

THE INDEPENDENT DOMINATION NUMBER OF A RANDOM GRAPH

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Abstract

We prove a two-point concentration for the independent domination number of the random graph $G_{n,p}$ provided $p^2 \ln(n) \geq 64 \ln((\ln n)/p)$.

Keywords: random graph, two-point concentration, independent domination.

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1. INTRODUCTION

Let G be a graph with vertex set $[n]$ and let $S \subset [n]$. If for every vertex $u \notin S$ there is a vertex $v \in S$ such that u and v are adjacent then S is called a *dominating set*. If further for every $v, w \in S$ there is no edge between v and w then S is called an *independent dominating set*. The *domination number*, $\gamma(G)$ is the smallest integer s such that there exists a dominating set of cardinality s . The *independent domination number*, $i(G)$ is the smallest integer s such that there exists an independent dominating set of cardinality s . $\mathcal{G}(n, p)$ is the set of all graphs $G_{n,p}$ with vertex set $[n]$ and edges chosen independently with probability $0 \leq p = p(n) \leq 1$. Hence, for each $G_{n,p}$ $P(G_{n,p}) = p^{e(G_{n,p})}(1-p)^{\binom{n}{2}-e(G_{n,p})}$. For a graph property A we say A occurs *asymptotically almost surely* (a.a.s.) if $P(G_{n,p} \text{ has property } A) \rightarrow 1$ as $n \rightarrow \infty$. See Bollobás [2] for notation and terminology.

Weber [7] showed if $p = 1/2$ then a.a.s. $\gamma(G_{n,p})$ is either $\lfloor \log_2 n - \log_2(\log_2 n \ln n) \rfloor + 1$ or $\lfloor \log_2 n - \log_2(\log_2 n \ln n) \rfloor + 2$ and a.a.s. $i(G_{n,p})$

is $\lfloor \log_2 n - \log_2(\log_2 n \ln n) \rfloor + 2$ or $\lfloor \log_2 n - \log_2(\log_2 n \ln n) \rfloor + 3$. Godbole and Wieland [4] extended Weber's result showing if p is constant or $p = p(n) \rightarrow 0$ such that $p^2 \ln n \geq 40 \ln((\ln^2 n)/p)$ then a.a.s. $\gamma(G_{n,p})$ is either $\lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor + 2$, where $b = 1/(1-p)$. Very recently Bonato and Wang [3] showed that if p is constant then a.a.s. $\lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor + 1 \leq i(G_{n,p}) \leq \lfloor \log_b n \rfloor$. In this paper we show that if p is constant or $p = p(n) \rightarrow 0$ such that $p^2 \ln(n) \geq 64 \ln(\ln(n)/p)$ then a.a.s. $i(G_{n,p})$ is either $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$. This extends Weber's result (the case $p = 1/2$) and immediately implies Bonato and Wang's result (the case p is constant). We then empirically explore the number of independent dominating sets of size k ranging on $[n]$ and make a conjecture about the distribution.

2. TWO-POINT CONCENTRATION

Throughout this section we will use p as the probability an edge exists in $G = G_{n,p}$, $q = 1 - p$ the probability an edge does not exist in G and $b = \frac{1}{q}$. We will also make extensive use of two inequalities,

$$(1) \quad 1 - x \leq \exp\{-x\}, x \in \mathbb{R},$$

$$(2) \quad 1 - x \geq \exp\left\{\frac{-x}{1-x}\right\}, x \in [0, 1).$$

We begin by defining the random variables X_k and Y_s as the number of independent dominating sets of cardinality k in G and the number of independent dominating sets of cardinality s or less in G respectively. Clearly $Y_s = \sum_{k=1}^s X_k$. It is now obvious that

$$E(X_k) = \binom{n}{k} (1 - q^k)^{n-k} q^{\binom{k}{2}}$$

and by linearity of expectation,

$$E(Y_s) = \sum_{k=1}^s E(X_k) = \sum_{k=1}^s \binom{n}{k} (1 - q^k)^{n-k} q^{\binom{k}{2}}.$$

We now state our first lemma.

