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PARITY VERTEX COLOURING OF GRAPHS

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Abstract

A parity path in a vertex colouring of a graph is a path along which each colour is used an even number of times. Let $\chi_p(G)$ be the least number of colours in a proper vertex colouring of G having no parity path. It is proved that for any graph G we have the following tight bounds $\chi(G) \leq \chi_p(G) \leq |V(G)| - \alpha(G) + 1$, where $\chi(G)$ and $\alpha(G)$ are the chromatic number and the independence number of G, respectively. The bounds are improved for trees. Namely, if T is a tree with diameter $\mathrm{diam}(T)$ and radius $\mathrm{rad}(T)$, then $\lceil \log_2 (2+\mathrm{diam}(T)) \rceil \leq \chi_p(T) \leq 1 + \mathrm{rad}(T)$. Both bounds are tight. The second thread of this paper is devoted to relationships between parity vertex colourings and vertex rankings, i.e. a proper vertex colourings with the property that each path between two vertices of the same colour q contains a vertex of colour greater than q. New results on graphs critical for vertex rankings are also presented.

Keywords: parity colouring, graph colouring, vertex ranking, ordered colouring, tree, hypercube, Fibonacci number.

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1. Introduction

The graphs we consider are finite, simple and undirected. Let the *usage* of a colour on a path be the number of times it appears along the path. The *parity path* is a path for which every colour has even usage. We define the *parity vertex colouring* to be a proper vertex colouring having no parity path. The *parity vertex chromatic number* $\chi_p(G)$ is the minimum number of colours in a parity vertex colouring of G. Observe that using distinct colours on all vertices produces a parity vertex colouring while paths on two vertices force $\chi_p(G) \geq \chi(G)$, where $\chi(G)$ is the chromatic number of G. For any terminology and notations not defined here the readers are referred to [4].

The work on parity colouring of graphs was initiated in [2] and it began by studying which graphs embed in the hypercube. Both problems are closely related and the question of embeddings is motivated by the fact that hypercube is a common and one of the most efficient architectures for parallel computation [8]. Inspired by the recent paper of Bunde, Milans, West and Wu [2], who studied parity edge colourings, we begin the study of parity vertex colourings. A significant part of this paper is also devoted to the intriguing relationship between parity colouring and *ranking of vertices*, i.e. a colouring with the property that each path between two vertices of the same colour q contains a vertex of colour greater than q. The study of vertex rankings is motivated by parallel Cholesky factorization of matrices [9] and applications in VLSI layout [10]. It is also worth pointing out that every parity colouring is a *conflict-free colouring*, i.e. a colouring in which every path uses some colour exactly once. Conflict-free colourings were recently studied due to their theoretical and practical importance, e.g. for frequency assignment in cellular networks [5].

2. Fundamental Bounds on the Parity Vertex Chromatic Number

Theorem 1. *Let $G = (V, E)$ be an n-vertex graph with the chromatic number $\chi(G)$ and the independence number $\alpha(G)$. Then*

$$\chi(G) \leq \chi_p(G) \leq n - \alpha(G) + 1.$$

Proof. Two adjacent vertices form a path, therefore they must have different colours, hence $\chi_p(G) \geq \chi(G)$. It is easy to see that $\chi_p(K_n) = n = \chi(K_n)$. Hence the lower bound is tight.

Let S be a maximum independent set of G, i.e. $|S| = \alpha(G)$. Let us colour the vertices of G in the following manner. The vertices of S are coloured with the same colour, say 1. The vertices of $V - S$ are coloured with different colours from the set $\{2, 3, \ldots, n - \alpha(G) + 1\}$. Because every path on at least two vertices contains a vertex from the set $V - S$ this colouring is a parity vertex colouring. ■

The tightness of the upper bound follows from Theorem 2.

Lemma 1. *For the union $G \cup H$ of any two vertex disjoint graphs G and H we have*

$$\chi_p(G \cup H) = \max\{\chi_p(G), \chi_p(H)\}\,.$$

Let $G + H$ be a *join of two graphs* G and H defined as follows $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$.

