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Note

A CHARACTERIZATION OF LOCATING-TOTAL DOMINATION EDGE CRITICAL GRAPHS

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Abstract

For a graph $G = (V, E)$ without isolated vertices, a subset D of vertices of V is a total dominating set (TDS) of G if every vertex in V is adjacent to a vertex in D. The total domination number $\gamma_t(G)$ is the minimum cardinality of a TDS of G. A subset D of V which is a total dominating set, is a locating-total dominating set, or just a LTDS of G, if for any two distinct vertices u and v of $V(G) \setminus D$, $N_G(u) \cap D \neq N_G(v) \cap D$. The locating-total domination number $\gamma_L^t(G)$ is the minimum cardinality of a locating-total dominating set of G. A graph G is said to be a locating-total domination edge removal critical graph, or just a γ_L^{t+}-ER-critical graph, if $\gamma_L^t(G - e) > \gamma_L^t(G)$ for all e non-pendant edge of E. The purpose of this paper is to characterize the class of γ_L^{t+}-ER-critical graphs.

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1. Introduction

Various types of criticality with respect to domination parameters (such as vertex and edge removal, vertex and edge addition) have been studied see for example [2] for surveys and references. In this paper we investigate graphs which are critical upon edge removal with respect to the locating total domination number.

Unless stated otherwise we follow the notation and terminology of [2]. Specifically, $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$ denoted the *open* and *closed neighborhood,* respectively, of a vertex v of a graph $G = (V(G), E(G))$. A vertex of degree one is called *a pendant vertex* (or *a leaf*) and its neighbor is called *a support vertex.* We denote by $S(G)$ (resp. $L(G)$) the set of support vertices (resp. leaves) of G and by $L_v(G)$ the set of leaves adjacent to a support vertex v. A support vertex v is *strong* (respectively, weak) if $|L_v| \geq 2$ (respectively, $|L_v| = 1$). An edge incident with a leaf is called a pendant edge. We call the *core* of G the subset $C(G) = V(G) \setminus (S(G) \cup L(G))$. The subgraph induced in G by a subset of vertices S is denoted $G[S]$. A subset S is an independent set if no edge exists between any two vertices of $G[S]$. We denote by $K_{1,p}$, $p \geq 1$ a star. Recall that a galaxy is a forest in which each component is a star, that is, every edge of a galaxy is a pendant edge. If confusion is unlikely we omit the (G) from the above notation.

For a graph $G = (V, E)$ without isolated vertices, a subset D of vertices of V is a *total dominating set* (TDS) of G if every vertex in V is adjacent to a vertex in D. The total domination number $\gamma_t(G)$ is the minimum cardinality of a TDS of G. A subset D of V which is a TDS is a *locating-total dominating set*, or just a LTDS of G, if for any two distinct vertices u and v of $V(G) \setminus D$, $N_G(u) \cap D \neq N_G(v) \cap D$. The *locating-total domination number* $\gamma_L^t(G)$ is the minimum cardinality of a LTDS of G. Note that locating-total domination was introduced by Haynes, Henning and Howard [4].

By $\mu(G)$-set of G, where $\mu(G)$ is a domination parameter, we mean a vertex-set of G realizing $\mu(G)$, e.g., a $\gamma_t(G)$-set of G is a TDS X of G with $|X| = \gamma_t(G)$.

In this paper, we study the effects on increasing the locating-total domination number when an edge is deleted. Such problems have been considered before for some domination parameters. Sumner and Blitch [3] were the first introducing edge critical graphs for domination number.

When we remove a non-pendant edge e from a graph G, $G - e$ remains without isolated vertices, the locating-total domination number can increase, decrease or remain unchanged, e.g., if G is a P_5 then $\gamma_L^t(G) = 3$ and $\gamma_L^t(G-e) = 4$ for all e non-pendant edge of E. If G is a clique K_4 then $\gamma_L^t(G) = 3$ and $\gamma_L^t(G-e) = 2$ for all $e \in E$. If G is a P_6 then $\gamma_L^t(G) = \gamma_L^t(G-e) = 4$ for all e non-pendant edge of E.

A graph G is said to be *a locating-total domination edge removal critical graph*, or just a γ_L^{t+}-ER-critical graph, *if* $\gamma_L^t(G-e) > \gamma_L^t(G)$ for all e non-pendant edge of E.