Lemma 2.1. *Let $s = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor$, then $E(Y_s) \rightarrow 0$ if p fixed or if $p \rightarrow 0$ as $n \rightarrow \infty$ and $p \geq \frac{\epsilon \ln^2 n}{n}$.*

Proof. Lemma 2 of [4] states the expected number of dominating sets of size less than or equal to $r = \lfloor \log_b n - \log_b(\log_b n \ln n) \rfloor$ goes to 0 if $p \geq \frac{\epsilon \ln^2 n}{n}$. Since every independent dominating set is a dominating set it is clear $E(Y_r) \rightarrow 0$ as $n \rightarrow \infty$. It remains to show,

$$\sum_{k=r+1}^s E(X_k) \rightarrow 0.$$

Using Stirling's inequality, inequality (1),

$$\begin{aligned} E(X_k) &= \binom{n}{k} (1 - q^k)^{n-k} q^{\binom{k}{2}} \\ &\leq \exp \left\{ k \ln n + 2k - k \ln k - nq^k + \frac{k^2}{2} \ln q - \frac{k}{2} \ln q \right\} \\ &:= \exp\{f(k)\}. \end{aligned}$$

Now,

$$f'(k) = \ln n + 1 - \ln k + nq^k \ln \left(\frac{1}{q} \right) - k \ln \left(\frac{1}{q} \right) - \frac{1}{2} \ln \left(\frac{1}{q} \right).$$

Note $f'(k)$ is decreasing for all positive value of k and $f'(\log_b n - \log_b(\log_b n \ln n) + \log_b 2) \geq 0$ for sufficiently large n . So for sufficiently large n , we have $f(k)$ increasing for all $k \leq \log_b n - \log_b(\log_b n \ln n) + \log_b 2$. Hence, setting $k = \log_b n - \log_b(\log_b n \ln n) + \log_b 2$ we have

$$\begin{aligned} E(Y_s) &\leq (k - r) \exp\{f(k)\} \\ &\leq (\log_b 2) \exp \left\{ k \ln n + 2k - k \ln k - nq^k + \frac{k^2}{2} \ln q - \frac{k}{2} \ln q \right\} \\ &\leq (\log_b 2) \exp \left\{ -k \ln k + 3k + \frac{k}{2} \ln \left(\frac{1}{q} \right) \right\} \\ &\rightarrow 0 \end{aligned}$$

since $k \ln k$ clearly dominates the other two terms in the exponent.

We now note that since $i(G)$ is always at least 1,

$$\log_b n - \log_b(\log_b n \ln n) + \log_b 2 \geq 1.$$

A condition satisfied if

$$p \geq \frac{e \ln^2 n}{2n}$$

which is easily seen after noting $p \leq \ln \frac{1}{q}$. However, the condition $p \geq \frac{e \ln^2 n}{n}$ used above is stronger so we must use it instead. ■

Lemma 2.2. *If p fixed or if $p \rightarrow 0$ and $\frac{p^2}{64} \geq \frac{\ln(\frac{\ln n}{p})}{\ln n}$ then $E(X_s) \rightarrow \infty$ for $s = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$.*

Proof. Using inequality (2), Stirling's Formula, and that for $k^2 = o(n)$ $(n)_k = (1 - o(1))n^k$

$$\begin{aligned} & E(X_k) \\ &= \binom{n}{k} (1 - q^k)^{n-k} q^{\binom{k}{2}} \\ &\geq \binom{n}{k} (1 - q^k)^n q^{\frac{k^2}{2}} \\ (3) \quad &\geq \binom{n}{k} \exp \left\{ \frac{-nq^k}{1 - q^k} + \frac{k^2}{2} \ln q \right\} \\ &\geq (1 - o(1)) \frac{n^k}{k!} \exp \left\{ \frac{-nq^k}{1 - q^k} + \frac{k^2}{2} \ln q \right\} \quad (\text{if } k^2 = o(n)) \\ &\geq (1 - o(1)) \left(\frac{ne}{k} \right)^k (2\pi k)^{-\frac{1}{2}} \exp \left\{ \frac{-nq^k}{1 - q^k} + \frac{k^2}{2} \ln q \right\} \quad (\text{if } k \rightarrow \infty) \\ &\geq (1 - o(1)) \exp \left\{ k \ln n + k - k \ln k - \frac{1}{2} \ln(2\pi k) - \frac{nq^k}{1 - q^k} + \frac{k^2}{2} \ln q \right\}. \end{aligned}$$