Theorem 2. *For any graphs G and H we have*

$$\chi_p(G + H) = \min\{\chi_p(G) + |V(H)|, \chi_p(H) + |V(G)|\}\,.$$

Proof. ($\leq$) Let $\chi_p(G) = r$. Whenever G is coloured using colours $1, \ldots, r$, it is possible to colour H with colours $r+1, \ldots, r+|V(H)|$. Consequently, $\chi_p(G+H) \leq \chi_p(G)+|V(H)|$. Similarly, we get $\chi_p(G+H) \leq \chi_p(H)+|V(G)|$. From the above it follows that $\chi_p(G+H) \leq \min\{\chi_p(G)+|V(H)|, \chi_p(H)+|V(G)|\}$.

($\geq$) If $\chi_p(G) = |V(G)|$ and $\chi_p(H) = |V(H)|$, then $\chi_p(G+H) = |V(H)|+|V(G)| \geq \min\{\chi_p(G)+|V(H)|, \chi_p(H)+|V(G)|\}$. Now, suppose that $\chi_p(G) < |V(G)|$. Then there exist two vertices of G, say u and v, coloured with the same colour. Therefore each vertex of H has to be coloured with a different colour. Otherwise, for any $x, y \in V(H)$ there would be a parity path (u, x, v, y). Moreover, because of join, colours used for $V(H)$ have to be different from all colours used for $V(G)$. Consequently, $\chi_p(G+H) = \chi_p(G)+|V(H)| \geq \min\{\chi_p(G)+|V(H)|, \chi_p(H)+|V(G)|\}$. ■

From Theorem 2 we immediately have the following corollary for complete bipartite graphs.

Corollary 1. *Let $K_{r,s}$ be a complete bipartite graph such that $r \leq s$. Then*

$$\chi_p(K_{r,s}) = r+1\,.$$

Similar reasoning can be applied to find the parity vertex chromatic number of complete k-partite graphs.

Corollary 2. *Let $K_{r_1,\ldots,r_k}$ be a complete k-partite graph. Then*

$$\chi_p(K_{r_1,\ldots,r_k}) = \sum_{i=1}^{k} r_i - \max_{1\leq i\leq k} r_i + 1\,.$$

Proof. By induction on k. ■

Theorem 3. *Let S be a cut-set of a graph G and let $H_1, \ldots, H_r$ be the components of $G[V-S]$. Then*

$$\chi_p(G) \leq \max_{1\leq i\leq r}\{\chi_p(H_i)\} + |S|.$$

Proof. Assuming that vertices of each H_i are coloured with consecutive colours starting from 1, it is enough to colour each vertex of S with a different colour, which has not been used in $V(H_1) \cup \cdots \cup V(H_r)$. ■

The bound is tight, i.e. there exist graphs for which the equality holds, e.g. $K_1 + (K_p \cup K_q)$, $p, q \geq 1$. Notice that the cut-set S in Theorem 3 may be assumed to be minimal. A recursive application of theorem gives an algorithm, which in general does not have to be efficient. However, for some classes of graphs the desired upper bound can be computed in polynomial time.

In [2] it was proved that for every path P_m on m vertices $\chi_p'(P_m) = \lceil \log_2 m \rceil$. Since P_n is the line graph of the path P_{n+1} we obtain the following.

Theorem 4. *Let P_n be an n-vertex path. Then*

$$\chi_p(P_n) = \left\lceil \log_2(n+1) \right\rceil .$$

To be able to formulate our next results, we recall some definitions. The *distance* between two vertices u and v of a graph G, $\operatorname{dist}(u, v)$, is defined to be the length of the shortest path between u and v. The *eccentricity* $e(v)$ of a vertex v in the graph G is the distance from v to a vertex furthest from v, i.e. $e(v) = \max\{\operatorname{dist}(v, u) | u \in V(G)\}$. The *radius* $\operatorname{rad}(G)$ of a connected graph G is defined as $\operatorname{rad}(G) = \min\left\{e(v) | v \in V(G)\right\}$, and the *diameter* $\operatorname{diam}(G)$ of a connected graph G is defined by $\operatorname{diam}(G) = \max\left\{e(v) | v \in V(G)\right\}$. A vertex c is called *central* if its eccentricity equals $\operatorname{rad}(G)$, i.e. $e(c) = \operatorname{rad}(G)$. The following theorem strengthens Theorem 1 for trees.

