PAIRED DOMINATION IN PRISMS OF GRAPHS

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Abstract

The paired domination number $\gamma_{pr}(G)$ of a graph $G$ is the smallest cardinality of a dominating set $S$ of $G$ such that $\langle S \rangle$ has a perfect matching. The generalized prisms $\pi G$ of $G$ are the graphs obtained by joining the vertices of two disjoint copies of $G$ by $|V(G)|$ independent edges. We provide characterizations of the following three classes of graphs: $\gamma_{pr}(\pi G) = 2 \gamma_{pr}(G)$ for all $\pi G$; $\gamma_{pr}(K_2 \Box G) = 2 \gamma_{pr}(G)$; $\gamma_{pr}(K_2 \Box G) = \gamma_{pr}(G)$.

Keywords: domination, paired domination, prism of a graph, Cartesian product.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

The paired domination number of a graph $G$ is the smallest cardinality of a dominating set $S$ of $G$ such that $\langle S \rangle$ has a perfect matching, and is denoted by $\gamma_{pr}(G)$. The paired domination number of the Cartesian product $G \Box H$ of two isolate-free graphs $G$ and $H$ was first investigated by Brešar, Henning and Rall [1], who obtained upper bounds on $\gamma_{pr}(G) \gamma_{pr}(H)$ in terms

∗Supported by the Natural Sciences and Engineering Research Council of Canada.
of \( \gamma_{pr}(G \Box H) \). They showed, i.e., that for any nontrivial tree \( T \) and any isolate-free graph \( H \), \( \gamma_{pr}(T)\gamma_{pr}(H) \leq 2\gamma_{pr}(T \Box H) \).

We compare the paired domination number of a graph \( G \) with the paired domination numbers of its generalized prisms \( \pi G \); i.e., the graphs obtained by joining the vertices of two disjoint copies of \( G \) by \( |V(G)| \) independent edges. Obviously, \( \gamma_{pr}(\pi G) \leq 2\gamma_{pr}(G) \). Graphs \( G \) for which \( \gamma_{pr}(\pi G) = 2\gamma_{pr}(G) \) regardless of how the two copies of \( G \) are joined are called universal \( \gamma_{pr} \)-doubler.

After providing background information in Section 2, we give necessary and sufficient conditions for a graph to be a universal \( \gamma_{pr} \)-doubler in Section 3. We also give necessary and sufficient conditions for a graph to be a prism \( \gamma_{pr} \)-doubler, i.e., \( \gamma_{pr}(K_2 \Box G) = 2\gamma_{pr}(G) \) (Section 4), and a prism \( \gamma_{pr} \)-fixer, i.e., \( \gamma_{pr}(K_2 \Box G) = \gamma_{pr}(G) \) (Section 5). Open problems related to this topic are mentioned in Section 6.

2. Definitions and Background

For any permutation \( \pi \) of \( V(G) \), the prism of \( G \) with respect to \( \pi \) is the graph \( \pi G \) obtained from two copies \( G_1 \) and \( G_2 \) of \( G \) by joining \( u \in V(G_1) \) and \( v \in V(G_2) \) if and only if \( v = \pi(u) \). If \( \pi \) is the identity \( 1_G \), then \( \pi G = K_2 \Box G \), the Cartesian product of \( G \) and \( K_2 \). The graph \( K_2 \Box G \) is called the prism of \( G \) and, in general, \( \pi G \) is a generalized prism of \( G \).

We shall abbreviate \( V(G) \), \( E(G) \) and \( V(G_i) \) to \( V \), \( E \) and \( V_i \), respectively. Let \( u \in V \) and \( S \subseteq V \). In \( \pi G \) we denote the counterparts of \( u \) (or \( S \)) in \( G_1 \) and \( G_2 \) by \( u_1 \) and \( u_2 \) (or \( S_1 \) and \( S_2 \)) respectively. Conversely, the vertex \( u_1 \) and set \( S_1 \) in \( G_1 \) (or \( u_2 \) and \( S_2 \) in \( G_2 \)) are denoted by \( u \) and \( S \) respectively when considered in \( G \).

For \( v \in V \), the open neighbourhood \( N(v) \) of \( v \) is defined by \( N(v) = \{ u \in V : u \in E \} \), and the closed neighbourhood \( N[v] \) of \( v \) is the set \( N(v) \cup \{ v \} \). For \( S \subseteq V \), \( N(S) = \bigcup_{s \in S} N(s) \), \( N[S] = \bigcup_{s \in S} N[s] \), \( N\{S\} = N[S] - S \). For \( v \in S \) we call \( w \in V - S \) an \( S \)-external private neighbour of \( v \) if \( N(w) \cap S = \{ v \} \). Denote the set of all \( S \)-external private neighbours of \( v \) by \( epn(v, S) \).

A set \( S \subseteq V \) dominates \( G \) or is a dominating set of \( G \) if every vertex in \( V - S \) is adjacent to a vertex in \( S \). The domination number \( \gamma(G) \) of \( G \) is defined by \( \gamma(G) = \min\{|S| : S \text{ dominates } G\} \). A dominating set \( S \) is a paired dominating set (PDS) if \( \langle S \rangle \) has a perfect matching. A vertex \( v \) is an \( \overline{M} \)-vertex of a matching \( M \) if \( v \) does not belong to any edge of \( M \). If \( S \)
is a PDS and $M$ is a perfect matching of $\langle S \rangle$, we call $M$ an $S$-matching. A $\gamma$-set of $G$ is a dominating set of $G$ of cardinality $\gamma(G)$; a $\gamma_{pr}$-set is defined similarly. We follow [9] for domination terminology.