The condition $k^2 = o(n)$ is satisfied if $p \gg \frac{\ln n}{n^2}$ and $k = \log_b n - \log_b(\log_b n \ln n) + \log_b 2 + \epsilon$, where $\epsilon > 0$. One can easily show

$\frac{d}{dk} \left(k \ln n + k - k \ln k - \frac{1}{2} \ln(2\pi k) - \frac{nq^k}{1 - q^k} + \frac{k^2}{2} \ln q \right) \geq 0$ as long as k is much smaller than nq^k , which is true for large n when assuming the just mentioned conditions. Substituting in (3) $k = s$ on the left and $k =$

$\log_b n - \log_b (\log_b n \ln n) + \log_b 2 + \frac{1}{2}$ on the right it is shown for sufficiently large n

$$\begin{aligned} E(X_s) &\geq (1 - o(1)) \exp \left\{ \frac{1}{2} \log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{1 - q^k} \right) + \log_b n \right. \\ &\quad - \log_b (\log_b n \ln n) \ln (\log_b n \ln n) + \frac{1}{2} \ln (\log_b n \ln n) \\ &\quad \left. - \log_b n \ln s - (1 + \log_b 2) \ln s - \frac{1}{2} \ln 2\pi - \frac{1}{8} \ln \frac{1}{q} \right\} \\ &\geq (1 - o(1)) \exp \{A - B\}, \end{aligned}$$

where

$$\begin{aligned} A &= \frac{1}{2} \log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{1 - q^s} \right) + \log_b n, \\ B &= \log_b (\log_b n \ln n) \ln (\log_b n \ln n) + \log_b n \ln (\log_b n), \\ &\quad + (1 + \log_b 2) \ln (\log_b n) + \frac{1}{2} \ln (2\pi e^L) \end{aligned}$$

and L is any constant bounding $\frac{1}{8} \ln \frac{1}{q}$, which exists since $\ln \left(\frac{1}{q} \right)$ is constant or $\ln \left(\frac{1}{q} \right) \rightarrow 0$. Since $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$ and $\log_b n \sim \frac{\ln n}{p}$ we have $p \gg \frac{\log_b n \ln n}{n}$. So for n sufficiently large,

$$\begin{aligned} A &= \frac{1}{2} \log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{(1 - q^s)} \right) + \log_b n \\ &= \frac{1}{2} \log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{\left(1 - \frac{q^{\frac{1}{2}} \log_b n \ln n}{2n} \right)} \right) + \log_b n \\ &\geq \frac{1}{2} \log_b n \ln n \left(1 - \frac{q^{\frac{1}{2}}}{\left(1 - \frac{pq^{\frac{1}{2}}}{2} \right)} \right) + \log_b n \\ &= \frac{1}{2} \log_b n \ln n \left(\frac{1 - \frac{pq^{\frac{1}{2}}}{2} - q^{\frac{1}{2}}}{1 - \frac{pq^{\frac{1}{2}}}{2}} \right) + \log_b n. \end{aligned}$$

Using the inequality, $\frac{x}{2} \leq 1 - (1 - x)^{\frac{1}{2}}$, we obtain

$$\begin{aligned}
A &\geq \frac{p}{4} \log_b n \ln n \left(\frac{1 - (1-p)^{\frac{1}{2}}}{1 - \frac{pq^{\frac{1}{2}}}{2}} \right) + \log_b n \\
&\geq \frac{p}{4} \log_b n \ln n \left(\frac{\frac{p}{2}}{1 - \frac{pq^{\frac{1}{2}}}{2}} \right) + \log_b n \\
&\geq \frac{p^2}{8} \log_b n \ln n + \log_b n.
\end{aligned}$$