Theorem 5. *Let T be a tree. Then*

$$\left\lceil \log_2(2 + \operatorname{diam}(T)) \right\rceil \leq \chi_p(T) \leq 1 + \operatorname{rad}(T) .$$

Proof. The lower bound is a consequence of the above Theorem 4 when considering the longest path in T. To show the upper bound it is sufficient to find a colouring with $1 + \operatorname{rad}(T)$ colours having the required property that every path in T uses some colour an odd number of times. It is well known that every tree T has exactly one central vertex c if $\operatorname{diam}(T)$ is even and exactly two central vertices c and c^* which are adjacent if $\operatorname{diam}(T)$ is odd. Define the following k-colouring $\varphi : V(T) \to \{1, \ldots, k\}$ of the vertices of T, where $k = 1 + \operatorname{rad}(T)$. In both cases we colour vertex c with the colour $\varphi(c) = k$ and every vertex v such that $\operatorname{dist}(c, v) = j$ with the colour $\varphi(v) = j$. Clearly, $j \leq k - 1 = \operatorname{rad}(T)$. Next, we need to show that every path Q in T uses some colour an odd number of times. In fact, we argue that on every path Q there is a vertex which uses some colour exactly once.

Namely, if Q is a path in T, then the colour of the vertex in Q that is closest to c appears only once along Q. ■

The tightness of the lower bound follows from Theorem 4 while the upper bound is tight by Theorem 6.

Theorem 6. *For all $r \in \mathbb{N}$ there is a tree T_r with the radius $r = \mathrm{rad}(T_r)$ such that $\chi_p(T_r) = r+1$. Moreover, for $r > 0$ there are infinitely many such trees.*

Proof. Define (by induction) a sequence $(T_r : r \in \mathbb{N})$ of trees in the following way:

1. T_0 consists of only one vertex (its root t_0) and no edge,
2. By induction, there exists a rooted tree T_{r-1} with radius $r-1$ and parity vertex chromatic number at least r. Let T_r be the rooted tree obtained from $2^{r-1}+1$ copies of T_{r-1} by introducing a new vertex t_r to serve as the root of T_r and adding edges between t_r and the roots of the copies of T_{r-1}.

We show that the parity vertex chromatic number of T_r is at least $r+1$. Suppose for a contradiction that φ is a parity vertex colouring of T_r that uses only r colours. Let $a = \varphi(t_r)$. We consider two cases.

First suppose that the colouring φ does not use a in any copy S of T_{r-1}. In this case, the induction hypothesis implies that φ uses at least r colours on vertices in S, none of which is a. It follows that φ uses at least $r+1$ colours on T_r, a contradiction.

Otherwise, φ uses a in each copy of T_{r-1}. For each $1 \le j \le 2^{r-1}+1$ let v_j be some vertex in the j-th copy of T_{r-1} with $\varphi(v_j) = a$ and let P_j be the shortest path from t_r to v_j. Because there are $2^{r-1}+1$ paths of the form P_j, the pigeonhole principle implies that there exist two paths P_p and P_q which agree in the parity of the usage of each colour, with the possible exception of a. Let P be a path formed by a concatenation of P_p with (the reverse of) P_q. A path P' obtained by the removal of one of the endvertices of P is a parity path. But now P' contradicts that φ is a parity vertex colouring of T_r.

