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INTERVAL EDGE COLORINGS OF SOME PRODUCTS OF GRAPHS

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Abstract

An edge coloring of a graph G with colors $1, 2, \ldots, t$ is called an interval t-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of G colored by i, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. A graph G is interval colorable, if there is an integer $t \geq 1$ for which G has an interval t-coloring. Let $\mathfrak{N}$ be the set of all interval colorable graphs. In 2004 Kubale and Giaro showed that if $G, H \in \mathfrak{N}$, then the Cartesian product of these graphs belongs to $\mathfrak{N}$. Also, they formulated a similar problem for the lexicographic product as an open problem. In this paper we first show that if $G \in \mathfrak{N}$, then $G[nK_1] \in \mathfrak{N}$ for any $n \in \mathbf{N}$. Furthermore, we show that if $G, H \in \mathfrak{N}$ and H is a regular graph, then strong and lexicographic products of graphs G, H belong to $\mathfrak{N}$. We also prove that tensor and strong tensor products of graphs G, H belong to $\mathfrak{N}$ if $G \in \mathfrak{N}$ and H is a regular graph.

Keywords: edge coloring, interval coloring, regular graph, products of graphs.

2010 Mathematics Subject Classification: Primary: 05C15, 05C76; Secondary: 05C70.

1. Introduction

An edge coloring of a graph G with colors $1, 2, \ldots, t$ is called an interval t-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of G colored by i, and the colors of edges incident to any vertex of G are distinct and form an interval of integers. Interval edge colorings naturally arise in scheduling problems and are related to the problem of constructing timetables without "gaps"for teachers and classes. The notion of interval edge colorings was introduced by Asratian and Kamalian [1] in 1987. In [1] they proved that if a triangle-free graph $G = (V, E)$ has an interval t-coloring, then $t \leq |V| - 1$. In [19] interval edge colorings of complete bipartite graphs and trees were investigated. Furthermore, Kamalian [20] showed that if G admits an interval t-coloring, then $t \leq 2|V| - 3$. Giaro, Kubale and Małafiejski [12] proved that this upper bound can be improved to $2|V| - 4$ if $|V| \geq 3$. For a planar graph G, Axenovich [5] showed that if G has an interval t-coloring, then $t \leq \frac{11}{6}|V|$. In general, it is an NP-complete problem to decide whether a given bipartite graph G admits an interval edge coloring [35]. In papers [2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 23, 29, 30, 32, 34] the problem of existence and construction of interval edge colorings was considered and some bounds for the number of colors in such colorings of some classes of graphs were given. Surveys on this topic can be found in some books [3, 18, 25].

The different products of graphs were introduced by Berge [6], Sabidussi [33] and Vizing [36]. There are many papers [17, 24, 26, 27, 28, 31, 38] devoted to edge colorings of various products of graphs. In this paper we investigate interval edge colorings of various products of graphs.

2. Definitions and Preliminary Results

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. The maximum degree of a vertex of G is denoted by $\Delta(G)$ and the chromatic index of G by $\chi'(G)$. A partial edge coloring of G is a coloring of some of the edges of G such that no two adjacent edges receive the same color. If α is a partial edge coloring of G and $v \in V(G)$ then $S(v, \alpha)$ denotes the set of colors of colored edges incident to v.

A graph G is interval colorable, if there is an integer $t \geq 1$, for which G has an interval t-coloring. Let $\mathfrak{N}$ be the set of all interval colorable graphs

[1, 20]. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The Cartesian product $G\Box H$ is defined as follows:

$$V(G\Box H) = V(G) \times V(H),\ E(G\Box H) = \{((u_1, v_1), (u_2, v_2))| \\ u_1 = u_2 \text{ and } (v_1, v_2) \in E(H) \textit{ or } v_1 = v_2 \text{ and } (u_1, u_2) \in E(G)\}.$$

The tensor (direct) product $G \times H$ is defined as follows:

$$V(G \times H) = V(G) \times V(H), \\ E(G \times H) = \{((u_1, v_1), (u_2, v_2))|\ (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H)\}.$$

The strong tensor (semistrong) product $G \otimes H$ is defined as follows:

$$V(G \otimes H) = V(G) \times V(H),\ E(G \otimes H) = \{((u_1, v_1), (u_2, v_2))| \\ (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H) \text{ or } v_1 = v_2 \text{ and } (u_1, u_2) \in E(G)\}.$$

The strong product $G \boxtimes H$ is defined as follows:

$$\begin{aligned} &V(G \boxtimes H) = V(G) \times V(H), \\ &E(G \boxtimes H) = \{((u_1, v_1), (u_2, v_2))|\ (u_1, u_2) \in E(G) \\ &\text{and } (v_1, v_2) \in E(H) \text{ or } u_1 = u_2 \\ &\text{and } (v_1, v_2) \in E(H) \text{ or } v_1 = v_2 \text{ and } (u_1, u_2) \in E(G)\}. \end{aligned}$$

The lexicographic product (composition) $G[H]$ is defined as follows:

$$\begin{aligned} &V(G[H]) = V(G) \times V(H), \\ &E(G[H]) = \{((u_1, v_1), (u_2, v_2))|\ (u_1, u_2) \in E(G) \\ &\text{or } u_1 = u_2 \text{ and } (v_1, v_2) \in E(H)\}. \end{aligned}$$

The terms and concepts that we do not define can be found in [37].