Define C as:

$$(4) \quad C = \frac{p^2 \log_b n \ln n}{8} + \log_b n.$$

We will now find p such that for n sufficiently large $\frac{C}{8}$ is larger than all terms in B . Hence

$$\begin{aligned}
(5) \quad (1 - o(1)) \exp \{A - B\} &\geq (1 - o(1)) \exp \{C - B\} \\
&\geq (1 - o(1)) \exp \{C/2\} \\
&\rightarrow \infty.
\end{aligned}$$

It is obvious that the third and fourth terms of B are dominated by the first so we will only compare the first and second terms to $C/8$. Comparing the first term,

$$C/8 \geq \frac{1}{2} \log_b (\log_b n \ln n) \ln (\log_b n \ln n)$$

if for sufficiently large n

$$(6) \quad \frac{p}{8} \geq \frac{\ln \left(\frac{\ln^2 n}{p} \right)}{\sqrt{2} \ln n}.$$

Comparing the second term,

$$C/8 \geq \log_b n \ln (\log_b n)$$

if for sufficiently large n

$$(7) \quad \frac{p^2}{64} \geq \frac{\ln \left(\frac{\ln n}{p} \right)}{\ln n}.$$

Clearly (7) implies (6) and the condition $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$ and the lemma is proven. ■

Lemma 2.3. *If p fixed or if $p \rightarrow 0$ and $\frac{p^2}{64} \geq \frac{\ln\left(\frac{\ln n}{p}\right)}{\ln n}$ then $\frac{\text{Var} X_s}{E^2(X_s)} \rightarrow 0$ for $s = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$.*

Proof. Following the proof of Lemma 3 in [4] it is easily derived that

$$\begin{aligned} \text{Var}(X_s) &\leq E(X_s) - E^2(X_s) \\ &\quad + \binom{n}{s} \sum_{m=0}^{s-1} \binom{s}{m} \binom{n-s}{s-m} (1 - 2q^s + q^{2s-m})^{n-2s+m} q^{2\binom{s}{2} - \binom{m}{2}}. \end{aligned}$$

We write $s = \log_b n - \log_b(\log_b n \ln n) + \log_b 2 + \epsilon$ where $\epsilon = \epsilon(n) = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2 - \log_b n + \log_b(\log_b n \ln n) - \log_b 2$ and observe that $1 \leq \epsilon \leq 2$.

It is immediately obvious for any s such that $E(X_s) \rightarrow \infty$,

$$E(X_s) = o(E^2(X_s)).$$

We will now show

$$(8) \quad \binom{n}{s} \binom{s}{0} \binom{n-s}{s} (1 - 2q^s + q^{2s})^{n-2s} q^{2\binom{s}{2}} - E^2(X_s) = o(E^2(X_s))$$

and

$$\begin{aligned} (9) \quad &\binom{n}{s} \sum_{m=1}^{s-1} \binom{s}{m} \binom{n-s}{s-m} (1 - 2q^s + q^{2s-m})^{n-2s+m} q^{2\binom{s}{2} - \binom{m}{2}} \\ &= o(E^2(X_s)). \end{aligned}$$

To show (8) note,

$$\begin{aligned} &\binom{n}{s} \binom{s}{0} \binom{n-s}{s} (1 - 2q^s + q^{2s})^{n-2s} q^{2\binom{s}{2}} - E^2(X_s) \\ &\leq E^2(X_s) ((1 - q^s)^{-2s} - 1) \\ &\leq E^2(X_s) \left(\exp \left\{ \frac{2sq^s}{1 - q^s} \right\} - 1 \right) \quad (\text{by (2)}). \end{aligned}$$

