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ON VERTEX STABILITY WITH REGARD TO COMPLETE BIPARTITE SUBGRAPHS

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Abstract

A graph G is called $(H;k)$-*vertex stable* if G contains a subgraph isomorphic to H ever after removing any of its k vertices. $Q(H;k)$ denotes the minimum size among the sizes of all $(H;k)$-vertex stable graphs. In this paper we complete the characterization of $(K_{m,n};1)$-vertex stable graphs with minimum size. Namely, we prove that for $m \geq 2$ and $n \geq m+2$, $Q(K_{m,n};1) = mn+m+n$ and $K_{m,n} * K_1$ as well as $K_{m+1,n+1} - e$ are the only $(K_{m,n};1)$-vertex stable graphs with minimum size, confirming the conjecture of Dudek and Zwonek.

Keywords: vertex stable, bipartite graph, minimal size.

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1. Introduction

We deal with simple graphs without loops and multiple edges. We use the standard notation of graph theory, cf. [1]. The following notion was introduced in [2]. Let H be any graph and k a non-negative integer. A graph G is called $(H;k)$-*vertex stable* if G contains a subgraph isomorphic to H ever after removing any of its k vertices. Then $Q(H;k)$ denotes minimum size among the sizes of all $(H;k)$-vertex stable graphs. Note that if H does not have isolated vertices then after adding to or removing from a $(H;k)$-vertex

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stable graph any number of isolated vertices we still have a $(H;k)$-vertex stable graph with the same size. Therefore, in the sequel we assume that no graph in question has isolated vertices.

There are two trivial examples of (H,k)-vertex stable graphs, namely $(k+1)H$ (a disjoint union of $(k+1)$ copies of H) and $H * K_k$ (a graph obtained from $H \cup K_k$ by joining all the vertices of H to all the vertices of K_k). Therefore,

Proposition 1. $Q(H;k) \leq \min\left\{(k+1)|E(H)|,\ |E(H)| + k|V(H)| + \binom{k}{2}\right\}$.

On the other hand, the following is easily seen.

Proposition 2. *Suppose that H contains k vertices which cover q edges. Then $Q(H;k) \geq |E(H)| + q$.*

Recall also the following

Proposition 3 ([2])**.** *Let δ_H be a minimal degree of a graph H. Then in any $(H;k)$-vertex stable graph G with minimum size, $\deg_G v \geq \delta_H$ for each vertex $v \in G$.*

The exact values of $Q(H;k)$ are known in the following cases: $Q(C_i;k) = i(k+1)$, $i = 3,4$, $Q(K_4;k) = 5(k+1)$, $Q(K_n;k) = \binom{n+k}{2}$ for n large enough, and $Q(K_{1,m};k) = m(k+1)$, $Q(K_{n,n};1) = n^2 + 2n$, $Q(K_{n,n+1};1) = (n+1)^2$, $n \geq 2$, see [2, 3]. In this paper we complete the characterization of $(K_{m,n};1)$ vertex stable graphs with minimum size. Namely, we prove the following theorem and hence confirm Conjecture 1 formulated in [3].

Theorem 1. *Let m,n be positive integers such that $m \geq 2$ and $n \geq m+2$. Then $Q(K_{m,n};1) = mn + m + n$ and $K_{m,n} * K_1$ as well as $K_{m+1,n+1} - e$, where $e \in E\left(K_{m+1,n+1}\right)$, are the only $(K_{m,n};1)$-vertex stable graphs with minimum size.*

2. Proof of the Main Result

***Proof of Theorem* 1.** Let $m \geq 2$ and $n \geq m+2$ be positive integers. Define $G_1 := K_{m,n} * K_1$ and $G_2 := K_{m+1,n+1} - e$ where $e \in E\left(K_{m+1,n+1}\right)$. Let $G = (V,E)$ be a $(K_{m,n},1)$-vertex stable graph with minimum size. Thus, by Proposition 1, $|E(G)| \leq mn + m + n$. Clearly G contains a subgraph