Asratian and Kamalian proved the following:

Theorem 1 [1]. *Let G be a regular graph. Then*

(1) *$G \in \mathfrak{N}$ if and only if $\chi'(G) = \Delta(G)$.*

(2) *If $G \in \mathfrak{N}$ and $\Delta(G) \le t \le W(G)$, then G has an interval t-coloring.*

Corollary 2. *If G is an r-regular bipartite graph, then $G \in \mathfrak{N}$ and $w(G) = r$.*

Kubale and Giaro proved the following:

Theorem 3 [25]**.** *If* $G, H \in \mathfrak{N}$, *then* $G\square H \in \mathfrak{N}$. *Moreover,* $w(G\square H) \leq w(G) + w(H)$ *and* $W(G\square H) \geq W(G) + W(H)$.

The k-dimensional grid $G(n_1, n_2, \dots, n_k)$, $n_i \in \mathbf{N}$ is the Cartesian product of paths $P_{n_1}\square P_{n_2}\square \cdots \square P_{n_k}$. The cylinder $C(n_1, n_2)$ is the Cartesian product $P_{n_1}\square C_{n_2}$ and the torus $T(n_1, n_2)$ is the Cartesian product $C_{n_1}\square C_{n_2}$, where C_{n_i} is the cycle of length n_i. For these graphs Kubale and Giaro proved the following:

Theorem 4 [10]**.** *If* $G = G(n_1, n_2, \dots, n_k)$ *or* $G = C(m, 2n)$, $m \in \mathbf{N}$, $n \geq 2$, *or* $G = T(2m, 2n)$, $m, n \geq 2$, *then* $G \in \mathfrak{N}$ *and* $w(G) = \Delta(G)$.

For the greatest possible number of colors in interval edge colorings of grid graphs Petrosyan and Karapetyan proved the following theorems:

Theorem 5 [29]**.** *If* $G = C(m, 2n)$, $m \in \mathbf{N}$, $n \geq 2$, *then*

$$W(G) \geq 3m + n - 2.$$

Theorem 6 [29]**.** *If* $G = T(2m, 2n)$, $m, n \geq 2$, *then*

$$W(G) \geq \max\{3m + n, 3n + m\}.$$

In [30] Petrosyan investigated interval edge colorings of complete graphs and n-dimensional cubes Q_n. In particular, he proved the following theorems:

Theorem 7. $W(Q_n) \geq \frac{n(n+1)}{2}$ *for any* $n \in \mathbf{N}$.

Theorem 8. *Let* $n = p2^q$, *where* p *is odd and* q *is nonnegative. Then*

$$W(K_{2n}) \geq 4n - 2 - p - q.$$

The Hamming graph $H(n_1, n_2, \dots, n_k)$, $n_i \in \mathbf{N}$ is the Cartesian product of complete graphs $K_{n_1}\square K_{n_2}\square \cdots \square K_{n_k}$. The graph H_n^k is the Cartesian product of the complete graph K_n by itself k times. It is easy to see that from Theorems 1, 3 and 8, we have the following result:

Theorem 9. *Let* $n = p2^q$, *where* p *is odd and* q *is nonnegative. Then*

(1) $H_{2n}^k \in \mathfrak{N}$,

(2) $w(H_{2n}^k) = (2n-1)k$,

(3) $W(H_{2n}^k) \geq (4n-2-p-q)k$.

It is known that there are graphs G and H for which $G\square H \in \mathfrak{N}$ ($G[H] \in \mathfrak{N}$), but $G \in \mathfrak{N}$, $H \notin \mathfrak{N}$ or $G, H \notin \mathfrak{N}$. For example, $K_2\square C_3 \in \mathfrak{N}$ and $K_{1,1,3}\square C_3 \in \mathfrak{N}$ ($K_2[C_5] \in \mathfrak{N}$ and $C_5[P] \in \mathfrak{N}$), but $K_{1,1,3}, C_3 \notin \mathfrak{N}$ ($P, C_5 \notin \mathfrak{N}$, where P is the Petersen graph). Moreover, general results can be obtained from the following theorems:

Theorem 10 (Kotzig [24], Pisanski, Shawe-Taylor, Mohar [31])**.** *If G and H are two regular graphs for which at least one of the following conditions holds:*

(1) *G and H contain a perfect matching,*

(2) *$\chi'(G) = \Delta(G)$,*

(3) *$\chi'(H) = \Delta(H)$,*

then $\chi'(G\square H) = \Delta(G\square H)$ and $\chi'(G[H]) = \Delta(G[H])$.

Theorem 11 (Kotzig [24], Pisanski, Shawe-Taylor, Mohar [31])**.** *Let G be a cubic graph. Then $\chi'(G\square C_n) = \Delta(G\square C_n) = 5$ and $\chi'(C_n[G]) = \Delta(C_n[G])$ for any $n \geq 4$.*

Corollary 12. *If G and H are two regular graphs for which at least one of the following conditions holds:*

(1) *G and H contain a perfect matching,*

(2) *$G \in \mathfrak{N}$,*

(3) *$H \in \mathfrak{N}$,*

then $G\square H, G[H] \in \mathfrak{N}$ and $w(G\square H) = \Delta(G\square H)$, $w(G[H]) = \Delta(G[H])$.