Since $p \gg \frac{\ln \frac{3}{2} n}{n^{\frac{1}{2}}}$, we know that $\frac{2sq^s}{1-q^s} \geq 0$ and approaches 0 as $n \rightarrow \infty$. Thus,

$$\left(\exp \left\{ \frac{2sq^s}{1-q^s} \right\} - 1 \right) \rightarrow 0.$$

To show (9) let

$$f(m) = \binom{s}{m} \binom{n-s}{s-m} (1 - 2q^s + q^{2s-m})^{n-2s+m} q^{2\binom{s}{2} - \binom{m}{2}}$$

and note for sufficiently large n

$$\begin{aligned} f(m) &\leq \binom{s}{m} \frac{n^{s-m}}{(s-m)!} (1 - 2q^s + q^{2s-m})^{n-2s+m} q^{2\binom{s}{2} - \binom{m}{2}} \\ &\leq 2 \binom{s}{m} \frac{n^{s-m}}{(s-m)!} (1 - 2q^s + q^{2s-m})^n q^{2\binom{s}{2} - \binom{m}{2}} \\ &\leq 2 \binom{s}{m} \frac{n^{s-m}}{(s-m)!} \exp(n(-2q^s + q^{2s-m})) q^{2\binom{s}{2} - \binom{m}{2}} \quad (\text{by (1)}) \end{aligned}$$

where the second inequality holds for $p \gg \frac{\ln \frac{3}{2} n}{n^{\frac{1}{2}}}$. Define

$$g(m) := 2 \binom{s}{m} \frac{n^{s-m}}{(s-m)!} \exp(n(-2q^s + q^{2s-m})) q^{2\binom{s}{2} - \binom{m}{2}}$$

and consider the the ratio of consecutive terms of $g(m)$.

$$(10) \quad h(m) := \frac{g(m+1)}{g(m)} = \frac{(s-m)^2}{nq^m(m+1)} \exp\{npq^{2s-m-1}\}.$$

We will show $h(m) \geq 1$ iff $m \geq m_0$ for some $m_0(n) \rightarrow \infty$, hence g is first decreasing and then increasing. Further we will show $g(1) \geq g(s-1)$, which implies $\sum_{m=1}^{s-1} f(m) \leq sg(1)$. Observe for sufficiently large n ,

$$\begin{aligned} h(1) &= \frac{(s-1)^2}{2nq} \exp\left\{ \frac{np}{q^2} q^{2s} \right\} \\ &\leq \frac{\log_b^2 n}{2nq} \exp\left\{ \frac{(\log_b n \ln n)^2 p}{4nq^{2-2\epsilon}} \right\} \end{aligned}$$

$$\leq \frac{\ln^2 n}{2np^2q} \exp \left\{ \frac{\ln^4 n}{4npq^{2-2\epsilon}} \right\} \quad (\text{by (1)})$$

$$\rightarrow 0$$

since $p \gg \frac{\ln n}{n^{\frac{1}{2}}}$ and

$$\begin{aligned} h(s-1) &= \frac{1}{nsq^{s-1}} \exp \{ npq^s \} \\ &\geq \frac{2q^{1-\epsilon}}{\log_b^2 n \ln n} \exp \left\{ \frac{pq^\epsilon \log_b n \ln n}{2} \right\} \\ &= \frac{2q^{1-\epsilon} \ln^2 \frac{1}{q}}{\ln^3 n} \exp \left\{ \frac{pq^\epsilon \ln^2 n}{-2 \ln q} \right\} \\ &\geq \frac{2q^{1-\epsilon} p^2}{\ln^3 n} \exp \left\{ \frac{q^{1+\epsilon} \ln^2 n}{2} \right\} \quad (\text{by (1), (2)}) \\ &\geq 1 \end{aligned}$$

provided $p \neq 1 - o(1)$. Also,

$$h(m) = \frac{(s-m)^2}{n(m+1)q^m} \exp \{ npq^{2s-m-1} \} \geq 1$$

iff

$$npq^{2s-m-1} \geq \ln \left(\frac{n(m+1)q^m}{(s-m)^2} \right)$$

iff

$$m \geq \log_b \left(\frac{\ln \left(\frac{n(m+1)q^m}{(s-m)^2} \right)}{npq^{2s-1}} \right)$$

iff

$$m \geq \log_b \left(\frac{4n \ln \left(\frac{n(m+1)q^m}{(s-m)^2} \right)}{p \log_b^2(n) \ln^2(n)} \right) + 2\epsilon - 1$$

iff

$$m \geq \log_b \left(\frac{4n}{p} \right) + \log_b \left(\ln \left(\frac{n(m+1)q^m}{(s-m)^2} \right) \right) - 2 \log_b (\ln n \log_b n) + 2\epsilon - 1.$$