Since all edges of a star $K_{1,p}$, $p \geq 1$ are pendant edges, we suppose in the following that the star is γ_L^{t+}-ER-critical.

The purpose of this paper is to give a descriptive characterization of the class of γ_L^{t+}-ER-critical graphs. In a similarly way, we have characterized in [1] the class of γ_L^{+}-ER-critical graphs.

2. Preliminary Results

The following results will be of use throughout the paper.

Observation 1. *For every graph G, the set $S(G)$ of all support vertices is contained in every $\gamma_L^t(G)$-set and for each $v \in S(G)$, every $\gamma_L^t(G)$-set contains at least $|L_v|$ vertices in $\{v\} \cup L_v$.*

Proposition 2. *If D is a $\gamma_L^t(G)$-set of a γ_L^{t+}-ER-critical graph $G = (V, E)$, then $V \setminus D$ is an independent set.*

Proof. If an edge e exists in $G[V \setminus D]$, then D is also a LTDS of $G - e$. Hence $\gamma_L^t(G-e) \leq \gamma_L^t(G)$, which contradicts that G is a γ_L^{t+}-ER-critical graph. ■

Proposition 3. *If D is a $\gamma_L^t(G)$-set of a γ_L^{t+}-ER-critical graph $G = (V, E)$, then $G[D]$ is a galaxy.*

Proof. Suppose to the contrary that $G[D]$ is not a galaxy. So, $G[D]$ contains a non-pendant edge e. Since $G[D]-e$ is a subgraph without isolated vertices, D is a LTDS of $G-e$ and $\gamma_L^t(G-e) \leq |D| = \gamma_L^t(G)$, which contradicts the criticality of G. ■

Definition 4. Let $H = (V, E)$ be a connected graph which satisfies the following conditions:

(1) $V = X \cup Y$.

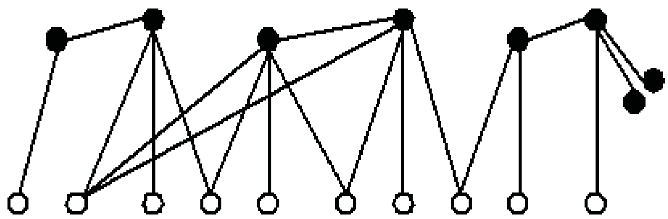
(2) $G[X]$ is a galaxy and Y is an independent set.

(3) For every y in Y and for every nonempty subset $X' \subseteq N(y)$ there exists a unique $y' \in Y$ such that $N(y') = X'$.

(4) If H is different from P_2, then every support vertex x in X verifies $N(x) \cap Y \neq \emptyset$.

Let $\mathcal{H}$ be the set of all such graphs.

Examples:
– $P_4, P_5 \in \mathcal{H}$.
– For $p \geq 1$, $K_{1,p} \in \mathcal{H}$. If $p = 1$ then $K_{1,p} = P_2 = G[X]$ and $Y = \emptyset$.



Graph of $\mathcal{H}$, black vertices define X and white define Y.

Remark 1. Let H be a graph in $\mathcal{H}$. By Definition 4 and Observation 1, X is a $\gamma_L^t(H)$-set. $C(H)$ is an independent set and $\forall x \in C(H)$, $N(x) \cap Y = \emptyset$.

3. Characterization

Lemma 5. *If $H \in \mathcal{H}$, then H is a γ_L^{t+}-ER-critical graph.*

Proof. By definition of a locating-total domination edge removal critical graph, every star $K_{1,p}, p \geq 1$ is a γ_L^{t+}-ER-critical graph. Let $H = (V, E)$ be a graph in $\mathcal{H}$ different from $K_{1,p}, p \geq 1$. Delete any non-pendant edge $e = xy$. Since Y is an independent set (see Definition 4-(2)), we have to consider only two cases.

Case 1. $x \in X$ and $y \in Y$.

By Definition 4-(3), there exists $y' \in Y; y' \neq y$ such that $N(y) \setminus \{x\} = N(y')$, so X is not a LTDS of $H - e$. Now by Observation 1, Remark 1 and since all neighbors of y are support vertices, $\gamma_L^t(H - e) \geq |X \cup \{z\}| = |X| + 1 = \gamma_L^t(H) + 1$, where z is y or y'. Hence, H is a γ_L^{t+}-ER-critical graph.