Corollary 13. *Let G be a cubic graph. Then $G\square C_n, C_n[G] \in \mathfrak{N}$ and $w(G\square C_n) = \Delta(G\square C_n) = 5$, $w(C_n[G]) = \Delta(C_n[G])$ for any $n \geq 4$.*

Theorem 14. *The torus $T(n_1, n_2) \in \mathfrak{N}$ if $n_1 \cdot n_2$ is even, $T(n_1, n_2) \notin \mathfrak{N}$ if $n_1 \cdot n_2$ is odd and the Hamming graph $H(n_1, n_2, \ldots, n_k) \in \mathfrak{N}$ if $n_1 \cdot n_2 \cdots n_k$ is even, $H(n_1, n_2, \ldots, n_k) \notin \mathfrak{N}$ if $n_1 \cdot n_2 \cdots n_k$ is odd.*

Proof. Since $T(n_1, n_2)$ and $H(n_1, n_2, \ldots, n_k)$ are regular graphs, by Theorem 1 and Corollary 12, we have $T(n_1, n_2) \in \mathfrak{N}$ when $n_1 \cdot n_2$ is even and $H(n_1, n_2, \ldots, n_k) \in \mathfrak{N}$ when $n_1 \cdot n_2 \cdots n_k$ is even.

Let us show that $T(n_1, n_2) \notin \mathfrak{N}$ when $n_1 \cdot n_2$ is odd and $H(n_1, n_2, \ldots, n_k) \notin \mathfrak{N}$ when $n_1 \cdot n_2 \cdots n_k$ is odd.

Since $T(n_1, n_2)$ and $H(n_1, n_2, \ldots, n_k)$ are regular graphs, we have

$$|E(T(n_1, n_2))| = 2n_1 \cdot n_2 \quad \text{and}$$
$$|E(H(n_1, n_2, \ldots, n_k))| = \frac{n_1 \cdot n_2 \cdots n_k \cdot \Delta(H(n_1, n_2, \ldots, n_k))}{2}.$$

If $\chi'(T(n_1, n_2)) = \Delta(T(n_1, n_2)) = 4$, then

$$|E(T(n_1, n_2))| \leq 2(n_1 \cdot n_2 - 1), \text{ since } n_1 \cdot n_2 \text{ is odd.}$$

This shows that $\chi'(T(n_1, n_2)) = \Delta(T(n_1, n_2)) + 1 = 5$ and, by Theorem 1, $T(n_1, n_2) \notin \mathfrak{N}$.

Similarly, if $\chi'(H(n_1, n_2, \ldots, n_k)) = \Delta(H(n_1, n_2, \ldots, n_k))$, then

$$|E(H(n_1, n_2, \ldots, n_k))| \leq \frac{(n_1 \cdot n_2 \cdots n_k - 1) \cdot \Delta(H(n_1, n_2, \ldots, n_k))}{2},$$

since $n_1 \cdot n_2 \cdots n_k$ is odd.

This shows that $\chi'(H(n_1, n_2, \ldots, n_k)) = \Delta(H(n_1, n_2, \ldots, n_k)) + 1$ and, by Theorem 1, $H(n_1, n_2, \ldots, n_k) \notin \mathfrak{N}$. ■

3. Main Results

First, we consider interval edge colorings of the tensor product of graphs. In [25] Kubale and Giaro noted that there are graphs $G, H \in \mathfrak{N}$, such that $G \times H \notin \mathfrak{N}$. Here, we prove that if one of the graphs belongs to $\mathfrak{N}$ and the other is regular, then $G \times H \in \mathfrak{N}$.

Theorem 15. *If $G \in \mathfrak{N}$ and H is an r-regular graph, then $G \times H \in \mathfrak{N}$. Moreover, $w(G \times H) \leq w(G) \cdot r$ and $W(G \times H) \geq W(G) \cdot r$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$, $V(H) = \{v_1, v_2, \ldots, v_m\}$ and

$$V(G \times H) = \left\{w_j^{(i)} \mid 1 \leq i \leq n, 1 \leq j \leq m\right\},$$

$$E(G \times H) = \left\{\left(w_p^{(i)}, w_q^{(j)}\right) \mid (u_i, u_j) \in E(G) \text{ and } (v_p, v_q) \in E(H)\right\}.$$

Let us consider the graph $K_2 \times H$. Clearly, $K_2 \times H$ is an r-regular bipartite graph, thus, by Corollary 2, $K_2 \times H \in \mathfrak{N}$ and $w(K_2 \times H) = r$. Let α be an interval t-coloring of the graph G, β be an interval r-coloring of the graph $K_2 \times H$ and

$$V(K_2 \times H) = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\},$$

$$E(K_2 \times H) = \{(x_i, y_j) \,|\, (v_i, v_j) \in E(H), 1 \le i \le m, 1 \le j \le m\}.$$

Define an edge coloring γ of the graph $G \times H$ in the following way: for every $\left(w_p^{(i)}, w_q^{(j)}\right) \in E(G \times H)$

$$\gamma\left(\left(w_p^{(i)}, w_q^{(j)}\right)\right) = (\alpha((u_i, u_j)) - 1) \cdot r + \beta((x_p, y_q)),$$

where $1 \le i \le n, 1 \le j \le n, 1 \le p \le m, 1 \le q \le m$.

It is not difficult to see that γ is an interval $t \cdot r$-coloring of the graph $G \times H$. By the definition of γ, we have $w(G \times H) \le w(G) \cdot r$ and $W(G \times H) \ge W(G) \cdot r$. ■

The Figure 1 shows the interval 6-coloring γ of the graph $P_4 \times C_5$ described in the proof of Theorem 15.

Note that from Theorems 1 and 15, we have the following result:

Corollary 16 (Pisanski, Shawe-Taylor, Mohar [31]). *If G is 1-factorable and H is a regular graph, then $G \times H$ is also 1-factorable.*

We showed that if $G \in \mathfrak{N}$ and H is regular, then $G \times H \in \mathfrak{N}$. Now we prove a similar result for the strong tensor product of graphs.