It is easy to see that $\gamma(G) \leq \gamma(\pi G) \leq 2\gamma(G)$ for all permutations $\pi$ of $V$. If $\gamma(K_2 \Box G) = \gamma(G)$, then $G$ is called a prism fixer, and if $\gamma(K_2 \Box G) = 2\gamma(G)$, then $G$ is a prism doubler. If $\gamma(\pi G) = \gamma(G)$ for all permutations $\pi$ of $V$, then $G$ is a universal fixer, and if $\gamma(\pi G) = 2\gamma(G)$ for all $\pi$, then $G$ is a universal doubler.

Prism fixers we first studied by Hartnell and Rall [7, 8] in connection with Vizing's conjecture on the domination number of the Cartesian product of graphs. Prism and universal doublers were studied in [3], while fixers and doublers for other domination parameters, such as total and paired domination, were investigated in [11]. The graphs $\overline{K}_n$, $n \geq 1$, are universal fixers because $\pi \overline{K}_n = nK_2$ for all permutations $\pi$ of $V$. Moreover, these graphs are the only universal fixers known to date. The following conjecture was formulated in [10] and also studied in [2, 4, 6].

**Conjecture 1.** The graphs $\overline{K}_n$, $n \geq 1$, are the only universal fixers.

It is obvious that $\gamma_{pr}(\pi G) \leq 2\gamma_{pr}(G)$ for any graph $G$ and any permutation $\pi$ of $V$. Unlike the case for the domination number, though, the paired domination number of $\pi G$ is not bounded below by the paired domination number of $G$. For the graph $G$ in Figure 1, $\gamma_{pr}(G) = 6$, but for any $\pi G$ obtained by adding enough edges to the graph shown, $\gamma_{pr}(\pi G) = 4$. However, if $\pi$ is the identity, then the above-mentioned lower bound follows from the work in [1]. We give a direct proof below.

![Figure 1. $\gamma_{pr}(\pi G) < \gamma_{pr}(G)$](image)
Proposition 1. For any isolate-free graph $G$, $\gamma_{pr}(G) \leq \gamma_{pr}(K_2 \Box G) \leq 2\gamma(G)$.

Proof. For the upper bound, note that if $D$ is a $\gamma$-set of $G$, then $D_1 \cup D_2$ is a PDS of $K_2 \Box G$. For the lower bound, let $W$ be a $\gamma_{pr}$-set of $K_2 \Box G$ with $X_1 = W \cap V_1$ and $D_2 = W \cap V_2$ and let $S = X \cup D$. Then $S$ dominates $G$ and $|S| = |X| + |D| - |X \cap D|$.

If $X \cap D = \emptyset$, then $(S)$ contains a perfect matching (the matching corresponding to the perfect matching of $(W)$) and $S$ is a PDS of $G$ with $|S| = |W|$, so we are done.

Assume $X \cap D \neq \emptyset$. Let $M$ be a maximum matching of $(S)$ and $Z = \{z^1, \ldots, z^k\}$ the set of $\overline{M}$-vertices; note that $k \leq |X \cap D|$. Let $S^0 = S$ and for $i = 1, \ldots, k$, construct $S^i$ recursively as follows.

- If $z^i$ is adjacent to $s^i \in V - S^{i-1}$, let $S^i = S^{i-1} \cup \{s^i\}$. Otherwise, $z^i$ is adjacent to $x \in S^{i-1}$ because $G$ is isolate-free; hence $N[z^i] \subseteq S^{i-1}$. Let $S^i = S^{i-1} - \{z^i\}$.

Then $S^k$ dominates $G$, $(S^k)$ has a perfect matching and thus $S^k$ is a PDS of $G$. Moreover, $|S^k| \leq |S| + |Z| \leq |X| + |D| = |W|$ and the result follows.

Corollary 2. If a graph $G$ is a prism $\gamma_{pr}$-doubler, then $\gamma_{pr}(G) = \gamma(G)$.

3. Universal Doublers

Suppose $D'$ is a $\gamma_{pr}$-set of a graph $G$ in which $u$ is paired with $v$, and $\text{epn}(v, D') = \emptyset$. Then $D = D' - \{v\}$ dominates $G$, and $D_1 \cup D_2$ is a $\gamma_{pr}$-set of $K_2 \Box G$ in which $u_1$ is paired with $u_2$. Thus $G$ is not a prism $\gamma_{pr}$-doubler and thus not a universal $\gamma_{pr}$-doubler. A similar argument (but with another permutation) shows that if $G$ has a $\gamma_{pr}$-set $D$ in which $|\text{epn}(v, D)|$ is small compared to $\gamma_{pr}(G)$ for some vertex $v \in D$, then $G$ is not a universal $\gamma_{pr}$-doubler. These cases suggest that vertices contained in $\gamma_{pr}$-sets of universal $\gamma_{pr}$-doublers have large degrees relative to $\gamma_{pr}(G)$, and hence that $\gamma_{pr}(G)$ is small compared to the order of $G$, which we denote throughout by $n$.

In this section we obtain necessary and sufficient conditions for a graph to be a universal $\gamma_{pr}$-doubler. These conditions easily lead to an upper bound on the paired domination number of a universal $\gamma_{pr}$-doubler $G$, and lower bounds on the degrees and number of external private neighbours of the vertices in $\gamma_{pr}$-sets of $G$. 
We begin with a simple lemma.

