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WEAK ROMAN DOMINATION IN GRAPHS

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Abstract

Let $G = (V, E)$ be a graph and f be a function $f : V \to \{0, 1, 2\}$. A vertex u with $f(u) = 0$ is said to be *undefended* with respect to f, if it is not adjacent to a vertex with positive weight. The function f is a *weak Roman dominating function* (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f^{'} : V \to \{0, 1, 2\}$ defined by $f^{'}(u) = 1, f^{'}(v) = f(v) - 1$ and $f^{'}(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The *weight of* f is $w(f) = \sum_{v \in V} f(v)$. *The weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G. In this paper, we characterize the class of trees and split graphs for which $\gamma_r(G) = \gamma(G)$ and find γ_r-value for a caterpillar, a $2 \times n$ grid graph and a complete binary tree.

Keywords: domination number, weak Roman domination number.

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1. Introduction

Cockayne *et al.* [1] defined a *Roman dominating function* (RDF) on a graph $G = (V, E)$ to be a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. For a real valued function $f : V \to R$, *the weight of* f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V, f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. *The Roman Domination number*, denoted by $\gamma_R(G)$ is the minimum weight of an RDF in G; that is $\gamma_R(G) = \min\{w(f) : f \text{ is a RDF in } G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function.

Let V_0, V_1 and V_2 be the sets of vertices assigned the values 0, 1 and 2 respectively, under f. There is a one to one correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus $f = (V_0, V_1, V_2)$.

Henning *et al.* [4] defined the *weak Roman dominating function* as follows. A vertex $u \in V_0$ is *undefended*, if it is not adjacent to a vertex in V_1 or V_2. The function f is a *weak Roman dominating function* if each vertex $u \in V_0$ is adjacent to a vertex $v \in V_1 \cup V_2$ such that the function $f' : V \to \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. *The weight* $w(f)$ *of* f is defined to be $|V_1| + 2|V_2|$. *The weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight of a WRDF in G; that is, $\gamma_r(G) = \min\{w(f) : f \text{ is a WRDF in } G\}$. A WRDF of weight $\gamma_r(G)$ is called a $\gamma_r(G)$-function. Roman domination and Weak Roman domination in graphs have been studied in $[1, 4 - 12]$.

Notice that in a WRDF, every vertex in V_0 is dominated by a vertex in $V_1 \cup V_2$, while in an RDF every vertex in V_0 is dominated by at least one vertex in V_2 (this is more expensive). Furthermore, in a WRDF, every vertex in V_0 can be defended without creating an undefended vertex.

It has been observed that $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$. In this paper, we focus our study on the relation $\gamma(G) \leq \gamma_r(G)$. We characterize the class of trees and split graphs for which $\gamma_r(G) = \gamma(G)$ and find γ_r-value for some specific graphs.

2. Notation

For notation and graph theoretic terminology we in general follow [2]. Throughout this paper, we only consider finite undirected graphs with neither loops nor multiple edges. Let $G = (V, E)$ be a graph with vertex set V of order n and edge set E, and let v be a vertex in V. *The open neighborhood of* v is $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood of v is $N[u] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its open neighborhood $N(S) = \bigcup_{v \in S} N(v)$ and its *closed neighborhood* $N[S] = N(S) \cup S$. A vertex u is called a *private neighbor of* v with respect to S, or simply *an S-pn of* v, if $N[u] \cap S = \{v\}$. The set $pn(v, S) = N[v] - N[S - \{v\}]$ of all *S-pns* of v is called the *private neighbor set of* v with respect to S. *The external private neighbor set of* v with respect to S is defined as $epn(v, S) = pn(v, S) - \{v\}$. Hence the set $epn(v, S)$ consist of all *S-pns* of v that belong to $V - S$.

