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CLIQUE GRAPH REPRESENTATIONS OF PTOLEMAIC GRAPHS

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Abstract

A graph is ptolemaic if and only if it is both chordal and distance-hereditary. Thus, a ptolemaic graph G has two kinds of intersection graph representations: one from being chordal, and the other from being distance-hereditary. The first of these, called a clique tree representation, is easily generated from the clique graph of G (the intersection graph of the maximal complete subgraphs of G). The second intersection graph representation can also be generated from the clique graph, as a very special case of the main result: *The maximal P_n-free connected induced subgraphs of the p-clique graph of a ptolemaic graph G correspond in a natural way to the maximal P_{n+1}-free induced subgraphs of G in which every two nonadjacent vertices are connected by at least p internally disjoint paths.*

Keywords: Ptolemaic graph, clique graph, chordal graph, clique tree, graph representation.

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1. Basic Concepts

For any graph G, denote the family of all *maxcliques* of G—meaning the inclusion-maximal complete subgraphs of G—as $\mathcal{C}(G)$, and denote the family of all inclusion-maximal induced connected subgraphs of G that are *cographs*—meaning they contain no induced path of length three—as $\mathcal{CC}(G)$.

(Of course $\mathcal{C}(G)$ can be equivalently described as the family of all maximal induced subgraphs of G that contain no induced path of length two.)

Let $\Omega(\mathcal{C}(G))$ [respectively, $\Omega(\mathcal{CC}(G))$] denote the *clique intersection graph* [or the $\mathcal{CC}$ *intersection graph*] of G, meaning the intersection graph that has the members of $\mathcal{C}(G)$ [or $\mathcal{CC}(G)$] as nodes, with two nodes adjacent if and only if their vertex sets have nonempty intersection. Let $\Omega^{\mathrm{w}}(\mathcal{C}(G))$ and $\Omega^{\mathrm{w}}(\mathcal{CC}(G))$ denote their weighted counterparts where, for S, S' in $\mathcal{C}(G)$ or in $\mathcal{CC}(G)$, the weight of the edge SS' equals $|V(S) \cap V(S')|$. Figure 1 shows an example.

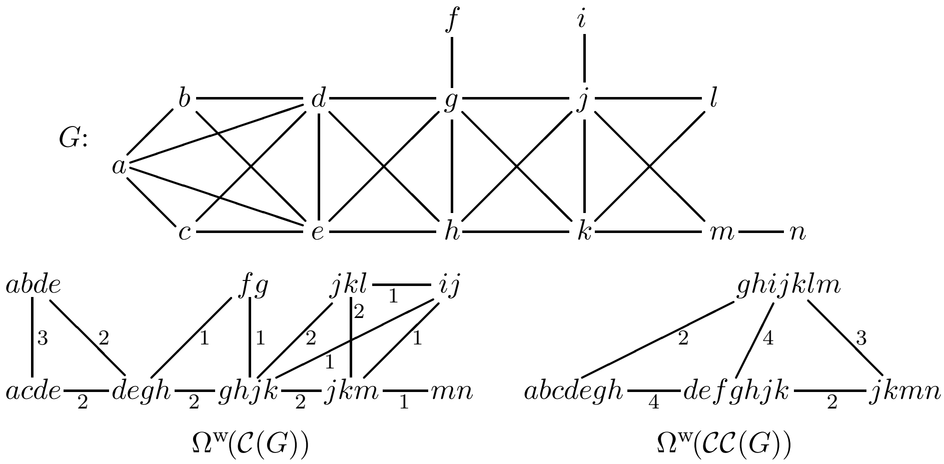

Figure 1. A graph G (at the top) with its weighted clique graph (lower left) and its weighted $\mathcal{CC}$ graph (lower right), where *abde* abbreviates the subgraph of G that is induced by $\{a, b, d, e\}$, etc.