Define

$$x(m) = \log_b \left(\frac{4n}{p} \right) + \log_b \left(\ln \left(\frac{n(m+1)q^m}{(s-m)^2} \right) \right) - 2 \log_b (\ln n \log_b n) + 2\epsilon - 1.$$

Now,

$$\frac{d}{dm} x(m) = \frac{\left(m^2 - \left(s - 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} \right) m - \left(1 - \frac{1}{\ln\left(\frac{1}{q}\right)} \right) s + \frac{2}{\ln\left(\frac{1}{q}\right)} \right)}{(m+1)(s-m) \left(\ln \left(\frac{n(m+1)q^m}{(s-m)^2} \right) \right)}$$

and the roots of the numerator are:

$$\begin{aligned} & \frac{s - 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} \pm \sqrt{\left(s - 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} \right)^2 + 4s \left(1 - \frac{1}{\ln\left(\frac{1}{q}\right)} \right) - \frac{8}{\ln\left(\frac{1}{q}\right)}}}{2} \\ &= \frac{s - 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} \pm (s+1) \sqrt{\left(1 - \frac{3}{(s+1)\ln\left(\frac{1}{q}\right)} \right)^2 - \frac{8}{(s+1)\ln^2\left(\frac{1}{q}\right)}}}{2}. \end{aligned}$$

Using Taylor Series with remainder about 0, one can show if $0 \leq z \leq 3 - 2\sqrt{2}$ then for any y such that $|y| \leq z$

$$1 - 3y - \frac{8z^2}{(1 - 6z + z^2)^{\frac{3}{2}}} \leq \sqrt{(1 - 3y)^2 - 8y^2} \leq 1 - 3y + \frac{8z^2}{(1 - 6z + z^2)^{\frac{3}{2}}}.$$

Letting $y = z = \frac{1}{(s-1)\ln\left(\frac{1}{q}\right)}$, we show

$$\frac{d}{dm} x(m) = \frac{\left(m + 1 - \frac{1}{\ln\left(\frac{1}{q}\right)} - \delta \right) \left(m - s + \frac{2}{\ln\left(\frac{1}{q}\right)} + \delta \right)}{(m+1)(s-m) \left(\ln \left(\frac{n(m+1)q^m}{(s-m)^2} \right) \right)}$$

$$\text{where } |\delta| \leq \frac{8}{(s+1)\ln^2\left(\frac{1}{q}\right) \left(1 - \frac{6}{(s+1)\ln\left(\frac{1}{q}\right)} + \frac{1}{(s+1)^2\ln^2\left(\frac{1}{q}\right)} \right)^{\frac{3}{2}}}.$$

Thus $\delta = \Theta\left(\frac{1}{p \ln n}\right) \rightarrow 0$ as $n \rightarrow \infty$ since $p \gg \frac{1}{\ln(n)}$.

So on $(-\infty, -1)$ and $(\ln^{-1}(\frac{1}{q}) - 1 + \delta, s - 2 \ln^{-1}(\frac{1}{q}) - \delta)$ $x(m)$ is decreasing and on $(-1, \ln^{-1}(\frac{1}{q}) - 1 + \delta)$ and $(s - 2 \ln^{-1}(\frac{1}{q}) - \delta, s)$ $x(m)$ is increasing. Thus $m_1 = \ln^{-1}(\frac{1}{q}) - 1 + \delta$ is a relative maximum and $m_2 = s - 2 \ln^{-1}(\frac{1}{q}) - \delta$ is a relative minimum of $x(m)$.