Distance between two vertices u and v is denoted as $d(u, v)$. For $k \geq 1$, the open neighborhood of a vertex $v \in V(T)$, denoted by $N_k(v)$ is the set of vertices in $V(T)$ different from v whose distance from v is at most k. That is $N_k(v) = \{w \in V(T) - \{v\} : d(v, w) \leq k\}$. The boundary of the open k-neighborhood of v, denoted by $\partial N_k(v)$ is the set of vertices in $V(T)$ whose distance from v is exactly k. That is $\partial N_k(v) = \{w \in V(T) : d(v, w) = k\}$. Note that $v \notin N_k(v), \partial N_k(v) \subseteq N_k(v)$ if $k \geq 1$.

A *star* $K_{1,n}$ has one vertex v of degree n and n vertices of degree one. A *split graph* is a graph $G = (V, E)$ whose vertices can be partitioned into two sets X and Y where the vertices in X are independent and vertices in Y form a complete graph. A *leaf* is a vertex whose degree is one. A *support* is a vertex which is adjacent to at least one leaf. A *weak support* is a vertex which is adjacent to exactly one leaf. A *strong support* is a vertex which is adjacent to at least two leaf vertices. *A rooted tree* is a tree in which one of the vertices is distinguished from others. The distinguished vertex is called the *root* of the tree. The length of the path from the root r to a vertex x is the *depth* of x in T. *A complete binary tree* is a 2-ary tree in which all leaves have the same depth and all internal vertices have degree 3, except the root. If T is a complete binary tree with root vertex v, the set of all vertices with depth k are called *vertices at level k*. A *caterpillar* is a tree whose removal of leaf vertices leaves a path which is called the *spine* of the caterpillar.

For arbitrary graphs G and H, the *Cartesian product* of G and H is defined to be the graph $G \square H$ with vertices $\{(u, v) : u \in G, v \in H\}$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \square H$ if and only if one of the following is true: $u_1 = u_2$ and v_1 is adjacent to v_2 in H; or $v_1 = v_2$ and u_1 is adjacent to u_2 in G. If $G = P_m$ and $H = P_n$, then the Cartesian product $G \square H$ is called the $m \times n$ grid graph and is denoted by $G_{m,n}$.

A set $S \subseteq V$ dominates a set $U \subseteq V$, if every vertex in U is adjacent to a vertex of S. If S dominates $V - S$, then S is called a *dominating set* of G. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$-*set*. The literature on Domination and its variations in graphs has been surveyed and detailed in the two books by Haynes *et al.* [2, 3].

We need the following results for our further discussion.

Theorem 2.1 [4]. *For any graph G,* $\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Theorem 2.2 [4]. *For* $n \geq 4, \gamma_r(C_n) = \gamma_r(P_n) = \lceil \frac{3n}{7} \rceil$.

Theorem 2.3 [4]. *For any graph* $G, \gamma(G) = \gamma_r(G)$ *if and only if there exists a* $\gamma(G)$*-set* S *such that*

(i) $pn(v, S)$ *induces a clique for every* $v \in S$.

(ii) *for every vertex* $u \in V(G) - S$ *that is not a private neighbor of any vertex of* S, *there exists a vertex* $v \in S$ *such that* $pn(v, S) \bigcup \{u\}$ *induces a clique.*

3. Properties of Weak Roman Domination Number

Theorem 3.1. *For any graph* $G, \gamma_r(G) = 1$ *if and only if* G *is complete.*

Theorem 3.2. *For any graph* G *of order* n, $n > 3$ *which is not complete,* $\gamma_r(G) = 2$ *and* $\gamma(G) = 1$ *if and only if* G *has a vertex of degree* $n - 1$.

Theorem 3.3. *For any graph* G *on* n *vertices,* $\gamma_r(G) = n$ *if and only if* $G = \overline{K_n}$.

We omit the proof of the above theorems as they are straightforward.

4. Classifying Graphs with $\gamma_r(G) = \gamma(G)$

In this section, we first characterize trees T for which $\gamma_r(T) = \gamma(T)$. For this purpose we introduce a family $\Im$ of trees as follows. A tree $T \in \Im$ if the following conditions hold.

(i) No vertex of T is a strong support.

(ii) If $u \in V(T)$ is a non support which is adjacent to a support, then $N(u)$ contains exactly one vertex which is neither a support nor adjacent to a support and all other members of $N(u)$ are either supports or adjacent to supports [see Figure 1].