Case 2. $x \in X$ and $y \in X$.
By Remark 1, we have to consider two subcases.

Subcase 2.1. x and y be support vertices in H and at least one is a weak support, without loss of generality, let x be a weak support such that $(N(x) \setminus \{y\}) \cap X = \emptyset$. One neighbor of x with x are in every LTDS of $H - e$, so by Observation 1 and Definition 4, $\gamma_L^t(H - e) \geq |X| + 1 = \gamma_L^t(H) + 1$. Hence, H is a γ_L^{t+}-ER-critical graph.

Subcase 2.2. y is a support vertex in H and x is a vertex in the core $C(H)$. By Definition 4, $N(x)$ is a set of weak support vertices and $(N(y) \setminus \{x\}) \cap X = \emptyset$, so X is not a LTDS of $G - e$. Now by Observation 1, Remark 1 and Definition 4, $\gamma_L^t(H - e) \geq |X \cup \{y'\}| = |X| + 1 = \gamma_L^t(H) + 1$ where y' is a vertex adjacent to y. Hence, H is a γ_L^{t+}-ER-critical graph. ■

Now, the following theorem characterizes the class of γ_L^{+}-ER-critical graphs.

Theorem 6. *A nontrivial connected graph $G = (V, E)$ is a γ_L^{t+}-ER-critical graph if and only if $G \in \mathcal{H}$.*

Proof. The "if " part follows from Lemma 5, so let us prove the "only if" part. If $G = K_{1,p}, p \geq 1$, then $G \in \mathcal{H}$. Let G be a connected γ_L^{t+}-ER-critical graph different from $K_{1,p}, p \geq 1$. Let X be a $\gamma_L^t(G)$-set of G. By Proposition 2 and Proposition 3, $Y = V \setminus X$ is an independent set and $G[X]$ is a galaxy. Hence, conditions (1) and (2) of the Definition 4 are proved. Now, it remains to prove Condition (3) and Condition (4).

Proof of Condition (3). For that, let $y \in Y$, $N(y) = \{x_1, \ldots, x_k\}, k \geq 1$ and $X' \subseteq N(y)$. We consider the following cases.

Case 1. $|X'| = k$. If $k = 1$, then $X' = N(y) = \{x_1\}$ and y is a pendant vertex. As X is a $\gamma_L^t(G)$-set of G , y is a unique vertex in Y such that $N(y) = X'$. If $k \neq 1$, then since X is a $\gamma_L(G)$-set of G, y is the unique vertex in Y such that $N(y) = X'$.

Case 2. $2 \leq |X'| \leq k-1$. Let $X' = \{x'_1, \ldots, x'_l\} \subseteq N(y)$ with $l \leq k-1$. If $l = k-1$, then there exists a unique vertex $y^l \in Y$ with $N(y^l) = \{x'_1, \ldots, x'_l\}$, for otherwise D is a LTDS of $G-e$ with $e = yv$ and $v \in N(y) - N(y^l)$ which contradicts that G is a γ_L^{t+}-ER-critical graph. We repeat this process for $y^j \in Y$ with $N(y^j) = \{x'_1, \ldots, x'_j\}$ where $l+1 \leq j \leq k-2$. Consequently, there exists $y^l \in Y$ with $N(w^l) = X'$.

Proof of Condition (4). Let x be a support vertex of G in $G[X]$. Suppose to the contrary that $N(x) \cap Y = \emptyset$. Thus, $N[x] \subset X$, since G is not a star, there exists a vertex $y \in N(x) \setminus L_x$. Let x' be a pendant vertex adjacent to x, $X - \{x'\}$ is a LTDS smaller than X, a contradiction. ■

Notice that a disconnected graph G is γ_L^{t+}-ER-critical graph if and only if each component of G is γ_L^{t+}-ER-critical graph. So we have the following result.

Corollary 1. *A graph $G = (V, E)$ without isolated vertices is γ_L^{t+}-ER-critical graph if and only if G is the union of graphs of $\mathcal{H}$.*

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