**Lemma 3.** If $\gamma(G) = \gamma_{pr}(G)$, then $n \geq 2\gamma_{pr}(G)$ and $G$ has a PDS of cardinality $\gamma_{pr}(G) + 2i$ for each $1 \leq i \leq \gamma_{pr}(G)/2$.

**Proof.** It is well known [9, Theorem 2.1] that $n \geq 2\gamma(G)$, so $n \geq 2\gamma_{pr}(G)$. The latter part of the statement follows because each pair of vertices in a $\gamma_{pr}$-set $X$ which is also a $\gamma$-set can be split into two pairs since each vertex of $X$ has an external private neighbour [9, Theorem 1.1].

We next define notation that will be used throughout this section. Let

\[
\begin{aligned}
X \subseteq V \text{ such that } 0 < |X| < \gamma_{pr}(G); \\
Y = V - N[X]; \\
M \text{ be a matching of } \langle X \rangle; \\
Z = X - V(M), \text{ i.e., } Z \text{ is the set of } M\text{-vertices in } X; \\
k = |Z|.
\end{aligned}
\]

We now characterize universal $\gamma_{pr}$-doublers in terms of the cardinalities of the sets $X$, $Y$ and $Z$ as defined in (1).

**Theorem 4.** A graph $G$ is a universal $\gamma_{pr}$-doubler if and only if, for each set $X \subseteq V$ with $0 < |X| < \gamma_{pr}(G)$, a maximum matching $M$ of $\langle X \rangle$, and $Y$ and $k$ as defined in (1),

\[|Y| \geq 2\gamma_{pr}(G) - |X| - k - 1.\]

**Proof.** Suppose that for some $X \subseteq V$ with $0 < |X| < \gamma_{pr}(G)$,

\[|Y| < 2\gamma_{pr}(G) - |X| - k - 1.\]

We consider two cases, depending on the parity of $k$.

**Case 1.** $k$ is even.

Then by definition of $Z$, $|X|$ is even. Choose a PDS $D$ of $G$ as follows.

(i) If $|Y| + k \leq \gamma_{pr}(G)$, then let $D$ be any $\gamma_{pr}$-set of $G$.

(ii) Otherwise, let $D$ be any PDS of $G$ with $|D| = |Y| + k$ if $|Y|$ is even, or $|D| = |Y| + k + 1$ if $|Y|$ is odd. (A PDS of this size exists by Lemma 3.)
Let \( \pi \) be any permutation of \( V \) such that \( \pi(Y \cup Z) \subseteq D \) and \( \langle \pi(Z) \rangle \) has a perfect matching \( M' \) that is contained in a \( D \)-matching. Then \( W = X_1 \cup D_2 \) dominates \( \pi G \) and \( \langle W \rangle \) has a \( W \)-matching in which each edge \( u_2v_2 \) in \( M'_2 \) is replaced by two edges \( z_1u_2 \) and \( z'_1v_2 \), where \( z, z' \in Z \). (See Figure 2.)

![Figure 2.](image)

Therefore \( W \) is a PDS of \( \pi G \). If \( D \) is a \( \gamma_{\text{pr}} \)-set of \( G \) (i.e., if \( D \) was defined in (i)), then

\[ |W| = |X| + |D| < 2\gamma_{\text{pr}}(G), \]

i.e., \( G \) is not a universal \( \gamma_{\text{pr}} \)-doubler. If \( D \) was defined in (ii), then

\[
|W| = |X| + |D| \\
\leq |X| + |Y| + k + 1 \\
< |X| + (2\gamma_{\text{pr}}(G) - |X| - k) + k + 1 \\
= 2\gamma_{\text{pr}}(G),
\]

and \( G \) is not a universal doubler in this case either.

**Case 2.** \( k \) is odd.

Then \( |X| \) is odd. If \( |X| = \gamma_{\text{pr}}(G) - 1 \), then \( |Y| \leq \gamma_{\text{pr}}(G) - k - 1 = |X - Z| \).
Let $\pi$ be any permutation of $V$ such that
$$\pi(Y) \subseteq X - Z, \quad \pi(Z) = Z \quad \text{and} \quad Y \subseteq \pi(X - Z).$$
Then $W = X_1 \cup X_2$ dominates $\pi(G)$ and it is easy to see that $\langle W \rangle$ has a perfect matching. Therefore $W$ is a PDS of $\pi G$ and
$$|W| = 2|X| = 2\gamma_{pr}(G) - 2,$$
so $G$ is not a universal doubler.
Thus we assume that $0 < |X| < \gamma_{pr}(G) - 2$. Similar to Case 1, we choose the PDS $D$ of $G$ as follows.

(iii) If $|Y| + k \leq \gamma_{pr}(G)$, let $D$ be any $\gamma_{pr}$-set of $G$.

(iv) Otherwise, let $D$ be any PDS of $G$ with $|D| = |Y| + k - 1$ if $|Y|$ is even, or $|D| = |Y| + k$ if $|Y|$ is odd.