(iii) For any vertex u of degree at least two, there exist at least one leaf v such that $d(u, v) \leq 3$.

(iv) Two vertices which are neither supports nor adjacent to supports are not adjacent.

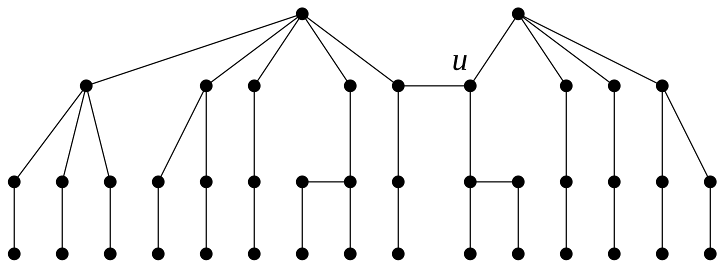

Figure 1. A tree $T \in \Im$.

We now prove the following lemmas.

Lemma 4.1. *Let T be a tree with $\gamma_r(T) = \gamma(T)$. Then there exists a $\gamma(T)$-set S such that for every u in $V - S$, there exists a $v \in S$ adjacent to u such that either $pn(v, S) = \emptyset$ or $pn(v, S) = \{u\}$.*

Proof. Follows directly from Theorem 2.3. ∎

Lemma 4.2. *Let T be a tree with $\gamma_r(T) = \gamma(T)$. Then no support of T is a strong support.*

Proof. Suppose not. Then there exists a strong support w in T and clearly $w \in S$ where S is any $\gamma(T)$-set and by Theorem 2.3, $pn(w, S)$ forms a clique, which is a contradiction. ∎

Lemma 4.3. *Let T be a tree with $\gamma_r(T) = \gamma(T)$ and S be a $\gamma(T)$-set. Then if $x_1, x_2 \in S$ are adjacent then both x_1 and x_2 are supports.*

Proof. Suppose not. Then the following cases arise.

Case (i). x_1 is a support and x_2 is not a support.
Clearly x_2 has a private neighbor z in $V - S$. For otherwise, $S_1 = S - \{x_2\}$ will be a $\gamma(T)$-set contradicting the minimality of S. Since x_2 is not a support, there exists a path (x_2, z, w, y) such that either $pn(y, S) = \emptyset$ or $pn(y, S) = \{w\}$ where $w \in V - S$ and $y \in S$. Now $S_1 = [S - \{x_2, y\}] \bigcup \{w\}$ is a $\gamma(T)$-set contradicting the minimality of S.

Case (ii). x_1 and x_2 are not supports.
Then as in case (i) corresponding to each $x_i, i = 1, 2$, there exists paths (x_i, z_i, y_i, w_i) $i = 1, 2$ such that either $pn(y_i, S) = \emptyset$ or $pn(y_i, S) = \{w_i\}$ and z_i is a private neighbor of x_i, $i = 1, 2$ where $w_i \in V - S$ and $y_i \in S$. Now $S_1 = S - \{x_1, x_2, y_1, y_2\} \bigcup \{w_1, w_2\}$ is a $\gamma(T)$-set, which is a contradiction. ■

Lemma 4.4. *Let T be a tree with $\gamma_r(T) = \gamma(T)$. If $u \in V(T)$ is a non support which is adjacent to a support, then $N(u)$ contains exactly one vertex which is neither a support nor adjacent to a support and all other members of $N(u)$ are either supports or adjacent to supports.*

Proof. Let $u \in V(T)$ be at a distance two from a leaf. By Lemma 4.1, there exists a $\gamma(T)$-set S such that for every $w \in V - S$, there exists a $v \in S$ adjacent to w such that either $pn(v, S) = \emptyset$ or $pn(v, S) = \{w\}$. By Lemma 4.1, $u \in V - S$. Now there exists a vertex $z_1 \in S$ which is adjacent to u such that $pn(z_1, S) = \emptyset$. Now we claim that each member of $N(u) - \{z_1\}$ is either a support or adjacent to a support. Suppose not. Let $u_1 \in N(u) - \{z_1\}$ be neither a support nor adjacent to a support.