H isomorphic to $K_{m,n}$. Let $H = (X, Y; E_H)$ with vertex bipartition sets X, Y such that $|X| = m$ and $|Y| = n$. Let $v \in X$. Since G is $(K_{m,n}; 1)$-vertex stable, $G - v$ contains a subgraph H' isomorphic to $K_{m,n}$. Let $H' = (X', Y'; E_{H'})$ with vertex bipartition sets X', Y' such that $|X'| = m$ and $|Y'| = n$. We denote $x_1 = |X \cap X'|$, $x_2 = |X \cap Y'|$, $y_1 = |Y \cap X'|$, $y_2 = |Y \cap Y'|$. Hence $x_1 + x_2 \leq m - 1$, $y_1 + y_2 \leq n$, $y_1 \leq m$. One can see that $|E(G)| \geq 2mn - x_1y_2 - x_2y_1$. Consider the following linear programming problem with respect to y_1 and y_2

$$\begin{cases} y_1 \leq m \\ y_1 + y_2 \leq n \\ y_1 \geq 0 \\ y_2 \geq 0 \\ c = x_1y_2 + x_2y_1 \to \max \end{cases}$$

where x_1 and x_2 are parameters such that $x_1, x_2 \geq 0, x_1 + x_2 \leq m - 1$.

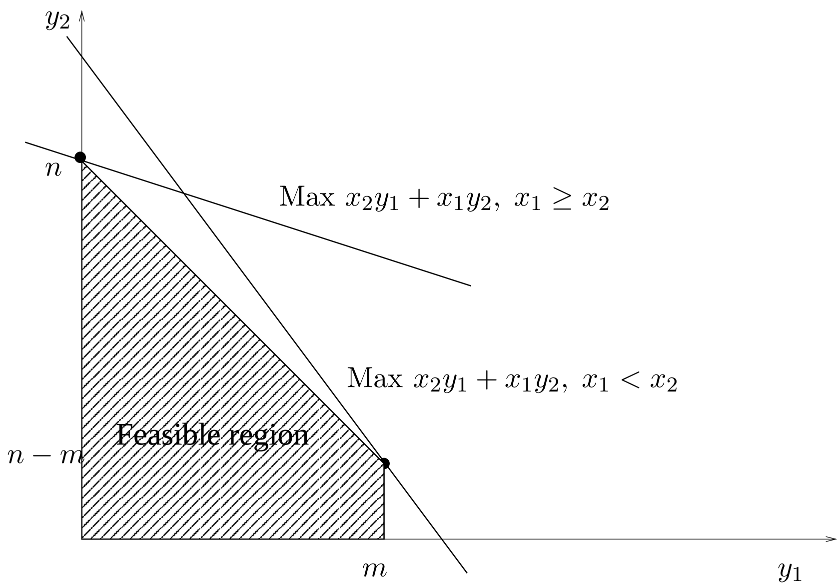

Figure 1. Geometrical interpretation of the linear programming problem.

The proof falls into two cases.

Case 1. $x_1 < x_2$.

In this case $y_1 = m, y_2 = n - m$, $c = x_2 m + x_1(n - m)$ is the unique optimal solution of the above linear programming problem. This can be easily checked using a geometrical interpretation of the linear programming problem, see Figure 1. Thus $|E(G)| \geq 2mn - x_2 m - x_1(n - m)$ and the inequality is strict if $y_1 \neq m$ or $y_2 \neq n - m$. We assume that $x_1 + x_2 = m - 1$ because otherwise the size of G may only increase. Then

$$|E(G)| \geq 2mn - m^2 + m + x_1(2m - n) := f(x_1).$$

Subcase 1a. $n > 2m$.
Then $f(x_1)$ is decreasing. Furthermore, $x_1 < \frac{m-1}{2}$ since $x_1 < x_2$. Thus

$$|E(G)| > f\left(\frac{m-1}{2}\right) = \frac{3}{2}mn + \frac{1}{2}n \geq mn + m + n.$$

Thus $|E(G)| > mn + m + n$, a contradiction.

Subcase 1b. $n < 2m$.
Then $f(x_1)$ is increasing. Thus

$$E(G) \geq f(0) = 2mn - m^2 + m \geq mn + m + n$$

with equality if and only if $m = 2$ and $n = 4$, which is not possible in this subcase.

Subcase 1c. $n = 2m$.
In this case

$$E(G) \geq mn + m + n$$

with equality if and only if $m = 2$, $n = 4$, $y_1 = y_2 = 2$. Recall that $x_1 < x_2$ whence $x_1 = 0$ and $x_2 = 1$. Let $u \in Y' \setminus (X \cup Y)$. Thus $|E(G)| \geq 12 + \deg u$. Hence $\deg u = 2$ and $|V(G)| = 7$ because otherwise $|E(G)| > mn + m + n$. However, then G is not $(K_{2,4}; 1)$-stable. Indeed let w be a neighbor of u. Then $G - w$ does not contain any subgraph isomorphic to $K_{2,4}$ since $G - w$ has 6 vertices and one of them has degree 1. Therefore Case 1 is not possible.