It is easy to construct an infinite family of trees of radius r for every $r > 0$. Take s copies of T_{r-1} for arbitrary $s > 2^{r-1}$ instead of $2^{r-1}+1$ and proceed as above. ■

3. d-dimensional Cubes

It would be interesting to determine the exact values of parity vertex chromatic number for some families of graphs. Very intriguing candidates are d-dimensional cubes $Q_d, d \geq 0$. It is easy to see that $\chi_p(Q_0) = 1$, $\chi_p(Q_1) = 2$ and $\chi_p(Q_2) = 3$. For the next value of d we have

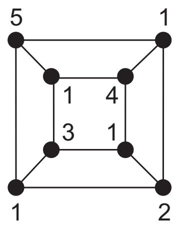

Figure 1. Parity vertex colouring of 3-dimensional cube Q_3.

Lemma 2. $\chi_p(Q_3) = 5$.

Proof. From Figure 1 it follows that $\chi_p(Q_3) \leq 5$. Because Q_3 contains P_8 as a subgraph, it follows from Theorem 4 that $\chi_p(Q_3) \geq 4$. Without loss of generality let us suppose that there is a parity vertex colouring using four colours. Since the graph Q_3 is hamiltonian and has exactly eight vertices there are at least three vertices of Q_3 coloured with the same colour, say 1. Let the colour 1 be used exactly three times. These vertices form an independent set in Q_3. Because any maximal independent set of Q_3 has four vertices, the fourth vertex of this set is coloured with different colour, say 2. If any other vertex has colour 2 it is only a neighbour of all three vertices coloured 1. The remaining three vertices of Q_3 must have mutually distinct colours. Otherwise, a parity cycle on four vertices appears. A contradiction that four colours are enough.

Let colour 1 be used four times. Vertices which are coloured with this colour form a maximal independent set of Q_3. If there is another colour used twice, then this colour cannot be used on the same cycle on four vertices. However, for each two vertices from remaining four ones, coloured with the same colour, there exists a common cycle of length 4 which contains a parity path P_4 as a subgraph. Hence the remaining vertices must be coloured with different colours, a contradiction. ∎

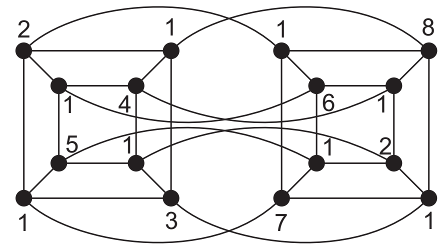

Figure 2. Parity vertex colouring of 4-dimensional cube Q_4.

Concerning graph Q_4. From Figure 2 we have that $\chi_p(Q_4) \leq 8$. Moreover, we have checked by computer that $\chi_p(Q_4) > 7$, hence $\chi_p(Q_4) = 8$. We strongly believe that the following question has an affirmative answer.

Problem 1. Is it true that $\chi_p(Q_d) = F_{d+2}$, where F_i is the i-th Fibonacci number?

4. Parity and Ranking

A *vertex ranking* of a graph is a proper vertex colouring by a linearly ordered set of colours such that for every path in the graph with end vertices of the same colour there is a vertex on this path with a higher colour (see [3] for a survey on rankings). The vertex ranking problem asks for a vertex ranking of a given graph G which has the minimum number of colours. This number denoted by $\chi_r(G)$ is the vertex *ranking number* of G. Notice that in any connected graph G there exists exactly one vertex coloured with the maximum colour $\chi_r(G)$.