Case (i). $u_1 \in S$.
Since u_1 is neither a support nor adjacent to a support, there is a path (u_1, u_2, u_3, u_4) such that $u_1, u_4 \in S$ and $u_2, u_3 \in V - S$. Now u_2 is a private neighbor of u_1 with respect to S. For otherwise $S_1 = S - \{z_1, u_1\} \bigcup \{u\}$ is a $\gamma(T)$-set, a contradiction. Further either $pn(u_4, S) = \{u_3\}$ or $pn(u_4, S) = \emptyset$. Hence $S_1 = S - \{z_1, u_1, u_4\} \bigcup \{u, u_3\}$ is a $\gamma(T)$-set, which is a contradiction.

Case (ii). $u_1 \notin S$.
Then there exists a path (u_1, u_2, u_3, u_4) such that $u_1, u_3 \in V - S$ and $u_2, u_4 \in S$ and $pn(u_4, S) = \emptyset$. Now $S_1 = S - \{z_1, u_2, u_4\} \bigcup \{u, u_3\}$ is a $\gamma(T)$-set, which is a contradiction. Hence in both the cases each member of $N(u) - \{z_1\}$ is a support. ■

Lemma 4.5. *Let T be a tree with $\gamma_r(T) = \gamma(T)$. For any vertex u of degree at least two, there exists at least one leaf v such that $d(u, v) \leq 3$.*

Proof. By Lemma 4.1, there exists a $\gamma(T)$-set such that for every u in $V - S$, there exists a $v \in S$ adjacent to u such that either $pn(v, S) = \emptyset$ or $pn(v, S) = \{u\}$. Let $v \in V(T)$ with $deg(v) \geq 2$. Suppose no leaf w exists such that $d(v, w) \leq 3$.

Case (i). $v \in S$.
Since $deg(v) \geq 2$, by Lemmas 4.1 and 4.3, there exists a path (v, v_1, v_2, v_3, v_4) such that $v_2, v_4 \in S$ and $v_1, v_3 \in V - S$ where $pn(v_i, S) = \emptyset$, $i = \{2, 4\}$. Now $S_1 = (S - \{v_2, v_4\}) \bigcup \{v_3\}$ is a dominating set, contradicting the minimality of S.

Case (ii). $v \notin S$.

Subcase (a). $pn(v_1, S) = \{v\}$.
Then as in case (i), there exists a path $(v_1, v_2, v_3, v_4, v_5)$ such that $v_1, v_3, v_5 \in S$ and $v_2, v_4 \in V - S$ with $pn(v_i, S) = \emptyset$, where $i = 3, 5$. Hence $S_1 = (S - \{v_3, v_5\}) \cup \{v_4\}$ is a dominating set, contradicting the minimality of S.

Subcase (b). $v \notin pn(v_1, S)$.
As in Subcase (a), we get a contradiction. ∎

Lemma 4.6. *Let T be a tree with $\gamma_r(T) = \gamma(T)$. Two vertices which are neither supports nor adjacent to supports are not adjacent.*

Proof. Proof follows from Lemmas 4.3, 4.4 and 4.5. ∎

As an immediate consequence of Lemmas 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6, we have the following characterization of trees T that satisfy $\gamma_r(T) = \gamma(T)$.

Theorem 4.7. *Let T be a tree, then $\gamma_r(T) = \gamma(T)$ if and only if $T \in \Im$.*

Proof. Suppose $T \in \Im$. Let $f : V(T) \to \{0, 1, 2\}$ be defined by $f(w) = 1$ if w is a support or not adjacent to a support and $f(w) = 0$ otherwise. Then clearly f is a γ_r-function with $V_2 = \emptyset$ and $|V_1| = \gamma(T)$. Hence $\gamma_r(T) = \gamma(T)$. Converse follows from Lemma 4.2, 4.4, 4.5 and 4.6. ∎

We now proceed to characterize the class of split graphs for which $\gamma_r(G) = \gamma(G)$.

