly, G is rooted at a vertex v. Using the Recoloring Lemma as cited above from [4], we shall prove that for every subdivision-path it is enough to use at most s different colors.

Let φ be a proper coloring of $\mathcal{H}$, and suppose that some subdivision-path Q is colored with at least $s+1$ colors. We choose such a Q which is nearest to the root.

Introduce the notation $Q = x_0, x_1, \ldots, x_k$, where the endpoint x_0 is nearer to the root than x_k. Then determine the smallest i for which the interval $[x_0, x_i]$ contains exactly $s+1$ colors. Moreover, consider the largest j such that $[x_j, x_i]$ contains all the $s+1$ colors. Hence, the interval $[x_{j+1}, x_{i-1}]$ has exactly $s-1$ colors, but involves no vertices with color $\alpha := \varphi(x_i)$ or $\beta := \varphi(x_j)$. Moreover, if a hyperedge starts before x_{j+1} and ends after x_{i-1}, it necessarily contains both colors α and β. Thus, the conditions of the Recoloring Lemma are satisfied for $B = [x_{j+1}, x_{i-1}]$, for C which is chosen as the vertex set of the subtree of G rooted in x_i, and for $A = X \setminus (B \cup C)$. Consequently, switching the colors α and β on the vertices of C, a proper coloring φ' is obtained.

Indeed, if for an edge $E_k \in \mathcal{E}$ at least one of the relations $E_k \subseteq A \cup B$ and $E_k \subseteq B \cup C$ holds, then $|\varphi'(E_k)| = |\varphi(E_k)|$; i.e., E_k remains properly colored, whilst on the edges intersecting both A and C, the number of colors can be decreased by 1, but it still remains at least s.

The colorings φ and φ' induce the same color partition on every division-path except Q, where the maximal starting interval having exactly s colors is longer by at least one in the new coloring. The recoloring procedure can be repeated as long as Q has more than s colors (this means at most $|Q| - s$ phases) and, finally, we obtain a proper coloring of $\mathcal{H}$ using exactly s colors on Q. Note that this coloring of Q will remain unchanged in the later phases.

For the next phase we choose again a subdivision-path colored with at least $s+1$ colors (if there exists such a path) nearest to the root, and the above procedure is repeated.

Finally, we get a coloring φ^* of $\mathcal{H}$, which colors each subdivision-path with at most s different colors. Since we considered the paths together with their endpoints, any two consecutive ones have at least one common color. Thus, we obtain

$$\chi(\mathcal{H}) \leq (s-1)|E(G_0)| + 1$$

because the algorithm generates a coloring with at most that many colors, independently of the sizes of edges.

Each phase of the algorithm takes linear time in the number $|X|$ of vertices, and not more than $|X|$ phases were needed. Therefore, the algorithm terminates in at most quadratic time. ■

4. Improvements and Modified Bounds

In the proofs of the previous two sections it was not our aim to prove estimates in their strongest forms. In this section we describe some ways to make bounds better.

4.1. General host graphs

Here we put some remarks concerning the results of Section 2. A general idea will be that with a careful choice of G_0 we may achieve better bounds for the lower chromatic number of a given hypergraph $\mathcal{H}$.

Assume first that the host graph G is disconnected, say with connected components $G_1, \ldots, G_k$. Then $\mathcal{H}$ can be decomposed into subhypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_k$ where an edge of $\mathcal{H}$ belongs to $\mathcal{H}_j$ $(1 \leq j \leq k)$ if and only if it is contained in $V(G_j)$. Then $\chi(\mathcal{H}) = \max\{\chi(\mathcal{H}_1), \ldots, \chi(\mathcal{H}_k)\}$.

From now on we assume that the host graph G is connected. If G is a cycle, then $\mathcal{H}$ is a circular hypergraph, and the results of [4] yield $\chi(\mathcal{H}) \leq 2s - 1$. This means a universal upper bound independent of r, for the case when G_0 is the 1-vertex graph with just one loop. This also includes the class of interval hypergraphs, corresponding to the case $G_0 = K_2$, for which the stronger result $\chi(\mathcal{H}) = \max_{1 \leq i \leq m} s_i \leq s$ was proved in [4].