Theorem 7. *Every vertex ranking of graph G is a parity vertex colouring of G and consequently we have $\chi(G) \leq \chi_p(G) \leq \chi_r(G)$.*

Proof. Assume on the contrary that φ is a vertex ranking of G, which is not a parity vertex colouring, i.e. G contains a parity path P. Let r be the maximum colour used on P. Since P contains at least two vertices coloured r, choose two of them, say u, v in such a way that no other vertex of colour r lies on P between u and v. Following the definition of ranking, a vertex of colour greater than r exists for each path between u and v, which contradicts the maximality of r. ■

From Theorem 7 it also follows that whenever G is a graph for which $\chi_r(G) = \chi(G)$, then $\chi_p(G) = \chi(G)$. Moreover, in [1] it was proved that any graph G for which $\chi_r(G) = \chi(G)$ satisfies $\chi(G) = \omega(G)$, where $\omega(G)$ denotes the clique number of G. Hence whenever $\chi_r(G) = \chi(G)$, then $\chi_p(G) = \omega(G)$. There exist graphs for which $\chi_p(G) = \omega(G)$ and $\chi_r(G) > \omega(G)$. Namely, the graph K_n^+ is obtained from K_n, having vertex set $\{v_1, v_2, \ldots, v_n\}$, by addition of n new vertices u_i and n new edges v_iu_i, $i \in \{1, 2, \ldots, n\}$. It is not hard to see that for $n \geq 3$ we have $\chi_p(K_n^+) = n = \omega(K_n^+) < \chi_r(K_n^+) = n+1$.

Trivially perfect graphs can be characterized by forbidding C_4 and P_4 as the induced subgraphs [6]. If we require the equality of χ_p and χ to hold for every subgraph, then we can prove the following theorem which is also known to hold for vertex rankings [1].

Theorem 8. *For a graph $G = (V, E)$ the following conditions are equivalent:*

(i) *G is trivially perfect,*

(ii) $\chi_r(G[A]) = \chi(G[A])$ *for every* $A \subseteq V(G)$,

(iii) $\chi_p(G[A]) = \chi(G[A])$ *for every* $A \subseteq V(G)$.

Proof. The equivalence of (i) and (ii) is proved in [1]. The implication (iii)⇒(i) is obvious since $\chi_p(C_4) = \chi_p(P_4) = 3$ while $\chi(C_4) = \chi(P_4) = 2$. It remains to prove the implication (i)⇒(iii). Let G be trivially perfect and assume that there exists a subset A such that $\chi_p(G[A]) > \chi(G[A])$. By the equivalence of (i) and (ii) we have $\chi_p(G[A]) > \chi_r(G[A])$, a contradiction by Theorem 7. ■

Note that arguments used within the proof hold also for conflict-free colouring.

The class of *cographs*, known also as the class of P_4-free graphs, is the smallest class of graphs fulfilling the following conditions:

1. The graph K_1 is a cograph.
2. If G_1, G_2 are vertex disjoint cographs, then
   (a) their union $G_1 \cup G_2$ is a cograph,
   (b) their join $G_1 + G_2$ is a cograph.

To see that the analogue of Theorem 2 holds for rankings, observe that whenever two vertices of G_1 have the same colour q in some ranking of G_1, then all vertices in $V(G_2)$ have to be coloured differently with colours larger than q. Now, since analogues of Lemma 1 and Theorem 2 hold as well for parity vertex colourings as for vertex rankings, we immediately have the following theorem.

Theorem 9. *If G is a cograph, then $\chi_p(G) = \chi_r(G)$.*

With the formulas given in Lemma 1 and Theorem 2 we can compute the parity vertex chromatic number of any cograph in polynomial time.

Further analysis of the relationships between ranking and parity chromatic number leads to the following intriguing problem which turned out to be challenging even for basic classes of graphs like trees.

Problem 2. For what classes of graphs there exists $c \in \mathbb{N}$ such that

$$\chi_r(G) - \chi_p(G) \leq c?$$

It seems that critical and minimal graphs could be used to solve this problem. Graphs critical for vertex ranking are analyzed in the next section and we use them to state the lower bound for trees.

5. Ranking Critical Graphs

A graph G is said to be *ranking k-critical* if $\chi_r(G) = k$ but $\chi_r(G - v) < k$ for every vertex $v \in V(G)$. *Ranking k-minimal* graphs are those graphs G for which $\chi_r(G) = k$ but $\chi_r(G - e) < k$ for any edge $e \in E(G)$. The *parity k-critical* and *parity k-minimal* graphs are defined analogously. Ranking minimal graphs were analyzed by Katchalski *et al.* in [7] (the authors called them k-critical). The following theorem strengthens the result of Katchalski *et al.* ([7] Proposition 2.1).