For each $p \geq 1$, the *p-clique graph* $K_p(G)$ of G is the graph that has the maxcliques of G as nodes, with two nodes Q and Q' adjacent in $K_p(G)$ if and only if $|V(Q) \cap V(Q')| \geq p$; see [6, section 6.1]. In other words, $K_p(G)$ is the graph that is formed by the edges of $\Omega^{\mathrm{w}}(\mathcal{C}(G))$ that have weight p or more. The *clique graph* of G is $K_1(G)$, typically abbreviated as $K(G)$; see [9]. For instance, $K(G)$ for the graph G in Figure 1 is $\Omega^{\mathrm{w}}(\mathcal{C}(G))$ without the edge weights; Figure 2 shows $K_2(G)$ for the same G.

A graph is *chordal* if every cycle of length four or more has a *chord* (meaning an edge that joins two vertices of the cycle that are not consecutive along the cycle). Among many characterizations in [3, 6], a graph G is

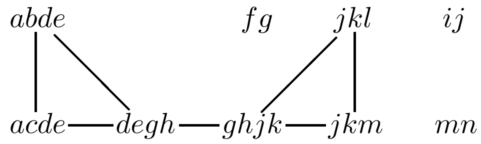

Figure 2. The 2-clique graph $K_2(G)$ of the graph G in Figure 1.

chordal if and only if G has a $\mathcal{C}(G)$ *tree* representation—typically called a *clique tree*—where this means a tree T whose nodes are the maxcliques of G such that, for each $v \in V(G)$, the subgraph T_v of T that is induced by those nodes of T that contain v is connected—in other words, each T_v is a subtree of T. A graph is chordal if and only if it is the intersection graph of a family of subtrees of some tree, and that family can always be taken to be the subtrees T_v of a clique tree T. The clique trees of a chordal graph G are exactly the maximum spanning trees of $\Omega^{\mathrm{w}}(\mathcal{C}(G))$; see [6] for a thorough discussion of all this. The graph shown in Figure 1 is chordal, and Figure 3 shows one of its clique trees.

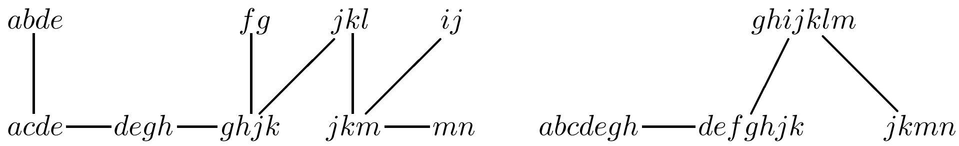

Figure 3. A clique tree (on the left) and a $\mathcal{CC}$ tree (on the right) for the graph G in Figure 1.

A set $S \subset V(G)$ is a *vertex separator* of a graph G if there are vertices v, w that are in a common component of G but different components of the subgraph induced by $V(G)-S$; such an S is also called a v, w*-separator*. If G is chordal with a clique tree T, then the inclusion-minimal vertex separators of G correspond exactly to $V(Q) \cap V(Q')$ where QQ' is an edge T; see [6, section 2.1] for details.

A graph G is *distance-hereditary* if the distance between vertices in a connected induced subgraph of G always equals their distance in G. Equivalently, G is distance-hereditary if and only if, for every $v, w \in V(G)$, all the induced v-to-w paths in G have the same length; see [3]. A graph G is distance-hereditary if and only if G has a *CC tree* T, where T is a spanning tree of $\Omega(\mathcal{CC}(G))$ such that each subgraph T_v (defined the same as for clique trees) is a subtree of T. Again, $\mathcal{CC}$ trees are the maximum spanning trees of $\Omega^{\mathrm{w}}(\mathcal{CC}(G))$; see [3, 5, 7] for details of all this. The graph shown in Figure 1 is distance-hereditary, and Figure 3 shows its $\mathcal{CC}$ tree (which is unique in this example).