Theorem 4.8. *For any split graph G with bipartition (X, Y) where X is independent and Y is complete, $\gamma_r(G) = \gamma(G)$ if and only if $deg(y) = n$, for every y in Y, where $|Y| = n$.*

Proof. Let G be a split graph satisfying the given conditions. Then the function $f = (V_0, V_1, V_2)$ defined by $V_1 = X, V_2 = \emptyset$ and $V_0 = V - S$ is a weak Roman dominating function and $S = X$ is the minimum dominating set. Hence $\gamma_r(G) = 2|V_2| + |V_1| = |X| = |S| = \gamma(G)$.

Conversely suppose that G is a split graph with bipartition (X, Y) where X is independent and Y is complete satisfying $\gamma_r(G) = \gamma(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_r-function of G and S be a γ-set of G. Since $\gamma_r(G) = \gamma(G)$, $V_2 = \emptyset$. Thus $S = V_1$ is a $\gamma(G)$-set.

First we claim that $deg(y) = n$, for every $y \in Y$. Let $y \in Y$ and $(y_1, y_2, \ldots, y_m)$ be the neighbors of $y \in X$.

Case (i). $y \in S$.
We claim that $y_i \in epn(y, S)$, $1 \leq i \leq m$. Suppose not. Then there exists a y_j for some j such that $y_j \notin epn(y, S)$. Then by Theorem 2.3, there exists a $w \in S$ such that $pn(w, S) \bigcup \{y_j\}$ induces a clique, which is a contradiction. Hence our claim, Further by Theorem 2.3, $pn(y, S)$ induces a clique which implies that $m = 1$. Therefore $deg(y) = n$ for every y in Y.

Case (ii). $y \notin S$.

Subcase (a). $y_i \notin S$, $1 \leq i \leq m$.
We claim that $m = 1$. Suppose not. Then corresponding to each y_i, there exists $z_i \in Y \cap S$, $1 \leq i \leq m$, $m \geq 2$ such that $z_i y_i \in E$ and $deg(z_i) = n$ (by Case (i)). Hence $S_1 = (S - \bigcup_{i=1}^{m} z_i) \bigcup \{y\}$ is a γ-set, which is a contradiction to the minimality of S. Therefore $m = 1$ and $deg(y) = n$, for every y in Y.

Subcase (b). $y_j \in S$ for some j.
We claim that $m = 1$. Suppose not. Then corresponding to each y_i, $i \neq j$, there exists a $z_i \in S$, $i \neq j$, $1 \leq i \leq m$, $m \geq 2$ such that $z_i y_i \in E$ and $deg(z_i) = n$ (by Case (i)). Hence $S_1 = (S - (\bigcup_{i=1}^{m} z_i)) \bigcup \{y_j\}$, $i \neq j$ is a γ-set, which is a contradiction to the minimality of S. Therefore $m = 1$ and $deg(y) = n$, for every y in Y. ∎

5. Specific Values of Weak Roman Domination Number

In this section we first determine the value of γ_r for a caterpillar T. For this purpose we proceed as follows.

Let $v_1, v_2, v_3, \ldots, v_k$ be the support vertices of T and n_i be the number of internal vertices of the (v_i, v_{i+1})-path, $1 \leq i \leq k-1$. Let $n_i \equiv j_i \pmod 7$. Now we consider a weak support $(\neq v_1)$ as an *artificial strong support* using the following procedure.

Let $v_r(\neq v_1)$ be the first weak support of the spine of T. It will be considered as an artificial strong support, if one of the following conditions hold.

(i) Both v_{r-1} and v_{r+1} are strong supports with $j_{r-1} \in \{2,4\}$ and $j_r \in \{2,4\}$.

(ii) v_{r-1} is a strong support with $j_{r-1} \in \{2,4\}$ and v_{r+1} is a weak support with $j_r \in \{1,3\}$.

(ii) v_{r-1} is a weak support with $j_{r-1} \in \{1,3\}$ and v_{r+1} is a strong support with $j_r \in \{2,4\}$.

Let v_s be the next weak support on the spine of T. Then it is considered as an artificial strong support if one of the following conditions hold.