Theorem 17. *If $G \in \mathfrak{N}$ and H is an r-regular graph, then $G \otimes H \in \mathfrak{N}$. Moreover, $w(G \otimes H) \le w(G) \cdot (r+1)$ and $W(G \otimes H) \ge W(G) \cdot (r+1)$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$, $V(H) = \{v_1, v_2, \ldots, v_m\}$ and

$$V(G \otimes H) = \left\{w_j^{(i)} \,|\, 1 \le i \le n, 1 \le j \le m\right\},$$

$$E(G \otimes H) = E(G \times H) \cup \left\{\left(w_p^{(i)}, w_p^{(j)}\right) \,|\, 1 \le p \le m \text{ and } (u_i, u_j) \in E(G)\right\}.$$

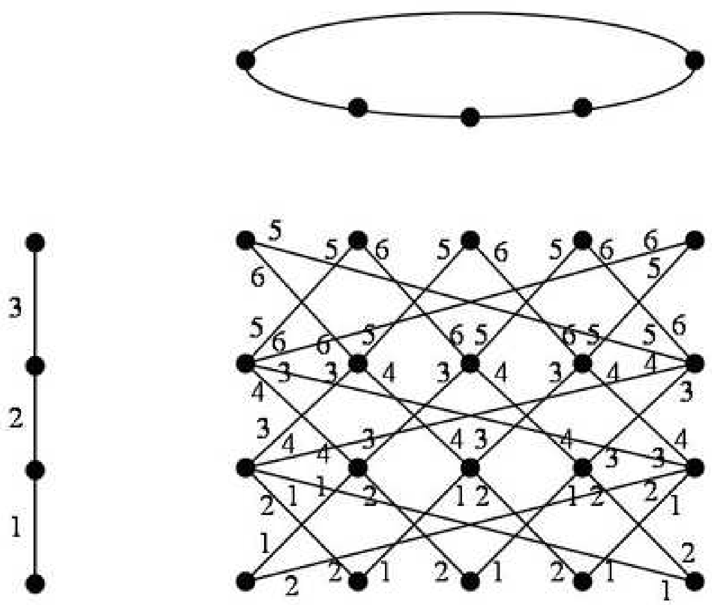

Figure 1. The interval 6-coloring γ of the graph $P_4 \times C_5$.

Let us consider the graph $K_2 \otimes H$. Clearly, $K_2 \otimes H$ is an $(r+1)$-regular bipartite graph, thus, by Corollary 2, $K_2 \otimes H \in \mathfrak{N}$ and $w\left(K_2 \otimes H\right) = r+1$. Let α be an interval t-coloring of the graph G, β be an interval $(r+1)$-coloring of the graph $K_2 \otimes H$ and

$$V\left(K_2 \otimes H\right) = \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m\},$$

$$E\left(K_2 \otimes H\right) = \{(x_i, y_i) \mid 1 \leq i \leq m\} \cup E\left(K_2 \times H\right).$$

Define an edge coloring γ of the graph $G \otimes H$ in the following way: for every $\left(w_p^{(i)}, w_q^{(j)}\right) \in E(G \otimes H)$

$$\gamma\left(\left(w_p^{(i)}, w_q^{(j)}\right)\right) = \left(\alpha\left((u_i, u_j)\right) - 1\right) \cdot (r+1) + \beta\left((x_p, y_q)\right),$$

where $1 \leq i \leq n, 1 \leq j \leq n, 1 \leq p \leq m, 1 \leq q \leq m$.

It is not difficult to see that γ is an interval $t \cdot (r+1)$-coloring of the graph $G \otimes H$. By the definition of γ, we have $w(G \otimes H) \leq w(G) \cdot (r+1)$ and $W(G \otimes H) \geq W(G) \cdot (r+1)$. ■

The Figure 2 shows the interval 9-coloring γ of the graph $P_4 \otimes C_5$ described in the proof of Theorem 17.

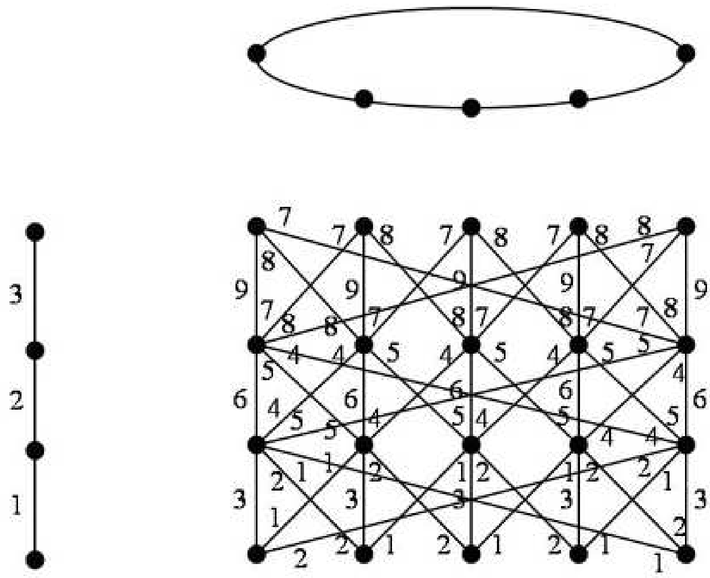

Figure 2. The interval 9-coloring γ of the graph $P_4 \otimes C_5$.