Let $w \in Z$ and let $\pi$ be any permutation of $V$ such that $\pi(Y \cup Z - \{w\}) \subseteq D$, $\pi(Z - \{w\})$ has a perfect matching $M'$ which is contained in a $D$-matching, and $\pi(w) = w' \in V - D$. Let $W = X_1 \cup D_2 \cup \{w_2'\}$. Since $X_1$ dominates $G_1 - Y_1$ and $D_2$ dominates $G_2$ and $Y_1$, it follows that $W$ dominates $\pi G$. Also, $\langle W \rangle$ has a perfect matching in which $w_1$ is paired with $w_2'$, and each edge $u_2v_2$ in $M_2'$ is replaced by two edges $z_1u_2$ and $z_2'v_2$, where $z, z' \in Z - \{w\}$. Therefore $W$ is a PDS of $G$. If $D$ was chosen in (iii) and thus is a $\gamma_{pr}$-set of $G$, then
$$|W| = |X| + |D| + 1 < \gamma_{pr}(G) - 2 + \gamma_{pr}(G) + 1 = 2\gamma_{pr}(G) - 1$$
and $G$ is not a universal $\gamma_{pr}$-doubler. On the other hand, if $D$ was chosen in (iv), then $|D| \leq |Y| + k$, so
$$|W| = |X| + |D| + 1$$
$$< |X| + 2\gamma_{pr}(G) - |X| - k - 1 + k + 1$$
$$= 2\gamma_{pr}(G)$$
and once again $G$ is not a universal $\gamma_{pr}$-doubler.

Conversely, let $\pi$ be a permutation of $V$ such that $\gamma_{pr}(\pi G) < 2\gamma_{pr}(G) - 1$ and consider any $\gamma_{pr}$-set $W$ of $\pi G$. Define
$$X_1 = W \cap V_1 \quad \text{and} \quad D_2 = W \cap V_2.$$
Assume without loss of generality that \(|X_1| < \gamma_{pr}(G)|. Let \(M'\) be a \(W\)-matching and let \(D'_2\) be the set of vertices in \(D_2\) which are not paired with another vertex in \(D_2\) under \(M'\). Say \(|D'_2| = k'\). Also, let \(k\) be the number of vertices not paired in a maximum matching of \(\langle X_1 \rangle\). Note that \(k \leq k'\).

If \(X_1 \neq \emptyset\), then \(|D_2| < 2\gamma_{pr}(G) - |X| - 1\) and each vertex of \(D_2 - D'_2\) dominates at most one vertex in \(Y_1\), while no vertex in \(D'_2\) dominates a vertex in \(Y_1\). Therefore \(|Y_1| \leq |D_2 - D'_2|\), which implies that

\[|Y| < 2\gamma_{pr}(G) - |X| - k' - 1 \leq 2\gamma_{pr}(G) - |X| - k - 1.\]

If \(X_1 = \emptyset\), then \(D_2\) dominates \(V_1\) and so \(D_2 = V_2\). Therefore \(n = |D_2| < 2\gamma_{pr}(G)|, so that by Lemma 3, \(\gamma(G) < \gamma_{pr}(G)\). Let \(X'\) be a \(\gamma\)-set of \(G\), \(Y' = V - N[X']\) and \(k'\) be the number of vertices not paired in a maximum matching of \(\langle X' \rangle\). Since \(k' \leq |X'| < \gamma_{pr}(G)\),

\[|Y'| = 0 < 2\gamma_{pr}(G) - |X'| - k' - 1.\]

As an example of universal \(\gamma_{pr}\)-doublers, consider the following family \(F\) of graphs. Form the graph \(F_{2n} \in F\) by joining each vertex of \(C_{2n}\) to \(2n - 1\) new vertices. Note that \(\gamma_{pr}(F_{2n}) = \gamma(F_{2n}) = 2n\). Figure 3 shows the graph \(F_4\).

![Figure 3. \(F_4 \in F\): An example of a universal \(\gamma_{pr}\)-doubler.](image)

By Theorem 4, to prove that \(F_{2n}\) is a universal \(\gamma_{pr}\)-doubler, we must show that for each pair of sets \(X, Y \subseteq V(F_{2n})\) as defined in (1), \(|Y| \geq 2\gamma_{pr}(F_{2n}) - |X| - k - 1\). Suppose \(|X| = 2n - d\), where \(1 \leq d \leq 2n - 1\). It is easy to see
that $|Y| \geq d(2n - 1)$. If $d = 1$, then $k \geq 1$, hence

$$2\gamma_{pr}(F_{2n}) - |X| - k - 1 \leq 4n - (2n - 1) - 1 - 1 = 2n - 1 \leq |Y|.$$ 

If $2 \leq d \leq 2n - 1$, then $k \geq 0$, hence

$$2\gamma_{pr}(F_{2n}) - |X| - k - 1 \leq 4n - (2n - d) - 1$$

$$= 2n + d - 1$$

$$\leq 2n + (2n - 1) - 1$$

$$= 2(2n - 1)$$

$$\leq d(2n - 1)$$

$$\leq |Y|.$$ 

Note that to construct a universal $\gamma_{pr}$-doubler $G$ from $C_{2n}$ by adding pendant edges at vertices of $C_{2n}$, at least $2n - 1$ pendant edges must be added at each vertex of $C_{2n}$. If some vertices of $C_{2n}$ are joined to more than $2n - 1$ new vertices, the resulting graph is also a universal $\gamma_{pr}$-doubler.

**Corollary 5.** If $\gamma(G) = \gamma_{pr}(G) = 2$, then $G$ is a universal $\gamma_{pr}$-doubler.

**Proof.** Suppose $\gamma(G) = \gamma_{pr}(G) = 2$. Let $x \in V$ and $Y = V - N[x]$. Since $\gamma(G) = 2$, $|Y| \geq 1$. The result follows from Theorem 4.

We use Theorem 4 to obtain the promised results on the degrees and number of external private neighbours of the vertices in $\gamma_{pr}$-sets of a universal $\gamma_{pr}$-doubler.