(a) Both v_{s-1} and v_{s+1} are weak supports with $j_{s-1} \in \{1,3\}$ and $j_s \in \{1,3\}$.

(b) v_{s-1} is a strong (artificial strong) support and v_{s+1} is a strong support with $j_{s-1} \in \{2,4\}$ and $j_s \in \{2,4\}$.

(c) v_{s-1} is a strong (artificial strong) support and v_{s+1} is a weak support with $j_{s-1} \in \{2,4\}$ and $j_s \in \{1,3\}$.

(d) v_{s-1} is a weak support and v_{s+1} is a strong support with $j_{s-1} \in \{1,3\}$ and $j_s \in \{2,4\}$.

We repeat this process of identifying artificial strong supports till all the support vertices in the spine are exhausted. Consider the caterpillar in Figure 2. v_2, v_5 and v_7 are artificial strong supports by (i), (a) and (d) respectively.

We now determine the value of γ_r for a caterpillar in the following theorem.

Theorem 5.1. *Let T be any caterpillar. Let $S = \{s : s$ is either a strong support or an artificial strong support$\}$ and $W = \{w : w$ is a weak support$\}$.*

Let $T_1 = T - (N[S] \bigcup W)$. *Let* Q_1, Q_2, $Q_3, \ldots, Q_k$ *be the components of* T_1. *Then* $\gamma_r(T) = 2|S| + \sum_{i=1}^{k} \gamma_r(Q_i)$.

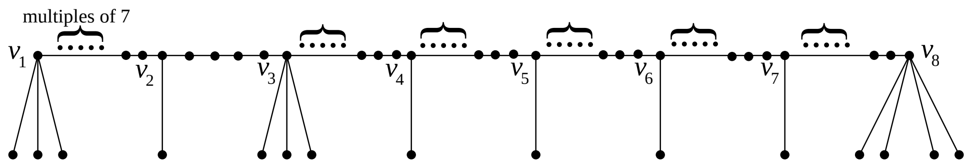

Figure 2

Proof. Let T be any caterpillar. Identify the artificial strong supports using the above said procedure. Let S and W be as defined in the theorem.

Let v be an artificial strong support. Let u_1 and u_2 be the supports that precede and succeed v on the spine. Let P be the (u_1, u_2) path. Let $w_1, w_2, w_3, \ldots, w_k$ be the internal vertices of the (u_1, v)-path and $z_1, z_2, z_3, \ldots, z_s$ be the internal vertices of the (v, u_2)-path.

Case (i). u_1 and u_2 are weak supports.
If one legion is posted at v, then $\lceil \frac{3k}{7} \rceil + \lceil \frac{3s}{7} \rceil + 3 = M_1$ legions are required to safeguard the vertices on the path P. But on the other hand, if two legions are posted at v, then $\lceil \frac{3(k-1)}{7} \rceil + \lceil \frac{3(s-1)}{7} \rceil + 4$ legions are required to safeguard the path P, which is less than M_1. Hence we assign two legions at v to safeguard $N[v]$.

Case (ii). u_1 is a weak support and u_2 is a strong support.
If one legion is posted at v, then $\lceil \frac{3k}{7} \rceil + \lceil \frac{3(s-1)}{7} \rceil + 4 = M_2$ legions are required to safeguard the path P. But on the other hand, if two legions are posted at v, then $\lceil \frac{3(k-1)}{7} \rceil + \lceil \frac{3(s-2)}{7} \rceil + 5$ legions are required to safeguard the path P, which is less than M_2. Hence we assign two legions at v to safeguard $N[v]$.

Case (iii). u_1 is a strong support and u_2 is a weak support.
If one legion is posted at v, then $\lceil \frac{3(k-1)}{7} \rceil + \lceil \frac{3s}{7} \rceil + 4 = M_3$ legions are required to safeguard the path P. But on the other hand, if two legions are posted at v, then $\lceil \frac{3(k-2)}{7} \rceil + \lceil \frac{3(s-1)}{7} \rceil + 5$ legions are required to safeguard the path P, which is less than M_3. Hence we assign two legions at v to safeguard $N[v]$.

Case (iv). Both u_1 and u_2 are strong supports.