Note that from Theorems 1 and 17, we have the following result:

Corollary 18 (Pisanski, Shawe-Taylor, Mohar [31]). *If G is 1-factorable and H is a regular graph, then $G \otimes H$ is also 1-factorable.*

Next, we consider interval edge colorings of the strong product of graphs. In [25] Kubale and Giaro noted that there are graphs $G, H \in \mathfrak{N}$, such that $G \boxtimes H \notin \mathfrak{N}$. Here, we prove that if two graphs belong to $\mathfrak{N}$ and one of them is regular, then $G \boxtimes H \in \mathfrak{N}$.

Theorem 19. *If $G, H \in \mathfrak{N}$ and H is an r-regular graph, then $G \boxtimes H \in \mathfrak{N}$. Moreover, $w(G \boxtimes H) \leq w(G) \cdot (r+1) + r$ and $W(G \boxtimes H) \geq W(G) \cdot (r+1) + r$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$, $V(H) = \{v_1, v_2, \ldots, v_m\}$ and

$$V(G \boxtimes H) = \bigcup_{i=1}^{n} V^i(H), \text{ where } V^i(H) = \left\{ w_j^{(i)} \mid 1 \leq j \leq m \right\},$$

$$E(G \boxtimes H) = E(G \otimes H) \cup \bigcup_{i=1}^{n} E^i(H), \text{ where}$$

$$E^i(H) = \left\{ \left(w_p^{(i)}, w_q^{(i)} \right) \mid \ (v_p, v_q) \in E(H) \right\}.$$

For $i = 1, 2, \ldots, n$, define a graph H_i as follows:

$$H_i = \left(V^i(H), E^i(H) \right).$$

First of all note that $\chi'(H) = \Delta(H) = r$ since $H \in \mathfrak{N}$ and H is an r-regular graph. This implies that there exists an interval r-coloring of the graph H. Let us consider the graph $K_2 \otimes H$. Clearly, $K_2 \otimes H$ is an $(r+1)$-regular bipartite graph, thus, by Corollary 2, $K_2 \otimes H \in \mathfrak{N}$ and $w\,(K_2 \otimes H) = r+1$. Let α be an interval t-coloring of the graph G, β be an interval $(r+1)$-coloring of the graph $K_2 \otimes H$.

Define an edge coloring γ of the graph $G \boxtimes H$ in the following way:

(1) for every $\left(w_p^{(i)}, w_q^{(j)} \right) \in E(G \otimes H)$

$$\gamma \left(\left(w_p^{(i)}, w_q^{(j)} \right) \right) = \left(\alpha\left((u_i, u_j) \right) - 1 \right) \cdot (r+1) + \beta\left((x_p, y_q) \right),$$

where $1 \leq i \leq n, 1 \leq j \leq n, 1 \leq p \leq m, 1 \leq q \leq m$.

(2) for $i = 1, 2, \ldots, n$, the edges of the subgraph H_i we color properly with colors

$$\max S\,(u_i, \alpha) \cdot (r+1) + 1, \max S\,(u_i, \alpha) \cdot (r+1) + 2, \ldots, \max S\,(u_i, \alpha) \cdot (r+1) + r$$

It is easy to see that γ is an interval $(t \cdot (r+1) + r)$-coloring of the graph $G \boxtimes H$. By the definition of γ, we have $w(G \boxtimes H) \leq w(G) \cdot (r+1) + r$ and $W(G \boxtimes H) \geq W(G) \cdot (r+1) + r$. ■

The Figure 3 shows the interval 11-coloring γ of the graph $P_4 \boxtimes C_4$ described in the proof of Theorem 19.

Note that there are graphs G and H for which $G \boxtimes H \in \mathfrak{N}$, but $G \in \mathfrak{N}, H \notin \mathfrak{N}$. For example, $K_2 \boxtimes C_3 \in \mathfrak{N}$, but $C_3 \notin \mathfrak{N}$. For regular graphs the following result was obtained by Zhou [38].

Theorem 20. *If G is 1-factorable and H is a regular graph, then $G \boxtimes H$ is also 1-factorable.*

Corollary 21. *Let G and H be two regular graphs and $G \in \mathfrak{N}$. Then $G \boxtimes H \in \mathfrak{N}$.*

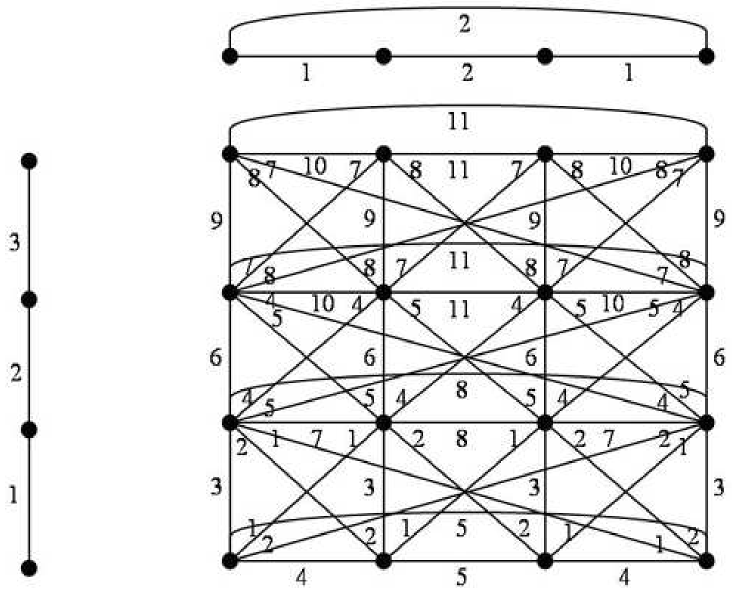

Figure 3. The interval 11-coloring γ of the graph $P_4 \boxtimes C_4$.