Theorem 10. *Let G be any connected graph such that $\chi_r(G) = k$ and let H_i, $i \in \{1, \ldots, p\}$ be all ranking k-critical subgraphs of G. Then $X = \bigcap_{i=1}^{p} V(H_i) \neq \emptyset$. Moreover, if $\widetilde{\varphi}(v) = k$ for some ranking $\widetilde{\varphi}$ then $v \in X$ and for any vertex $w \in X$ there exists a ranking φ such that $\varphi(w) = k$.*

Proof. If there existed ranking k-critical subgraphs H_i and H_j such that $V(H_i) \cap V(H_j) = \emptyset$, then for any ranking φ of G we would have two vertices $v_i \in V(H_i)$ and $v_j \in V(H_j)$ coloured with the maximum colour k, a contradiction. For the same reason any vertex coloured k must belong to X.

Now, observe that whenever $v \in X$, $\chi_r(H_i - v) = k - 1$ for all $1 \leq i \leq p$ and consequently $\chi_r(G - v) = k - 1$. Any $(k-1)$-ranking φ of $G - v$ can be easily extended to a k-ranking of G using colour k for the vertex v. ■

The operation of adding the edge between vertices of two disjoint ranking $(k-1)$-minimal graphs G_1, G_2 results in the ranking k-minimal graph G ([7] Lemma 2.1). In what follows we prove even a stronger statement concerning critical graphs.

Theorem 11. *Let G_1, G_2 be vertex disjoint connected graphs such that $\chi_r(G_i) = k - 1$, $i \in \{1, 2\}$ and let G be a graph obtained from $G_1 \cup G_2$ by addition of an edge v_1v_2 between some vertices $v_1 \in V(G_1)$, $v_2 \in V(G_2)$. Then*

(a) $\chi_r(G) = k$,

(b) *G is ranking k-critical if and only if G_1, G_2 are ranking $(k-1)$-critical.*

Proof. (a) Since $\chi_r(G_i) = k-1$, $i \in \{1, 2\}$ there exist vertices $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ coloured $k-1$, which following the last part of Theorem 10 may be any vertices of the appropriate $(k-1)$-critical subgraphs $H_i \leq G_i$ and hence are assumed to be different from v_1 and v_2. Therefore there exists a path $(x_1, \ldots, v_1, v_2, \ldots, x_2)$, which spoils ranking as long as some vertex coloured k appears between x_1 and x_2. Hence $\chi_r(G) \geq k$. On the other hand to see that $\chi_r(G) \leq k$, let φ_i be a k-ranking of G_i, $i \in \{1, 2\}$ and let the ranking φ of G be defined as follows: $\varphi(v) = \varphi_1(v)$ for $v \in V(G_1)$, $\varphi(v) = \varphi_2(v)$ for $v \in V(G_2) - \{v_2\}$ and $\varphi(v_2) = k$.

(b) ($\Leftarrow$) From (a) it follows that $\chi_r(G) = k$. Let $v \in V(G)$; without loss of generality assume v to be from $V(G_1)$. We argue that $\chi_r(G - v) = k - 1$. Since G_1 is $(k-1)$-critical, $G_1 - v$ has a $(k-2)$-ranking φ_1'. By Theorem 10 graph G_2 has such a $(k-1)$-ranking φ_2 that $\varphi_2(v_2) = k - 1$. Notice that colouring φ defined as $\varphi(v) = \varphi_1'(v)$ for $v \in V(G_1 - v)$ and $\varphi(v) = \varphi_2(v)$ for $v \in V(G_2)$ is a $(k-1)$-ranking of $G - v$. Hence G is ranking k-critical.