Case 2. $x_1 \geq x_2$.
In this case $c = x_1 n$ is the optimal solution of the above linear problem, see Figure 1. Therefore, $|E(G)| \geq 2mn - x_1 n$. If $x_1 \leq m - 2$ then $|E(G)| \geq 2mn - (m - 2)n = mn + 2n > mn + m + n$. Hence we may assume that

$x_1 = m - 1$ and $x_2 = 0$. Thus there is only one vertex, say u, such that $u \in X' \setminus X$.

Subcase 2a. $y_2 = n$.
Thus, u have n neighbors in Y. Note that $|V(G)| \leq m + n + 2$. Indeed, otherwise by Proposition 3, $|E(G)| \geq mn + n + 2m - 1 > mn + m + n$. Consider now a graph $G'' := G - w$ where $w \in Y$. Clearly $G - w$ contains a subgraph H'' isomorphic to $K_{m,n}$. Let $H'' = (X'', Y''; E_{H''})$ with vertex bipartition sets X'', Y'' such that $|X''| = m$ and $|Y''| = n$. Let $x'_1 = |X \cap X''|$, $x'_2 = |X \cap Y''|$, $y'_1 = |Y \cap X''|$, $y'_2 = |Y \cap Y''|$.

Suppose first that $|V(G)| = m+n+2$ and $u, u_1 \in V(G) \setminus (X \cup Y)$. Since $|E(G)| \leq mn+m+n$, $\deg u_1 = m$ and $\deg u \leq n+1$. In particular, $u_1 \notin X''$ and u has no neighbor in X. Furthermore, $|E(G)| \geq mn+n+m+x'_1x'_2+y'_1y'_2$. Thus, $x'_1 = 0$ or $x'_2 = 0$, and $y'_1 = 0$ or $y'_2 = 0$. We distinguish two possibilities

1. $x'_1 = 0$. Then $y'_1 \neq 0$. Indeed, otherwise $X'' = \{u, u_1\}$, a contradiction with previous observation that $u_1 \notin X''$. Hence, $y'_2 = 0$. Thus, $x'_2 = m$ and $u, u_1 \in Y''$ (so $n = m + 2$). Consequently, $y'_1 = m$. However, then G is not $(K_{m,m+2}; 1)$-stable. Indeed, let w_1 be a neighbor of u_1, $w_1 \in X'' \subset Y$. Then $G - w_1$ consists of a subgraph isomorphic to $K_{m+1,m+1}$ plus one vertex (namely u_1) and $m - 1$ edges incident to it. Therefore, $G - w_1$ does not contain any subgraph isomorphic to $K_{m,m+2}$.

2. $x'_1 \neq 0$. Then $x'_2 = 0$ and $u \notin Y''$. Consequently, $u_1 \in Y''$ and $y'_2 \neq 0$. Hence $y'_1 = 0$. It is easy to see now that $G \cong G_2$.

Assume now that $|V(G)| = m+n+1$. Hence $x'_1+x'_2 = m$ and $y'_1+y'_2 = n-1$. We have the next two possibilities.

3. $x'_1 + y'_1 = m$. Then $|E(G)| \geq mn + x'_1x'_2 + y'_1y'_2 + \deg u \geq mn + x'_1x'_2 + y'_1y'_2 + n + x'_1$. Hence

$$|E(G)| \geq mn + (m - x'_1)(n - 1 - m + 2x'_1) + n + x'_1 =: f_1(x'_1),\ 0 \leq x'_1 \leq m.$$

It is not difficult to see that $f_1(x'_1)$ obtains the smallest value for $x'_1 = 0$ or $x'_1 = m$ only. Thus, $|E(G)| \geq \min\{f_1(0), f_1(m)\}$. Note that $f_1(0) = 2mn + n - m - m^2 \geq mn + m + n$ with equality if and only if $n = m + 2$. However, then there is a vertex $y \in Y''$ such that $G - y \cong K_{m+1,m+1}$ so $G - y$ does not contain any subgraph isomorphic to $K_{m,m+2}$. Furthermore, $f(m) \geq mn + n + m$. Thus, $|E(G)| \geq mn + m + n$ with equality if and only $x_1 = m$. Then $G \cong G_1$.