As a principle for simplification, we may also assume without loss of generality that G_0 has no vertices of degree two. Indeed, if x is a vertex of G_0 adjacent to x' and x'', then we may remove x and insert a new edge $x'x''$. (This may create a multiple edge if x' and x'' are adjacent in G_0.) Denoting by G_0' the graph obtained, it is clear that G is a subdivision of G_0' as well, and G_0' has fewer vertices and edges than G_0.

Having assumed that G_0 is connected, we have $|V(G_0)| \leq |E(G_0)| + 1$ and $s \leq r$. Hence, the bound given in Theorem 1 can be modified to a dependence only on r and $|E(G_0)|$, namely

$$\chi(\mathcal{H}) \leq s + |V(G_0)| + (r-1) \cdot |E(G_0)| \leq r(|E(G_0)| + 1) + 1.$$

Another way to improve the bound on $f(G_0, r, s)$ is to take into consideration that every vertex $x \in X$ of degree one in G can be omitted from X_0.

Therefore, if ℓ_0 denotes the number of pendant vertices in G_0, the improved bound

$$\chi(\mathcal{H}) \le s + |V(G_0)| + (r-1) \cdot |E(G_0)| - \ell_0 \left\lceil \frac{r}{2} \right\rceil$$

is valid.

4.2. Hypertrees

Here we put some remarks concerning Theorem 3.

For small values $s = 1$ and $s = 2$, the upper bound on $\chi(\mathcal{H})$ can be given independently of the structure of tree G_0. Indeed, if $\max s_i = 1$ then $\chi(\mathcal{H}) = 1$, whilst in a colorable hypertree with $\max s_i = 2$ we can first contract each edge E_i with $t_i = 1$ to a single vertex and then color alternately the contracted host tree with two colors. In particular, if $\mathcal{H}$ is a colorable *mixed* hypertree with a set of non-monochromatic edges $\mathcal{D} \neq \emptyset$, then $\chi(\mathcal{H}) = 2$, independently of the features of its host tree [12].

But if $s \ge 3$ holds and the structure of the tree G_0 is not prescribed, then the lower chromatic number of $\mathcal{H}$ is not bounded. As a simple example, let us consider the hypertree $\mathcal{H}_\ell = (X_\ell, \mathcal{E}_\ell)$, where $X_\ell = \{v, x_1, \ldots, x_\ell\}$ and

$$\mathcal{E}_\ell = \{\{v, x_i, x_j\} : 1 \le i < j \le \ell\}$$

and every edge $E_k \in \mathcal{E}_\ell$ has color-bounds $s_k = t_k = 3$. It is easy to see that $\chi(\mathcal{H}_\ell) = \ell + 1$ for every $\ell \ge 2$, whilst $s = 3$ remains valid. In this case $\chi(\mathcal{H}_\ell)$ depends on $\Delta(G_0)$ because G_0 is isomorphic to a star on $l + 1$ vertices.

Moreover, we note that the lower chromatic number cannot be bounded above in terms of s and the maximum vertex degree Δ of G_0, either. (If $\Delta = 2$ then we get interval hypergraphs, and the lower chromatic number equals $\max s_i$). In the previous example $\mathcal{H}_\ell$, one can replace v with a tree T' of maximum degree 3 on a vertex set X', and extend it to a host tree T by joining at most two of the x_i to each leaf of T'. (To do this, we need $|X'| \ge \ell - 2$.) The edges of $\mathcal{H}_\ell$ then become $x_i - x_j$ paths in T, keeping the condition $\boldsymbol{s} = \boldsymbol{t} = 3$. Taking X' as a further edge with $\boldsymbol{s}(X') = \boldsymbol{t}(X') = 1$, the lower chromatic number gets arbitrarily large as ℓ grows.

5. OPEN PROBLEMS

Here we raise some related problems, and in the last subsection we propose a case study which seems to be of interest on its own right, too.

Problem 1. Find estimates on the coefficient of linearity for algorithms guaranteed by Theorem 2.

Problem 2. Give necessary and/or sufficient conditions on G_0 such that $f(G_0, r, s)$ can be bounded above by a function $f(G_0, s)$ of s, independent of r.

Motivated by the study of hypertrees, we raise the following variant of this question.

Problem 3. For which classes of graph G_0 does there exist a universal upper bound $f(\Delta(G_0), r)$ in terms of the maximum degree of the host graph and the rank of the hypergraph? Do all trees belong to this class?