($\Rightarrow$) Let us assume that G_1 is not $(k-1)$-critical and let $v \in V(G_1)$. Since G is k-critical, $\chi_r(G - v) \leq k - 1$ and since $\chi_r(G_2) = k - 1$, the only vertex coloured $k - 1$ must belong to $V(G_2)$. Accordingly, $G_1 - v$ has a $(k-2)$-ranking, i.e. $\chi_r(G_1 - v) \leq k - 2$ and it follows that G_1 is $(k-1)$-critical. By symmetry the same reasoning applies to G_2. ■

6. Parity and Ranking on Trees

In what follows we use the *canonical trees* which can be defined recursively. A graph K_1 is the first canonical tree T_1 with the only vertex as its root. The canonical tree T_k is obtained by taking two disjoint copies of trees T_{k-1} and joining their roots by an edge, then taking the root of the second copy to be the root of T_k.

Lemma 3. *The canonical tree T_k is ranking k-critical.*

Proof. By Theorem 11 for any two disjoint ranking $(k-1)$-critical graphs G_1 and G_2, the graph $G = (G_1 \cup G_2) + v_1v_2$ is ranking k-critical for any $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Obviously, $K_2 \simeq T_2$ is ranking 2-critical. Hence following the above mentioned result $T_3 \simeq P_4 = (K_2 \cup K_2) + v_1v_2$ is ranking 3-critical. Assume T_{k-1} to be ranking $(k-1)$-critical. It follows by induction that T_k is ranking k-critical. ■

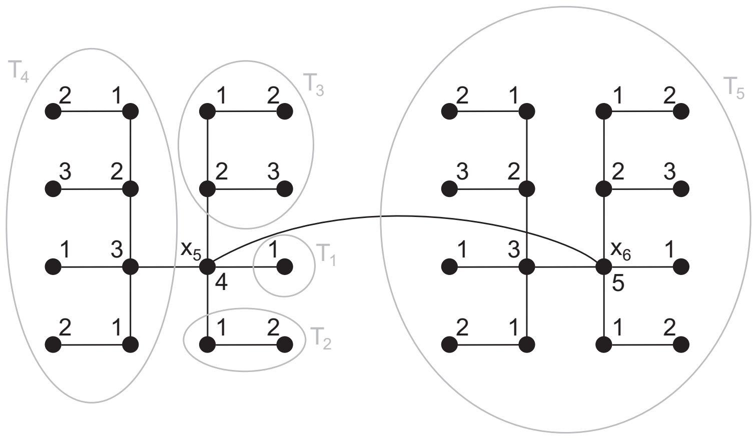

Figure 3. Parity vertex colouring of the cannonical tree T_6.

In order to obtain a k-ranking of T_k, observe that removing all vertices of degree 1 results in T_{k-1}. Now, colour greedily the subsequent vertices ordered non-decreasingly with respect to their degrees, which produces a vertex k-ranking of T_k.

Theorem 12. *For any canonical tree T_k, $k \geq 4$, we have*

$$\chi_r(T_k) - \chi_p(T_k) \geq 1.$$

Proof. It is not hard to see that for $k \in \{1,2,3\}$ we have $\chi_p(T_k) = k$ while $\chi_p(T_4) = 3$ (see Figure 3). The crucial property of canonical trees is that removing the root-vertex x_k from T_k gives a forest consisting of the $k-1$ components H_i isomorphic to the appropriate T_i, $i \in \{1,\ldots,k-1\}$ respectively. Assume that $\chi_p(T_i) \le i-1$ holds for all T_i, $4 \le i \le k-1$. It is enough to colour the root x_k of T_k using colour $k-1$ to obtain a $(k-1)$-parity vertex colouring of T_k. Hence by induction on k, we have $\chi_p(T_k) \le k-1$ for $k \ge 4$ and since by Lemma 3, $\chi_r(T_k) = k$ for $k \ge 1$, the theorem follows. ■

We strongly believe that it is possible to prove the following

Conjecture 1. For any tree T we have $\chi_r(T) - \chi_p(T) \le 1$.

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