**Corollary 6.** Let $G$ be a universal $\gamma_{pr}$-doubler and $D$ any $\gamma_{pr}$-set of $G$. Then $|epn(v, D)| \geq \gamma_{pr}(G) - 1$ for each $v \in D$.

**Proof.** Let $X = D - \{v\}$. Then $X \neq \emptyset$ because $\gamma_{pr}(G) \geq 2$, and $k = 1$ because there is only one vertex in $X$ that is not paired. By Theorem 4,

$$|V - N[X]| \geq 2\gamma_{pr}(G) - |X| - k - 1 = \gamma_{pr}(G) - 1.$$ 

Since $D$ is a dominating set, $v$ dominates $V - N[X]$. Moreover, $v \notin V - N[X]$ because $v$ is dominated by its partner in $D$. Hence $epn(v, D) = V - N[X]$ and the result follows.
The converse of Corollary 6 is shown to be false by the counterexample in Figure 4. The black vertices form the set $D$, which is the only $\gamma_{pr}$-set of $G$, and for all $v \in D$, $|\text{epn}(v, D)| = 3 = \gamma_{pr}(G) - 1$. Let $X$ consist of the circled vertices. Then

$$|Y| = |V - N[X]| = 2 < 2\gamma_{pr}(G) - |X| - k - 1 = 3,$$

so by Theorem 4, $G$ is not a universal $\gamma_{pr}$-doubler.

![Figure 4](image)

**Figure 4.** A counterexample to the converse of Corollary 6.

**Corollary 7.** If $G$ is a universal $\gamma_{pr}$-doubler and $v \in V$ is contained in a $\gamma_{pr}$-set of $G$, then $\deg v \geq \gamma_{pr}(G)$.

**Proof.** Suppose $D$ is a $\gamma_{pr}$-set of $G$ and $v \in D$. By Corollary 6, $|\text{epn}(v, D)| \geq \gamma_{pr}(G) - 1$. Since $v$ is paired with some vertex in $D$, the result follows.

The complete graphs of order at least three show that the converse of Corollary 7 is not true.

**Corollary 8.** If $G$ is a universal $\gamma_{pr}$-doubler of order $n$, then $\gamma_{pr}(G) \leq \sqrt{n}$.

**Proof.** By Corollary 7, $\deg v \geq \gamma_{pr}(G)$ for any vertex $v$ of any $\gamma_{pr}$-set $D$ of $G$. Hence $n \geq \lceil \gamma_{pr}(G) \rceil^2$.

We conclude this section by obtaining a sufficient condition for regular graphs to be universal $\gamma_{pr}$-doublers. This allows us to construct a family of universal $\gamma_{pr}$-doublers.

The PDS $D$ is an *efficient paired dominating set (EPDS)* if $N(u) \cap N(v) = \emptyset$ for any two vertices $u, v \in D$.

**Lemma 9.** If $G$ is regular and has an EPDS $D$, then $\gamma_{pr}(G) = |D|$.
Proof. Let \( X \) be a \( \gamma_{pr} \)-set of \( G \). Then \( |X| \leq |D| \) and by regularity, \( n \leq r|X| \). Since \( D \) is an EPDS, \( n = r|D| \). Hence \( |D| \leq |X| \) and so \( D \) is a \( \gamma_{pr} \)-set of \( G \).

Corollary 10. If \( G \) is \( r \)-regular with \( r \geq \gamma_{pr}(G) \) and \( G \) has an EPDS, then \( G \) is a universal \( \gamma_{pr} \)-doubler.

Proof. Let \( X \subseteq V \) with \( 0 < |X| < \gamma_{pr}(G) \) and define \( Y \) and \( k \) as in (1). Then \( |N[X]| \leq r|X| + k \). Since \( G \) has an EPDS, \( n = r\gamma_{pr}(G) \). Then

\[
|Y| \geq r\gamma_{pr}(G) - r|X| - k \geq \gamma_{pr}(G)(\gamma_{pr}(G) - |X|) - k.
\]

If \( |X| = \gamma_{pr}(G) - 1 \), then

\[
|Y| \geq \gamma_{pr}(G) - k = 2\gamma_{pr}(G) - |X| - k - 1,
\]

and if \( |X| \leq \gamma_{pr}(G) - 2 \), then

\[
|Y| \geq 2\gamma_{pr}(G) - k.
\]

In either case the hypothesis of Theorem 4 is satisfied and it follows that \( G \) is a universal \( \gamma_{pr} \)-doubler.

Corollary 10 allows us to construct a family \( \mathcal{H} \) of regular universal \( \gamma_{pr} \)-doublers. Label the vertices of \( C_{2m} \) consecutively by \( u_1, v_1, u_2, v_2, \ldots, u_m, v_m \). Construct each \( H_{2m,r} \in \mathcal{H} \) by replacing alternate edges \( u_i v_i \), \( i = 1, \ldots, m \), of \( C_{2m} \) by a copy of \( B_i \cong K_{r-1,r-1} \), \( r \geq 2m \), joining \( u_i \) to each vertex in one partite set, and \( v_i \) to each vertex in the other partite set of \( B_i \). See Figure 5 for \( H_{4,4} \).

Figure 5. The 4-regular universal \( \gamma_{pr} \)-doubler \( H_{4,4} \).
Clearly, $H_{2m,r}$ is $r$-regular. It is also easy to see that $\bigcup_{i=1}^{m}\{u_i, v_i\}$ forms an efficient $\gamma_{pr}$-set of $H_{2m,r}$ (in which each $v_i$ is partnered by $u_{i+1 \mod m}$). By Corollary 10, $H_{2m,r}$ is a universal $\gamma_{pr}$-doubler.