Let P_n and C_n denote, respectively, a path or cycle with n vertices. Let $v \sim w$ denote that vertices v and w are adjacent, and let $N(v) = \{x : v \sim x \in V(G)\}$. Define a *gem* to be a graph that consists of a cycle of length five together with two chords with a common endpoint. For any graph H, a graph G is said to be *H-free* if G has no induced subgraph isomorphic to H. For any graph G with induced subgraph H and vertex $v \in V(G) - V(H)$, let H^{+v} denote the subgraph of G induced by $V(H) \cup \{v\}$.

A graph is *ptolemaic* if it is both chordal and distance-hereditary; see [3, 4] for history and details. Being ptolemaic is equivalent to being both gem-free and chordal, and also to being both C_4-free and distance-hereditary. Ptolemaic graphs therefore have two kinds of tree representations: both a clique tree because of being chordal, and a $\mathcal{CC}$ tree because of being distance-hereditary. Corollary 5 will show how the clique graph of a ptolemaic graph G also determines $\mathcal{CC}(G)$ and thereby the $\mathcal{CC}$ trees of G. But first, Theorem 1 will further characterize ptolemaic graphs and Theorem 4 will show how subgraphs of a ptolemaic graph G can be identified in the clique graph of G.

Theorem 1. *Each of the following is equivalent to a chordal graph G being ptolemaic*:

(1.1) *Every edge in $K(G)$ is contained in some clique tree for G.*

(1.2) *For every $p \geq 1$, every induced path in $K_p(G)$ is contained in some clique tree for G.*

Proof. From [4, Theorem 2.4], a graph G is ptolemaic if and only if every nonempty intersection of two maxcliques of G is an inclusion-minimal vertex separator of G. Recalling that the inclusion-minimal vertex separators of G correspond exactly to the edges of a clique tree for G, and that every maximum spanning tree of $\Omega^{\mathrm{w}}(\mathcal{C}(G))$ is a clique tree for G, it follows that being ptolemaic is equivalent to condition (1.1). Also, the $p = 1$ case of (1.2) implies (1.1).

To finish the proof, suppose G is ptolemaic, condition (1.1) holds, and $p \geq 1$ [toward proving condition (1.2)]. Let $\mathcal{C}(G) = \{Q_1, \ldots, Q_c\}$ where $\Pi = Q_1, \ldots, Q_n$ is an induced path in $K_p(G)$, and let

$$\mu = \max\{|V(Q_i) \cap V(Q_j)| : 1 \leq i < j \leq n\}.$$

Using (1.1), let T_2 be a clique tree that contains the edge Q_1Q_2. Each node Q_i of T_2—equivalently, each maxclique Q_i of G—with $i \notin \{1,2\}$ will contain a vertex $v_i \notin V(Q_1) \cup V(Q_2)$. Form a new graph G_2 from G by creating a set S_2 of $\mu - |V(Q_1) \cap V(Q_2)| + 1$ new vertices that are adjacent precisely to each other and to the vertices in $Q_1 \cup Q_2$. For $i \in \{1,2\}$, let Q_i^2 be the subgraph induced by $V(Q_i) \cup S_2$ in G_2; for $i \notin \{1,2\}$, let $Q_i^2 = Q_i$. The maxcliques of G_2 will be precisely $Q_1^2, \dots, Q_c^2$ (since each $N(v_i) \cap S_2 = \emptyset$).

To show that G_2 is chordal, suppose C were a chordless cycle of G_2 with length four or more such that C contained a vertex $s \in S_2$ [arguing by contradiction]. Then C would consist of edges sq_1 and sq_2 with $q_1 \in V(Q_1) - V(Q_2)$ and $q_2 \in V(Q_2) - V(Q_1)$, together with an induced q_1-to-q_2 path π within G. Because Q_1Q_2 is an edge of the clique tree T_2, the set $V(Q_1) \cap V(Q_2)$ will be a q_1, q_2-separator, and so the path π must contain an internal vertex $w \in V(Q_1) \cap V(Q_2)$, making $w \sim s$ [contradicting that C was chordless].