4. $x'_2 + y'_2 = n$. Then $|E(G)| \geq mn + x'_1x'_2 + y'_1y'_2 + n + x'_2$. Hence,

$$|E(G)| \geq mn+(m-x'_2)x'_2+(x'_2-1)(n-x'_2)+n+x'_2 =: f_2(x_2),\ 1 \leq x'_2 \leq m.$$

One can see that $f_2(x'_2)$ obtains the smallest value for $x'_2 = 1$ or $x'_2 = m$ only. Thus, $|E(G)| \geq \min\{f_2(1), f_2(m)\}$. Note that $f_2(1) = mn + n + m$. Then $G \cong G_1$. On the other hand, $f_2(m) = 2mn + 2m - m^2 > mn + m + n$.

Subcase 2b. $y_2 < n$.
Thus, there is a vertex $z \in Y'$ such that $z \in V(G) \setminus (X \cup Y)$. This clearly forces $m - 1$ neighbours of z in $X \setminus \{v\}$. Consider now a graph $G - v_1$, $v \neq v_1 \in X$. We repeat all preceding arguments to the graph $G - v_1$. Consequently, $G \cong G_i$, $i = 1, 2$, or there is a vertex $z_1 \in V(G) \setminus (X \cup Y)$ which has $m - 1$ neighbors in $X \setminus \{v_1\}$. If $z = z_1$ then z has m neighbors in X and $G \cong G_1$ if $u \in Y$ or $G \cong G_2$ otherwise. If $z \neq z_1$ then either $\deg z + \deg z_1 \geq 2m + 1$ if both vertices z and z_1 are involved in a $K_{m,n}$ contained in $G - v$ or $G - v_1$, or $\deg u \geq n + 1$ otherwise. Thus, $|E(G)| \geq mn + 2m - 1 + n > mn + m + n$. ■

3. Concluding Remarks

In [2, 3] it is proved that $Q(K_{1,n}; k) = (k + 1)n$. However, for $n \geq 3$ the extremal graphs are not characterized.

Proposition 4. *Let G be a $(K_{1,n}; k)$-vertex stable graph with minimum size, $n \geq 3$. Then $G = (k + 1)K_{1,n}$.*

Proof. The proof is by induction on k. The thesis is obvious for $k = 0$. Assume that $k > 0$. Let G be a $(K_{1,n}; k)$-vertex stable graph with minimum size. Hence, $|E(G)| = (k+1)n$. Note that each $(K_{1,n}; k)$-vertex stable graph contains $k + 1$ vertices of degree at least n. Let $v \in V(G)$ with $\deg v \geq n$. Thus, $G-v$ is $(K_{1,n}; k-1)$-vertex stable graph with $|E(G-v)| \leq kn$. Hence, $|E(G-v)| = kn$ and $\deg v = n$. By the induction hypothesis $G-v = kK_{1,n}$. Note that v is not a neighbor of any vertex of degree n. Suppose on the contrary, that $uv \in E(G)$ and $\deg u = n$. Then $G - u$ contains only $k - 1$ vertices of degree greater than or equal to n whence is not $(K_{1,n}; k - 1)$-vertex stable, a contradiction. Thus, G contains $k + 1$ independent vertices of degree exactly n. We will show that these vertices have pairwise disjoint sets of neighbors. Indeed, otherwise let x be a common neighbor of two vertices of degree n. Thus, again, $G - x$ has only $k - 1$ vertices of degree greater than or equal to n, a contradiction. ∎

In the following table we present the complete characterization of $(K_{m,n}; 1)$-vertex stable graphs with minimum size.

$m; n$	$Q(K_{m,n}; 1)$		All $(K_{m,n}; 1)$-vertex stable graphs with minimum size	
$m = 1, n = 1$	2	[3]	$2K_{1,1}$	[3]
$m = 1, n = 2$	4	[3]	$K_{2,2}$, $2K_{1,2}$	[3]
$m = 1, n \geq 3$	2n	[2]	$2K_{1,n}$	
$m = 2, n = 2$	8	[3]	$K_{2,2} * K_1$, $K_{3,3} - e$, $2K_{2,2}$	[3]
$m \geq 2, n = m + 1$	$(m + 1)^2$	[3]	$K_{m+1,m+1}$	[3]
$m \geq 3, n = m$	$m^2 + 2m$	[3]	$K_{m,m} * K_1$, $K_{m+1,m+1} - e$	[3]
$m \geq 2, n \geq m + 2$	$mn + m + n$		$K_{m,n} * K_1$, $K_{m+1,n+1} - e$	

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