4. Prism Doublers

It is reasonable to expect that there are graphs that are prism $\gamma_{pr}$-doublers but not universal $\gamma_{pr}$-doublers. In this section we first supply necessary and sufficient conditions in Theorem 11, and then a simpler sufficient condition in Proposition 12, for a graph to be a prism doubler. The latter result combined with Corollary 6 allows us to construct prism $\gamma_{pr}$-doublers that are not universal $\gamma_{pr}$-doublers.

**Theorem 11.** A graph $G$ is a prism $\gamma_{pr}$-doubler if and only if for each set $X \subseteq V$ with $0 < |X| < \gamma_{pr}(G)$, any matching $M$ of $\langle X \rangle$, and $Y$ and $k$ as defined in (1), either

(i) $|Y| \geq 2\gamma_{pr}(G) - |X| - k - 1$, or

(ii) $|Y| = 2\gamma_{pr}(G) - |X| - k - d - 1$, where $d \geq 1$, and if $A \subseteq N[X] - Z$ dominates $N[X] - N[Y] - N[Z]$ and $\langle A \cup Y \rangle$ has a perfect matching, then $|A| \geq d$.

**Proof.** Assume $\gamma_{pr}(K_2 \Box G) = 2\gamma_{pr}(G)$ and consider any pair of sets $X, Y$ as defined in (1) and a matching $M$ of $\langle X \rangle$. If $|Y| \geq 2\gamma_{pr}(G) - |X| - k - 1$ then we are done, so assume $|Y| = 2\gamma_{pr}(G) - |X| - k - d - 1$ for some $d \geq 1$.

Suppose to the contrary that there exists a set $A \subseteq N[X] - Z$ such that $A$ dominates $N[X] - N[Y] - N[Z]$ and $\langle A \cup Y \rangle$ has a perfect matching $M^*$, but $|A| \leq d - 1$. Define the set $W \subseteq V(K_2 \Box G)$ by $W = X_1 \cup Y_2 \cup A_2 \cup Z_2$. By the definition of $X$ and $Y$, $X_1 \cup Y_2$ dominates $G_1$. Since $A_2$ dominates $N[X_2] - N[Y_2] - N[Z_2]$, $W$ also dominates $G_2$. Thus $W$ dominates $K_2 \Box G$. Moreover, $M \cup M^* \cup \{z_1,z_2 : z \in Z\}$ is a $W$-matching, so $W$ is a PDS of $K_2 \Box G$. But

$$|W| = |X| + |Y| + |Z| + |A|$$

$$\leq |X| + (2\gamma_{pr}(G) - |X| - k - d - 1) + k + (d - 1)$$

$$= 2\gamma_{pr}(G) - 2,$$

a contradiction. Thus (ii) holds.
Conversely, assume $\gamma_{pr}(K_2 \Box G) < 2\gamma_{pr}(G) - 1$ and let $W = X_1 \cup D_2$ be a $\gamma_{pr}$-set of $K_2 \Box G$. We may assume without loss of generality that $|X| < \gamma_{pr}(G)$.

We consider two cases, depending on whether $X = \emptyset$ or $X \neq \emptyset$.

**Case 1.** $X = \emptyset$.

Then $D_2 = V_2$ to dominate $G_1$. Therefore

$$|W| = |D| = n \leq 2\gamma_{pr}(G) - 2.$$

By Lemma 3, $\gamma(G) < \gamma_{pr}(G)$. Let $X'$ be a $\gamma$-set of $G$, $M'$ be a maximum matching of $\langle X' \rangle$, $Z'$ the set of $\overline{M'}$-vertices in $X'$ and $k' = |Z'|$. Then $k' > 0$ because $X'$ is not a PDS of $G$, and $Y' = V - N[X'] = \emptyset$ because $X'$ dominates $G$. But

$$2\gamma_{pr}(G) - |X'| - k' - 1 \geq 2\gamma_{pr}(G) - 2|X'| - 1 > 0 = |Y'|$$

and so (i) does not hold. Hence there exists a positive integer $d$ such that

$$0 = |Y'| = 2\gamma_{pr}(G) - |X'| - k' - d - 1,$$

i.e.,

$$d = 2\gamma_{pr}(G) - |X'| - k' - 1.$$

Let $A' = X' - Z'$. Then $A' \subseteq N[X'] - Z'$, $A'$ dominates $N\{X'\} - N[Y'] - N[Z']$ and, since $Y' = \emptyset$, $M'$ is a perfect matching of $\langle A' \cup Y' \rangle$. But

$$|A'| = |X'| - k' = 2|X'| - |X'| - k' < 2\gamma_{pr}(G) - |X'| - k' - 1 = d,$$

thus (ii) also does not hold.