Finally, we turn our attention to interval edge colorings of the lexicographic product of graphs. In [25] Kubale and Giaro posed the following question:

Problem 1. Does $G[H] \in \mathfrak{N}$ if $G, H \in \mathfrak{N}$?

We start by focusing on the special case of this problem, when $G \in \mathfrak{N}$ and $H = nK_1$ for any $n \in \mathbf{N}$.

Theorem 22. *If $G \in \mathfrak{N}$, then $G[nK_1] \in \mathfrak{N}$ for any $n \in \mathbf{N}$. Moreover, $w(G[nK_1]) \leq w(G) \cdot n$ and $W(G[nK_1]) \geq (W(G)+1) \cdot n - 1$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_m\}$ and

$$V\left(G[nK_1]\right) = \left\{v_j^{(i)} \mid 1 \leq i \leq m, 1 \leq j \leq n\right\},$$

$$E\left(G[nK_1]\right) = \left\{\left(v_p^{(i)}, v_q^{(j)}\right) \mid \ (u_i, u_j) \in E(G) \ \text{ and } \ p, q = 1, 2, \ldots, n\right\}.$$

Let α be an interval t-coloring of the graph G.

Define an edge coloring β of the graph $G[nK_1]$ in the following way: for every $\left(v_p^{(i)}, v_q^{(j)}\right) \in E\left(G[nK_1]\right)$

$$\beta\left(\left(v_p^{(i)}, v_q^{(j)}\right)\right) = \begin{cases} (\alpha((u_i,u_j)) - 1)\cdot n + p + q - 1 \pmod n, \\ \qquad\qquad\qquad\qquad \text{if } \ p+q \neq n+1, \\ \alpha((u_i,u_j))\cdot n, \ \text{if } \ p+q = n+1. \end{cases}$$

where $1 \le i \le m, 1 \le j \le m, 1 \le p \le n, 1 \le q \le n$.

It can be verified that β is an interval $t\cdot n$-coloring of the graph $G[nK_1]$. By the definition of β, we have $w(G[nK_1]) \le w(G)\cdot n$.

Now we show that $W(G[nK_1]) \ge (W(G)+1)\cdot n - 1$.

Let ϕ be an interval $W(G)$-coloring of the graph G.

Define an edge coloring ψ of the graph $G[nK_1]$ in the following way: for every $\left(v_p^{(i)}, v_q^{(j)}\right) \in E\left(G[nK_1]\right)$

$$\psi\left(\left(v_p^{(i)}, v_q^{(j)}\right)\right) = (\phi((u_i,u_j)) - 1)\cdot n + p + q - 1,$$

where $1 \le i \le m, 1 \le j \le m, 1 \le p \le n, 1 \le q \le n$.

It is easy to see that ψ is an interval $(W(G)\cdot n + n - 1)$-coloring of the graph $G[nK_1]$. ■

The Figure 4 shows the interval 6-coloring β of the graph $(K_{1,3}+e)[2K_1]$ described in the proof of Theorem 22.

Corollary 23 (Kamalian, Petrosyan [22])**.** *If k is even, then $C_k[nK_1] \in \mathfrak{N}$ and*

$$W(C_k[nK_1]) \ge 2n + \tfrac{n\cdot k}{2} - 1.$$

Corollary 24 (Kamalian, Petrosyan [23])**.** *Let $k = p2^q$, where p is odd and $q \in \mathbf{N}$. Then $K_k[nK_1] \in \mathfrak{N}$ and*

$$W(K_k[nK_1]) \ge (2k - p - q)\cdot n - 1.$$

Now we show that $G[H] \in \mathfrak{N}$ if $G, H \in \mathfrak{N}$ and H is regular.

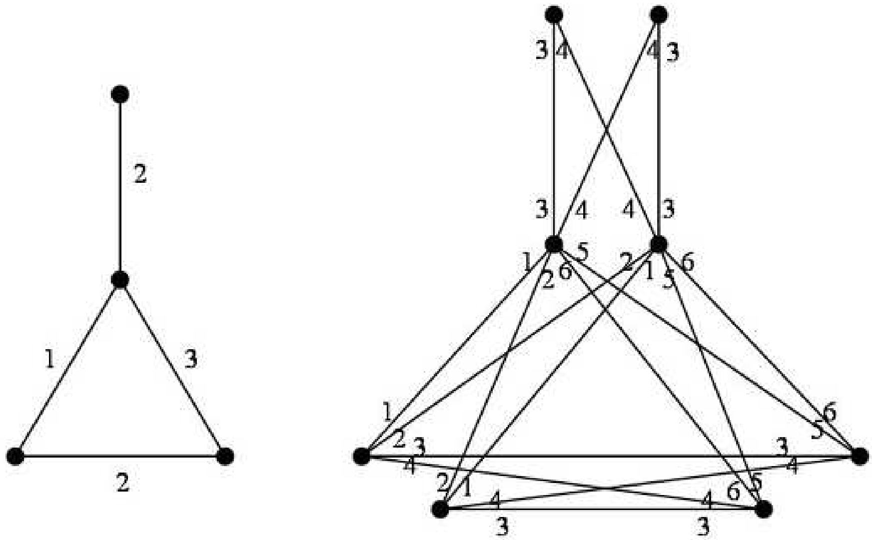

Figure 4. The interval 6-coloring β of the graph $(K_{1,3}+e)[2K_1]$.