If one legion is posted at v, then $\lceil \frac{3(k-1)}{7} \rceil + \lceil \frac{3(s-1)}{7} \rceil + 5 = M_4$ legions are required to safeguard the path P. But on the other hand, if two legions are posted at v, then $\lceil \frac{3(k-2)}{7} \rceil + \lceil \frac{3(s-2)}{7} \rceil + 6$ legions are required to safeguard the path P, which is less than M_4. Hence we assign two legions at v to safeguard $N[v]$.

Hence in all the cases we see that two legions are needed at v to safeguard $N[v]$.

Let $T_1 = T - (N[S] \bigcup W)$. Let Q_i, $1 \leq i \leq k$ be the components of T_1. Now we define a function $f : V \to \{0, 1, 2\}$ by $f(u) = 2$ when $u \in S$, $f(u) = 0$ when $u \in N(S)$ and $f(u) = f_i(u)$ if $u \in Q_i, 1 \leq i \leq k$ where f_i is a γ_r-function of Q_i. Hence $\gamma_r(T) = 2|S| + \sum_{i=1}^{k} \gamma_r(Q_i)$. ■

In the following two theorems we determine the values of γ_r for a $2 \times n$ grid graph $G_{2,n}$ and a complete binary tree.

Theorem 5.2. *For any $2 \times n$ grid graph $G_{2,n}$,*

$$\gamma_r(G_{2,n}) = \begin{cases} \left\lfloor \dfrac{4n}{5} \right\rfloor & if\ n \equiv 0 \pmod 5, \\ \left\lfloor \dfrac{4n}{5} \right\rfloor + 1 & otherwise. \end{cases}$$

Proof. Let $f = (V_0, V_1, V_2)$ be a weak Roman dominating function for $G_{2,n}$. Then any vertex of V_2 can dominate at most four vertices, while two vertices in V_1 can dominate at most five vertices. Thus in order to safeguard $G_{2,n}$, we must have $V_2 = 0$ and $\frac{5}{2}|V_1| \geq 2n$. Therefore $f(V) = 2|V_2| + |V_1| \geq \lfloor \frac{4n}{5} \rfloor$.

When $n = 5k$, $k \geq 1$, clearly $4k$ legions are needed to safeguard $10k$ vertices. Therefore $\gamma_r(G_{2,n}) = \lfloor \frac{4n}{5} \rfloor$. When $n = 5k + i$, $k \geq 0$, $4k$ legions can safeguard only $10k$ vertices. Therefore $\gamma_r(G_{2,n}) > \lfloor \frac{4n}{5} \rfloor$.

We show that $\gamma_r(G_{2,n}) = \lfloor \frac{4n}{5} \rfloor + 1$ by construction (see Figure 3). Let the vertices of $G_{2,n}$ be $v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_{1,n}$ and $v_{2,1}, v_{2,2}, v_{2,3}, \ldots, v_{2,n}$. Now we define a weak Roman dominating function g as follows. When $n = 5k+i$, $0 \leq i \leq 4$, $g(v_{1,5r+j}) = 1$, $j \in \{2, 5\}$ and $g(v_{2,5r+j}) = 1$, $j \in \{1, 4\}$, $0 \leq r \leq k$. When $n = 5k + 3$, $g(v_{2,n}) = 1$.

For all the remaining vertices u, let $g(u) = 0$. It is easily seen that

$$g(V) = \begin{cases} \left\lfloor \dfrac{4n}{5} \right\rfloor & \text{if } n \equiv 0 \pmod 5, \\ \left\lfloor \dfrac{4n}{5} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