Note $m_1 \in [1, s-1]$ iff $p \leq 1 - e^{-\frac{1}{2-\delta}}$ and $m_2 \in [1, s-1]$ iff $p \leq 1 - e^{-\frac{2}{1-\delta}}$. Also for n sufficiently large, $x(m)$ is continuous on $[1, s-1]$, for every $m \in [1, s-1]$ $x(m) \in [1, s-1]$, and $s-1 > x(1) > x(s-1) > 1$.

If $p > 1 - e^{-\frac{2}{1-\delta}}$, on $[1, s-1]$ $x(m)$ has an absolute maximum at 1 and an absolute minimum at $s-1$. So by the above information and the intermediate value theorem there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x(m_0)$ and $x(m_0) > x(s-1)$.

If $1 - e^{-\frac{1}{2-\delta}} < p \leq 1 - e^{-\frac{2}{1-\delta}}$, on $[1, s-1]$ $x(m)$ has an absolute maximum at 1 and an absolute minimum at m_2 . So by the above information and the intermediate value theorem there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x(m_0)$. Further, one can show by iteration that $x(m_0) \geq x(s-1)$.

If $p \leq 1 - e^{-\frac{1}{2-\delta}}$ or $p \rightarrow 0$, on $[1, s-1]$ $x(m)$ has an absolute maximum at m_1 and an absolute minimum at m_2 . So by the above information and the intermediate value theorem there exists a unique $m_0 \in [1, s-1]$ such that $m_0 = x(m_0)$. Further, one can show by iteration that $x(m_0) \geq x(s-1)$.

Thus, in any of the three cases there exists a unique $m_0 \in [1, s-1]$ such that $\forall m \geq m_0 = x(m_0) \geq x(m)$.

Now, for n sufficiently large

$$\ln \left(\frac{n(m_0 + 1)q^{m_0}}{(s - m_0)^2} \right) \geq \ln(nsq^{s-1}) \geq \ln \left(\frac{\log_b(n) \ln(n)s}{4q^{1-\epsilon}} \right)$$

which goes to infinity as n goes to infinity. Also, $\log_b\left(\frac{4n}{p}\right) \gg 2 \log_b(\log_b(n) \ln(n))$ and $2\epsilon - 1$ is bounded, thus $m_0 \rightarrow \infty$. Therefore, $h(m) \geq 1$ iff $m \geq m_0 \rightarrow \infty$ as $n \rightarrow \infty$.

Also, $g(1) \geq g(s-1)$ iff

$$\frac{n^{s-1}}{(s-1)!} \exp\{nq^{2s-1}\} \geq n \exp\{nq^{s+1}\} q^{-\binom{s-1}{2}}$$

iff

$$\frac{n^s}{s!} \exp\{-n(q^{s+1} - q^{2s-1})\} q^{\binom{s-1}{2}} \geq \frac{n^2}{s}$$

which is true since

$$\frac{n^s}{s!} \exp\{-n(q^{s+1} - q^{2s-1})\} q^{\binom{s-1}{2}} \geq E(X_s) \geq (1 - o(1)) \exp\{C/2\}$$

where $C = \frac{p^2 \log_b n \ln n}{8} + \log_b n$ and $(1 - o(1)) \exp\{C/2\} \geq n^2/s$ if $p \geq \frac{24}{\ln n}$, a condition clearly satisfied by our hypothesis. Hence we have shown,

$$\binom{n}{s} \sum_{m=1}^{s-1} \binom{s}{m} \binom{n-s}{s-m} (1 - 2q^s - q^{2s-m})^{n-2s+m} q^{2\binom{s}{2} - \binom{m}{2}} \leq s \binom{n}{s} g(1).$$