Theorem 25. *If $G, H \in \mathfrak{N}$ and H is an r-regular graph, then $G[H] \in \mathfrak{N}$. Moreover, if $|V(H)| = n$, then $w(G[H]) \leq w(G) \cdot n + r$ and $W(G[H]) \geq W(G) \cdot n + r$.*

Proof. Let $V(G) = \{u_1, u_2, \dots, u_m\}$, $V(H) = \{v_1, v_2, \dots, v_n\}$ and

$$V(G[H]) = \bigcup_{i=1}^{m} V^i(H), \text{ where } V^i(H) = \{w_j^{(i)} \mid 1 \leq j \leq n\},$$

$$E(G[H]) = \left\{ \left(w_p^{(i)}, w_q^{(j)} \right) \mid (u_i, u_j) \in E(G) \text{ and } p, q = 1, 2, \dots, n \right\}$$
$$\cup \bigcup_{i=1}^{m} E^i(H), \text{ where } E^i(H) = \left\{ \left(w_p^{(i)}, w_q^{(i)} \right) \mid (v_p, v_q) \in E(H) \right\}.$$

Let α be an interval t-coloring of the graph G and

$$H_i = \left(V^i(H), E^i(H) \right) \text{ for } i = 1, 2, \dots, m.$$

Note that $\chi'(H) = \Delta(H) = r$ since $H \in \mathfrak{N}$ and H is an r-regular graph. This implies that there exists an interval r-coloring of the graph H.

Define an edge coloring β of the graph $G[H]$ in the following way:

(1) for every $\left(w_p^{(i)}, w_q^{(j)}\right) \in E(G[H])$

$$\beta\left(\left(w_p^{(i)}, w_q^{(j)}\right)\right) = \begin{cases} r + (\alpha((u_i, u_j)) - 1) \cdot n + p + q - 1 \pmod{n}, \\ \qquad\qquad\qquad\qquad\qquad \text{if } p + q \neq n + 1, \\ r + \alpha((u_i, u_j)) \cdot n, \text{ if } p + q = n + 1, \end{cases}$$

where $1 \leq i \leq m, 1 \leq j \leq m, i \neq j, 1 \leq p \leq n, 1 \leq q \leq n$.

(2) for $i = 1, 2, \ldots, m$, the edges of the subgraph H_i we color properly with colors

$$(\min S(u_i, \alpha) - 1) \cdot n + 1, (\min S(u_i, \alpha) - 1) \cdot n + 2, \ldots,$$
$$(\min S(u_i, \alpha) - 1) \cdot n + r.$$

It can be verified that β is an interval $(t \cdot n + r)$-coloring of the graph $G[H]$. By the definition of β, we have $w(G[H]) \leq w(G) \cdot n + r$ and $W(G[H]) \geq W(G) \cdot n + r$. ■

The Figure 5 shows the interval 9-coloring β of the graph $K_4[K_2]$ described in the proof of Theorem 25.

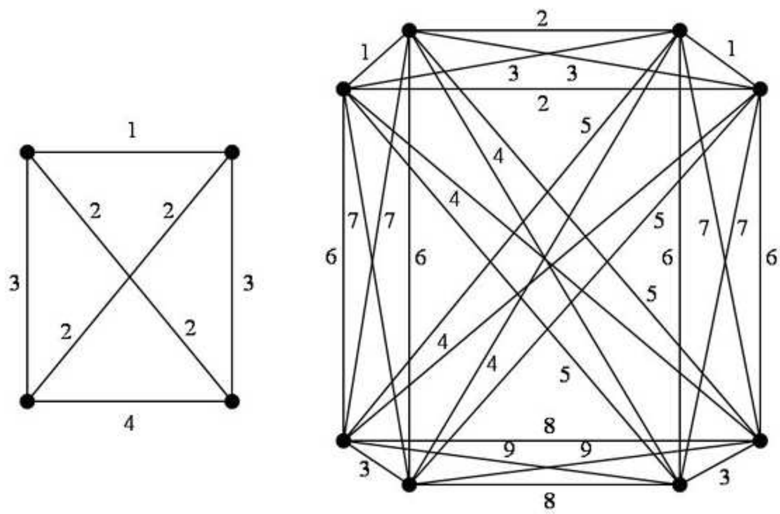

Figure 5. The interval 9-coloring β of the graph $K_4[K_2]$.

4. Problems

We conclude with the following problems on interval edge colorings of products of graphs.

Problem 2. Are there graphs $G, H \notin \mathfrak{N}$, such that $G \times H \in \mathfrak{N}$?

Problem 3. Are there graphs $G, H \notin \mathfrak{N}$, such that $G \otimes H \in \mathfrak{N}$?

Problem 4. Are there graphs $G, H \notin \mathfrak{N}$, such that $G \boxtimes H \in \mathfrak{N}$?

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References

[1] A.S. Asratian and R.R. Kamalian, *Interval colorings of edges of a multigraph*, Appl. Math. **5** (1987) 25–34 (in Russian).

[2] A.S. Asratian and R.R. Kamalian, *Investigation on interval edge-colorings of graphs*, J. Combin. Theory (B) **62** (1994) 34–43.

[3] A.S. Asratian, T.M.J. Denley and R. Häggkvist, Bipartite Graphs and their Applications (Cambridge University Press, Cambridge, 1998).

[4] A.S. Asratian, C.J. Casselgren, J. Vandenbussche and D.B. West, *Proper path-factors and interval edge-coloring of* $(3,4)$*-biregular bigraphs*, J. Graph Theory **61** (2009) 88–97.