■

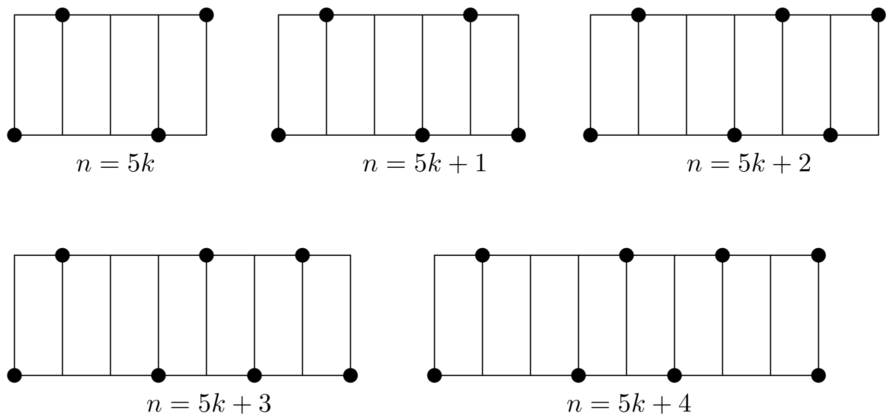

Figure 3. The construction for $G_{2,n}$, where $n = 5k + i$, $0 \leq i \leq 4$. Filled in circles denote vertices in V_1.

Theorem 5.3. *For any complete binary tree T with level k, $\gamma_r(T) = 2^m(1+2^3 + 2^6 + \cdots + 2^{k-1})$, where $k \equiv m \pmod 3$.*

Proof. Let T be a k-level complete binary tree rooted at v. We define a function $f : V(T) \to \{0, 1, 2\}$ as follows

Case (i). $k \equiv 0 \pmod 3$.
For each j such that $3j + 2 < k$, $j \geq 0$, $f(\partial N_{3j+2}(v)) = 2$, $f(v) = 1$ and $f(w) = 0$, if $w \in V - (\{v\} \cup \partial N_{3j+2}(v))$. Then $|V_2| = 2^{k-1} + 2^{k-4} + \cdots + 2^5 + 2^2$, $|V_1| = 1$. Clearly f is a γ_r function and

$$\begin{aligned} \gamma_r(T) &= 2|V_2| + |V_1| \\ &= 2(2^2 + 2^5 + \cdots + 2^{k-1}) + 1 \\ &= 1 + 2^3 + 2^6 + \cdots + 2^k \\ &= 2^0(1 + 2^3 + 2^6 + \cdots + 2^k). \end{aligned}$$

Hence $\gamma_r(T) = 2^m(1 + 2^3 + 2^6 + \cdots + 2^k)$ where $m = 0$.

Case (ii). $k \equiv 1 \pmod 3$.
For each j such that $0 \leq 3j \leq k-1$, $j \geq 0$, $f(\partial N_{3j}(v)) = 2$ and $f(w) = 0$, if $w \in V - \partial N_{3j}(v)$. Then $|V_2| = 1 + |\partial N_3(v)| + |\partial N_6(v)| + \cdots + |\partial N_{k-1}(v)| = 1 + 2^3 + 2^6 + \cdots + 2^{k-1}$ and $|V_1| = 0$. Clearly f is a γ_r function and

$$\begin{aligned}\gamma_r(T) &= 2|V_2| + |V_1| \\ &= 2(1 + 2^3 + 2^6 + \cdots + 2^{k-1})\end{aligned}$$

Hence $\gamma_r(T) = 2^m(1 + 2^3 + 2^6 + \cdots + 2^{k-1})$ where $m = 1$.

Case (iii). $k \equiv 2 \pmod 3$.
For each j such that $1 \leq 3j+1 \leq k-1$, $j \geq 0$, $f(\partial N_{3j+1}(v)) = 2$ and $f(w) = 0$, for all $w \in V - \partial N_{3j+1}(v)$. Then $|V_2| = |\partial N_1(v)| + |\partial N_4(v)| + \cdots + |\partial N_{k-1}(v)|$ and $|V_1| = 0$. Clearly f is a γ_r function and

$$\begin{aligned}\gamma_r(T) &= 2|V_2| + |V_1| \\ &= 2(2 + 2^4 + 2^7 + \cdots + 2^{k-1}) \\ &= 2^2(1 + 2^3 + 2^6 + \cdots + 2^{k-2})\end{aligned}$$

Hence $\gamma_r(T) = 2^m(1 + 2^3 + 2^6 + \cdots + 2^{k-m})$ where $m = 2$. ■

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