It is known that the possible feasible sets over the class of mixed hypergraphs are almost unrestricted [9], and nearly the same is true for r-uniform mixed hypergraphs [2]. On the other hand, in mixed interval hypergraphs, hypertrees and more generally in mixed hypergraphs over host graphs in which all cycles are mutually vertex-disjoint, the feasible sets are gap-free [9, 11]. The situation is not so clear, however, for color-bounded hypergraphs, except that the non-2-colorable ones have a rich family of feasible sets [4].

Problem 4. Characterize those graphs G_0 for which the feasible set of a color-bounded hypergraph is gap-free whenever its host graph is a G_0-subdivision.

A construction in [4] yields a hypertree of maximum degree 5 with gap in its feasible set. Let us recall further the following problem from the same paper.

Problem 5. What is the time complexity of deciding colorability and determining the upper chromatic number of color-bounded interval hypergraphs, that is those with $G_0 = K_2$?

The decision problem of colorability is intractable already on 3-uniform color-bounded hypertrees [4]. Although bounded rank makes a difference concerning G_0-subdivisions, for unrestricted edge sizes it still may be of essence in the last part of Theorem 3 that a proper coloring be given in the input. For *mixed* hypertrees the upper chromatic number is hard to determine [10], whereas colorability can be decided efficiently and the lower chromatic number can be computed in polynomial time [12]. The following question is open:

Problem 6. What is the time complexity of determining the lower chromatic number of colorable 3-uniform color-bounded hypertrees?

Remark 1. The analogous problem for 4-uniform hypertrees is already NP-complete, as shown by the following simple reduction from the Hypergraph 2-Coloring problem. To any 3-uniform hypergraph $\mathcal{H} = (X, \mathcal{E})$ we adjoin a new vertex w and modify $\mathcal{H}$ to the hypergraph $\mathcal{H}'$ whose edges are the quadruples $E' := E \cup \{w\}$ for all $E \in \mathcal{E}$. This $\mathcal{H}'$ clearly is a hypertree over the host star centered at w. Let all extended edges have $s = 3$ and $t = 4$. Then, whatever color we assign to w, in any coloring of $\mathcal{H}'$ the triples of $\mathcal{H}$ should still get at least two further colors. Hence, if $\mathcal{H}'$ has lower chromatic number 3, then $\mathcal{H}$ is 2-colorable in the standard sense. Also conversely, every proper 2-coloring of $\mathcal{H}$ can be extended to a 3-coloring of $\mathcal{H}'$ by assigning a third color to w. The reduction requires linear time, which proves intractability.

Simplifying to the case $\boldsymbol{s}(E) \in \{1, 2\}$ and $\boldsymbol{t}(E) \in \{|E|-1, |E|\}$ for all $E \in \mathcal{E}$, we ask:

Problem 7. Consider the class of r-uniform mixed hypergraphs whose host graph is a graph derived from a graph G_0. What are the inclusion-wise minimal graphs G_0 for which the maximum lower chromatic number of these r-uniform mixed hypergraphs tends to infinity with r? In particular, what is the smallest such G_0 ?

5.1. Example: Petersoid hypergraphs

In graph theory, there is a huge literature already about the Petersen graph alone. For this reason we propose here a related case study to be pursued in later research.

Let G_0 be the Petersen graph. We call a color-bounded hypergraph *Petersoid* if it is derived from G_0 ; that is, its host graph is a subdivision of the Petersen graph. These structures may have many interesting properties, which we propose to explore in detail. More formally, below we explicitly formulate some questions about their lower chromatic numbers.

Problem 8. Describe the complete characterization of $\chi(\mathcal{H})$ for all Petersoid hypergraphs $\mathcal{H}$. In particular, determine the exact value of $f(G_0, r, s)$ for all pairs r, s of integers.

Special attention is paid to *mixed hypergraphs*, what means $s_i \in \{1, 2\}$ and $t_i \in \{|E_i| - 1, |E_i|\}$ for all edges E_i. In this case s is omitted from the universal upper bound, i.e., we have a function $f(G_0, r)$. (If $s = 1$, then trivially the lower chromatic number is equal to 1.) This leads to the following subproblem.

Problem 9. Determine the exact value of $f(G_0, r)$ over the class of Petersoid mixed hypergraphs.

Currently we do not even known whether $f(G_0, r)$ can be arbitrarily large or there exists an upper bound $f(G_0)$ independent of r.

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