[5] M.A. Axenovich, *On interval colorings of planar graphs*, Congr. Numer. **159** (2002) 77–94.

[6] C. Berge, Theorie des Graphes et ses Applications (Dunod, Paris, 1958).

[7] M. Bouchard, A. Hertz and G. Desaulniers, *Lower bounds and a tabu search algorithm for the minimum deficiency problem*, J. Comb. Optim. **17** (2009) 168–191.

[8] Y. Feng and Q. Huang, *Consecutive edge-coloring of the generalized* θ*-graph*, Discrete Appl. Math. **155** (2007) 2321–2327.

[9] K. Giaro, *The complexity of consecutive* Δ*-coloring of bipartite graphs:* 4 *is easy,* 5 *is hard*, Ars Combin. **47** (1997) 287–298.

[10] K. Giaro and M. Kubale, *Consecutive edge-colorings of complete and incomplete Cartesian products of graphs*, Congr. Numer. **128** (1997) 143–149.

[11] K. Giaro, M. Kubale and M. Małafiejski, *On the deficiency of bipartite graphs*, Discrete Appl. Math. **94** (1999) 193–203.

[12] K. Giaro, M. Kubale and M. Małafiejski, *Consecutive colorings of the edges of general graphs*, Discrete Math. **236** (2001) 131–143.

[13] K. Giaro and M. Kubale, *Compact scheduling of zero-one time operations in multi-stage systems*, Discrete Appl. Math. **145** (2004) 95–103.

[14] H.M. Hansen, Scheduling with minimum waiting periods, Master's Thesis (Odense University, Odense, Denmark, 1992) (in Danish).

[15] D. Hanson and C.O.M. Loten, *A lower bound for interval coloring of bi-regular bipartite graphs*, Bull. ICA **18** (1996) 69–74.

[16] D. Hanson, C.O.M. Loten and B. Toft, *On interval colorings of bi-regular bipartite graphs*, Ars Combin. **50** (1998) 23–32.

[17] P.E. Himmelwright and J.E. Williamson, *On 1-factorability and edge-colorability of Cartesian products of graphs*, Elem. Der Math. **29** (1974) 66–67.

[18] T.R. Jensen and B. Toft, Graph Coloring Problems (Wiley Interscience Series in Discrete Mathematics and Optimization, 1995).

[19] R.R. Kamalian, Interval colorings of complete bipartite graphs and trees, preprint, Comp. Cen. of Acad. Sci. of Armenian SSR, Erevan, 1989 (in Russian).

[20] R.R. Kamalian, Interval Edge Colorings of Graphs (Doctoral Thesis, Novosibirsk, 1990).

[21] R.R. Kamalian and A.N. Mirumian, *Interval edge colorings of bipartite graphs of some class*, Dokl. NAN RA **97** (1997) 3–5 (in Russian).

[22] R.R. Kamalian and P.A. Petrosyan, *Interval colorings of some regular graphs*, Math. Probl. Comp. Sci. **25** (2006) 53–56.

[23] R.R. Kamalian and P.A. Petrosyan, *On interval colorings of complete k-partite graphs K_n^k*, Math. Probl. Comp. Sci. **26** (2006) 28–32.

[24] A. Kotzig, *1-Factorizations of cartesian products of regular graphs*, J. Graph Theory **3** (1979) 23–34.

[25] M. Kubale, ed., Graph Colorings (American Mathematical Society, 2004).

[26] E.S. Mahamoodian, *On edge-colorability of Cartesian products of graphs*, Canad. Math. Bull. **24** (1981) 107–108.

[27] B. Mohar and T. Pisanski, *Edge-coloring of a family of regular graphs*, Publ. Inst. Math. (Beograd) **33** (1983) 157–162.

[28] B. Mohar, *On edge-colorability of products of graphs*, Publ. Inst. Math. (Beograd) **36** (1984) 13–16.

[29] P.A. Petrosyan and G.H. Karapetyan, *Lower bounds for the greatest possible number of colors in interval edge colorings of bipartite cylinders and bipartite tori*, in: Proceedings of the CSIT Conference (2007) 86–88.

[30] P.A. Petrosyan, *Interval edge-colorings of complete graphs and n-dimensional cubes*, Discrete Math. **310** (2010) 1580–1587.

[31] T. Pisanski, J. Shawe-Taylor and B. Mohar, 1-*Factorization of the composition of regular graphs*, Publ. Inst. Math. (Beograd) **33** (1983) 193–196.

[32] A.V. Pyatkin, *Interval coloring of* $(3,4)$*-biregular bipartite graphs having large cubic subgraphs*, J. Graph Theory **47** (2004) 122–128.

[33] G. Sabidussi, *Graph multiplication*, Math. Z. **72** (1960) 446–457.

[34] A. Schwartz, *The deficiency of a regular graph*, Discrete Math. **306** (2006) 1947–1954.

[35] S.V. Sevast'janov, *Interval colorability of the edges of a bipartite graph*, Metody Diskret. Analiza **50** (1990) 61–72 (in Russian).

[36] V.G. Vizing, *The Cartesian product of graphs*, Vych. Sis. **9** (1963) 30–43 (in Russian).

[37] D.B. West, Introduction to Graph Theory (Prentice-Hall, New Jersey, 1996).

[38] M.K. Zhou, *Decomposition of some product graphs into* 1*-factors and Hamiltonian cycles*, Ars Combin. **28** (1989) 258–268.

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