**Case 2.** $X \neq \emptyset$.

Let $M^*$ be a $W$-matching, let $M_1$ be the matching of $\langle X_1 \rangle$ induced by $M^*$, and let $Z_1$ be the set of vertices in $X_1$ which are paired with vertices in $D_2$ (i.e., the vertices in $Z_2$) under $M^*$. Then in $G$, $Z$ is the set of $\overline{M}$-vertices in $X$, and $Z \subseteq D$. Define $Y$ and $k$ as in (1). Since $D_2$ dominates $Y_1, Y_2 \subseteq D_2$ and so $Y \subseteq D$. Moreover, $Y \cap Z = \emptyset$. Hence

$$|Y| \leq |D| - |Z| < 2\gamma_{pr}(G) - |X| - 1 - k.$$

Therefore (i) does not hold.
Let $A = D - Z - Y$. Then $A \subseteq N[X] - Z$. Since $D_2$ dominates all vertices of $G_2$ except possibly the vertices in $X_2 - D_2$, $D$ dominates $N\{X\}$, and so $A$ dominates $N\{X\} - N[Y] - N[Z]$. Moreover, $A \cup Y = D - Z$ and so $\langle A \cup Y \rangle$ has a perfect matching (corresponding to the edges of $M^*$ with both endvertices in $D_2$). (See Figure 6, where the black vertices indicate $X_1$ in $G_1$, $D_2$ in $G_2$, and $A$ in $G$, the grey vertices indicate $Y$ in $G$, and the dark edges indicate the matching $M^*$ in $K_2 \Box G$ and the perfect matching in $A \cup Y$.) Since

$$Y = D - Z - A, \quad A \cup Z \subseteq D \quad \text{and} \quad A \cap Z = \emptyset,$$

it follows that

$$|Y| = |D| - |Z| - |A| < 2\gamma_{pr}(G) - 1 - \left|X\right| - k - |A|.$$  

Thus

$$|Y| = 2\gamma_{pr}(G) - \left|X\right| - k - d - 1$$

for some $d > |A|$, and so (ii) also does not hold.

The following proposition enables us to describe classes of prism $\gamma_{pr}$-doublers that are not universal $\gamma_{pr}$-doublers.

**Proposition 12.** If every vertex that is contained in a $\gamma_{pr}$-set of $G \neq K_2$ is adjacent to at least one leaf, then $G$ is a prism $\gamma_{pr}$-doubler.
**Proof.** It is obvious that any support vertex of a graph $G$ is contained in each PDS of $G$. Thus, if $G$ satisfies the hypothesis, then $\gamma_{pr}(G) = k$, where $k$ is the number of support vertices of $G$. Say $u \in V$ is adjacent to the leaf $v$. Then in $K_2 \Box G$, $u_1, v_1, v_2, u_2$ is an induced 4-cycle, and $\deg v_1 = \deg v_2 = 2$. Thus any PDS of $K_2 \Box G$ contains at least two of these vertices, so that $\gamma_{pr}(K_2 \Box G) \geq 2k$, and the result follows.

Now let $H$ be a graph of order $k \geq 4$ that has a perfect matching and let $G$ be any graph obtained by joining each vertex of $H$ to at least one leaf, and some vertex $v$ to at most $k - 2$ leaves. By Proposition 12, $G$ is a prism $\gamma_{pr}$-doubler with $\gamma_{pr}(H) = k$. However, by Corollary 6, $G$ is not a universal $\gamma_{pr}$-doubler, because $|epn(v, V(H))| \leq k - 2 < \gamma_{pr}(G) - 1 = k - 1$.

5. **Prism Fixers**

Since $\gamma_{pr}(K_2 \Box G) \leq 2\gamma(G)$ for any graph $G$, it is immediately clear that if $\gamma_{pr}(G) = 2\gamma(G)$, then $G$ is a prism $\gamma_{pr}$-fixer. Examples of such graphs include nontrivial complete graphs, $P_5$, $C_5$ and $C_6$. We now extend this result to determine a necessary and sufficient condition for a graph to be a prism $\gamma_{pr}$-fixer.

Let $S \subseteq V$ such that $\langle S \rangle$ has a perfect matching $M$. A **paired partition** of $S$ is a partition $S_1, \ldots, S_k$ such that each edge of $M$ is contained in $\langle S_i \rangle$ for some $i$. A **weak** paired partition is a paired partition in which some of the sets may be empty. A **split** of $S$ is a partition $S = S_1 \cup S_2$ such that each edge of $M$ has one endvertex in $S_1$ and the other one in $S_2$.

In our next theorem we consider a weak paired partition $S = D \cup Y \cup Z$ of a $\gamma_{pr}$-set $S$ of $G$, and define $U = (V - S) \cap N[D] \cap N[Z]$ and $X = V - S - U$. Note that each vertex in $U$ is adjacent to a vertex in $D$ and to a vertex in $Z$, each vertex in $X$ is adjacent to vertices in at most one of $D$ and $Z$, and any vertex of $G - S$ may or may not be adjacent to a vertex in $Y$. See Figure 7, where $S$ consists of the black vertices, $U$ of the grey vertices and $X$ of the white vertices, and where the vertices in $D$ are indicated by circles, those in $Z$ by squares, and those in $Y$ by triangles.

**Theorem 13.** A graph $G$ is a prism $\gamma_{pr}$-fixer if and only if $G$ has a $\gamma_{pr}$-set $S$ with a weak paired partition $S = D \cup Y \cup Z$ in which $Y$ has a split $Y = Y' \cup Y''$ such that $Y'$ dominates $X = V - S - (N[D] \cap N[Z])$. 

Figure 7. Examples of weak paired partitions.