Finally, we show $s \binom{n}{s} g(1) = o(E^2(X_s))$,

$$\begin{aligned} \frac{s \binom{n}{s} g(1)}{E^2(X_s)} &= \frac{2s^2 n^{s-1} \exp\{n(q^{2s-1} - 2q^s)\}}{\binom{n}{s} (1 - q^s)^{2(n-s)} (s-1)!} \\ &\leq \frac{2s^3 \exp\{n(q^{2s-1} - 2q^s)\}}{(1 - o(1))n(1 - q^s)^{2n}} \quad (s^2 = o(n)) \\ &\leq \frac{2s^3}{(1 - o(1))n} \exp\left\{n \left(q^{2s-1} - 2q^s + \frac{2q^s}{1 - q^s}\right)\right\} \quad (\text{by (2)}) \\ &\leq \frac{2s^3}{(1 - o(1))n} \exp\left\{\frac{nq^{2s-1}(1 + 2q)}{1 - q^s}\right\} \\ &\leq \frac{2 \log_b^3 n}{(1 - o(1))n} \exp\left\{\frac{3 \log_b^2 n \ln^2 n}{4n(1 - q^s)} q^{2\epsilon-1}\right\} \\ &\rightarrow 0 \end{aligned}$$

since $p \gg \frac{\ln n}{n^{\frac{1}{3}}}$.

We have thus shown if $s = \log_b n - \log_b(\log_b n \ln n) + \log_b 2 + \epsilon = \lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$ then $\text{Var}(X_s) = o(E^2(X_s))$ provided $\frac{p^2}{64} \geq \frac{\ln(\frac{\ln n}{p})}{\ln n}$. \blacksquare

We now can state our main result.

Theorem 2.4. *Let p be fixed or $\frac{p^2}{64} \geq \frac{\ln(\frac{\ln n}{p})}{\ln n}$ then $i(G)$ is equal to $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 1$ or $\lfloor \log_b n - \log_b(\log_b n \ln n) + \log_b 2 \rfloor + 2$ a.a.s.*

Proof. By Markov's Inequality and Lemma 2.1 if $s = \lfloor \log_b n - \log_b (\log_b n \ln n) + \log_b 2 \rfloor$ then

$$P(i(G) \leq s) = P(Y_s \geq 1) \leq E(Y_s) \rightarrow 0$$

and by Chebyshev's Inequality, Lemma 2.2, and Lemma 2.3 if $s = \lfloor \log_b n - \log_b (\log_b n \ln n) + \log_b 2 \rfloor + 2$ then

$$P(i(G) > s) \leq P(X_s = 0) \leq P(|X_s - E(X_s)| \geq E(X_s)) \leq \frac{\text{Var}(X_s)}{E^2(X_s)} \rightarrow 0. \quad \blacksquare$$

3. EMPIRICAL DATA

In this section we used a standard random number generator which we verified to return a nearly uniform distribution for samples of size 10^6 lending some credibility to the empirical results.

We generated $N = 1000$ graphs G of order n , by randomly assigning edges with probability p , using a standard uniform random number generator. We then ran an algorithm of our design to count the number of graphs with an independent domination number of size k for each $k \in [n]$. The results are presented in the chart below for $n = 16, 32, 64$, and 128 with probabilities $p = .5$ and $.75$.

n, p	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
16, .5	0	522	476	2	0
32, .5	0	36	962	2	0
64, .5	0	0	693	307	0
64, .75	0	998	2	0	0
128, .75	0	419	561	0	0

We generated $N = 1000$ graphs G of order n , by randomly assigning edges with probability p , using a standard uniform random number generator. We then ran an algorithm of our design to calculate the average number of independent dominating sets of size k contained in all G for each $k \in [n]$. The results are presented in the chart below for $n = 12, 16, 20, 24$ and $p = .5$.

n	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
12	0.1	1.96	7.87	4	0.56	.11	0	0
16	0	1.12	13.02	12.98	3.91	.14	0	0
20	0	0.55	13.08	26.33	8.83	1.39	0	0
24	0	0.17	15.81	45.94	21.6	2.66	0.05	0.01

This data appears to imply the following conjecture:

Conjecture 3.1. $Y = X_1 + X_2 + \cdots + X_n$ suitably normalized converges weakly to $N(0, 1)$.

We have no proof to date. The random variables $\{X_k\}$ are highly dependent posing difficulty in verifying the conditions needed in many central limit theorems for dependent sums.

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