Proof. Suppose $G$ is a prism $\gamma_{\text{pr}}$-fixer and let $W$ be a $\gamma_{\text{pr}}$-set of $K_2 \square G$. Say $D'_1 = W \cap V_1$ and $Z'_2 = W \cap V_2$. Let $M^*$ be a $W$-matching in which as few vertices as possible are matched with their own image. Let

\[
S' = D' \cup Z', \\
Y' = D' \cap Z', \\
M' \text{ be the matching of } \langle S' \rangle \text{ induced by } M^*, \\
R \text{ be the set of } M'-\text{vertices.}
\]

Then $S'$ dominates $G$, $R \subseteq Y'$, and if $u \in R$, then $u_1u_2 \in E(M^*)$. Say $R = \{u^1, \ldots, u^k\}$, let $S^0 = S'$ and for $i = 1, \ldots, k$, construct $S^i$ recursively as follows.

(i) If $u^i$ is adjacent to $s^i \in V - S^{i-1}$, let $S^i = S^{i-1} \cup \{s^i\}$.

(ii) Otherwise, $u^i$ is adjacent to some vertex in $S^{i-1}$ because $G$ is isolate-free, hence $N[u^i] \subseteq S^{i-1}$; let $S^i = S^{i-1} - \{u^i\}$.

Then $S^k$ dominates $G$, $\langle S^k \rangle$ has a perfect matching and thus $S^k$ is a PDS of $G$. Moreover,

\[
|S^k| \leq |D'| + |Z'| - |Y'| + |R| \leq |W|.
\]

But $G$ is a prism $\gamma_{\text{pr}}$-fixer, so equality holds in (3). In particular, $R = Y'$ and each $S^i$ is constructed as described in (i). Moreover, $Y'$ is independent, for if $u, v \in Y'$ and $uv \in E$, then $u_1u_2, v_1v_2 \in E(M^*)$ (since $u$ and $v$ are $M'$-vertices) and $(M^* - \{u_1u_2, v_1v_2\}) \cup \{u_1v_1, u_2v_2\}$ is a $W$-matching in which
fewer vertices are mapped to their own images than in $M^*$, contradicting the choice of $M^*$.


Then $D \cup Z \cup Y$ is a weak paired partition of $S$ and $Y' \cup Y''$ is a split of $Y$ and we only need to prove that $Y'$ dominates $X$. Suppose $x \in X$. We assume that $x \not\in N[D]$; the case $x \not\in N[Z]$ is similar. Since $x \not\in S$, $x_1 \not\in D'_1$ and $x_2 \not\in Z'_2$. Thus $x_1$ is dominated in $G_1$ by a vertex in $D'_1 - D_1$, i.e., by a vertex in $Y'_1$. Therefore $x$ is dominated by a vertex in $Y'$ as required.

Conversely, assume $G$ has a $\gamma_{pr}$-set $S$ that satisfies the conditions of the theorem. Then $D_1 \cup Y'_1$ dominates $(G_1 - Z_1) \cup D_2$, and $Z_2 \cup Y'_2$ dominates $(G_2 - D_2) \cup Z_1$. Hence $W = D_1 \cup Z_2 \cup Y'_1 \cup Y'_2$ is a PDS of $K_2 \square G$ and $|W| = |S| = \gamma_{pr}(G)$. By Proposition 1, $W$ is a $\gamma_{pr}$-set of $K_2 \square G$.

The three graphs in Figure 7 are examples of prism $\gamma_{pr}$-fixers. Other examples of prism fixers include $K_n$ for $n \geq 2$, $P_n$ for $n \in \{3, 5, 6, 9\}$ and $C_n$ for $n \in \{5, 6, 9\}$. (This list contains all paths and cycles that are prism $\gamma_{pr}$-fixers.)

6. Problems

We conclude with open problems related to the above material. The graph $G$ in Figure 1 illustrates that the paired domination number of a graph may exceed the paired domination number of some of its generalized prisms. Note that this graph is $\gamma_{pr}$-edge-critical, i.e., $\gamma_{pr}(\pi G + e) < \gamma_{pr}(G)$ for each edge $e \in E(G)$. (See [5], for example.)

Problem 1.

(i) Characterize the class of graphs $G$ with $\gamma_{pr}(\pi G) < \gamma_{pr}(G)$ for some permutation $\pi$ of $V$.

(ii) If $\gamma_{pr}(\pi G) < \gamma_{pr}(G)$ for some permutation $\pi$ of $V$, what is $\max_{\pi \in S_n} \{\gamma_{pr}(\pi G)\}$?

(iii) What is $\min_{\pi \in S_n} \{\gamma_{pr}(\pi G)/\gamma_{pr}(G)\}$?

(iv) If $\gamma_{pr}(\pi G) < \gamma_{pr}(G)$ for some permutation $\pi$ of $V$, does it follow that $G$ is $\gamma_{pr}$-edge-critical? (The converse is not true—consider $C_5$.)
For the usual domination number $\gamma$, it is still an open problem to find a nontrivial connected universal fixer, or to show that no such graph exists. The corresponding problem for the paired domination number (for graphs $G$ with $\gamma_{pr}(G) \leq \gamma_{pr}(\pi G)$ for all permutations $\pi$ of $V$) has not been studied at all. It is easy to see that nontrivial complete graphs are universal $\gamma_{pr}$-fixers, but none of the other graphs listed at the end of Section 5 is a universal $\gamma_{pr}$-fixer.

**Problem 2.** Prove or disprove Conjecture 1: The graphs $\overline{K_n}$, $n \geq 1$, are the only universal $\gamma$-fixers.

**Problem 3.**

(i) Characterize the class of universal $\gamma_{pr}$-fixers.

(ii) Failing (i), find examples of noncomplete universal $\gamma_{pr}$-fixers.

**References**


Received 29 January 2009
Revised 27 July 2009
Accepted 27 July 2009