To show that G_2 is gem-free, suppose $\{a,b,c,d,e,a\}$ induced a gem in G_2 [arguing by contradiction], where a,b,c,d,e,a is a cycle that has exactly the two chords be and ce. If $a \in S_2$, then $a,b,e \in V(Q_i^2)$ where $i \in \{1,2\}$ and $c,d \notin V(Q_i^2)$; then there would exist a $v \in V(Q_i)$ with $c \not\sim v \not\sim d$, which would make $\{v,b,c,d,e\}$ induce a gem in G [contradicting that G is ptolemaic]. The case $d \in S_2$ is similar. If $b \in S_2$, then (without loss of generality) vertex a is in $Q_1 - Q_2$, vertex c is in $Q_2 - Q_1$, vertex d is not in $Q_1 \cup Q_2$, and vertices b and e are in $Q_1 \cap Q_2$; then there would exist a $v \in V(Q_1) \cap V(Q_2) - \{e\}$, which would make $\{a,v,c,d,e\}$ induce a gem in G [contradicting that G is ptolemaic]. The case $c \in S_2$ is similar. Note that $e \notin S_2$, since e is in at least three maxcliques of G.

Therefore, G_2 is ptolemaic.

Repeat the G_2 construction to form new ptolemaic graphs G_i—from G_{i-1} using $\mu - |V(Q_{i-1}) \cap V(Q_i)| + 1$ new vertices adjacent precisely to each other and to the vertices in $V(Q_{i-1}) \cup V(Q_i)$—whenever $3 \leq i \leq n$. The final ptolemaic graph G_n will have maxcliques $Q_1^n, \dots, Q_c^n$ that contain $Q_1, \dots, Q_c$ respectively, where $Q_1^n, \dots, Q_n^n$ forms an induced path Π_n of maximum-weight edges of $K_{\mu+1}(G_n)$. Let T_n be a maximum spanning tree of $\Omega^{\mathrm{w}}(\mathcal{C}(G_n))$ that contains Π_n. This T_n will be a clique tree for G_n and, by suppressing all the vertices in $V(G_n) - V(G)$, this T_n will correspond to a clique tree of G that contains the edges of Π. ■

The following consequence of Theorem 1 will be used several times in Section 2.

Lemma 2. *If G is ptolemaic with $p \geq 1$ and $n \geq 2$ and if $Q_1, \ldots, Q_n$ is an induced path in $K_p(G)$, then there exist $v_0, \ldots, v_n \in V(G)$ such that $v_0 \in V(Q_j)$ exactly when $j = 1$, each $1 \leq i \leq n-1$ has $v_i \in V(Q_j)$ exactly when $j \in \{i, i+1\}$, and $v_n \in V(Q_j)$ exactly when $j = n$.*

Proof. Suppose G is ptolemaic with $p \geq 1$ and $n \geq 2$, suppose $\Pi = Q_1, \ldots, Q_n$ is an induced path in $K_p(G)$ and, within this proof, identify each Q_i with $V(Q_i)$. Therefore $|i-j| = 1$ implies $|Q_i \cap Q_j| \geq p$ and $|i-j| > 1$ implies $|Q_i \cap Q_j| < p$. The existence of the desired $v_0 \in Q_1$ and $v_n \in Q_n$ follows from $Q_1 \not\subseteq Q_2$ and $Q_n \not\subseteq Q_{n-1}$ (since maxcliques of any graph have incomparable vertex sets). The existence of the desired $v_1 \in Q_1 \cap Q_2 - Q_3$ follows from $Q_1 \cap Q_2 \not\subseteq Q_2 \cap Q_3$ (since $|Q_1 \cap Q_3| < p$); the existence of $v_{n-1} \in Q_n \cap Q_{n-1} - Q_{n-2}$ follows similarly.

Suppose $1 < i < n-1$ [toward showing the existence of $v_i \in (Q_i \cap Q_{i+1}) - (Q_{i-1} \cup Q_{i+2})$]. Suppose instead that $Q_i \cap Q_{i+1} \subseteq Q_{i-1} \cup Q_{i+2}$ [arguing by contradiction]. By Theorem 1, Π is a path in some clique tree T for G. Because Π is an induced path, the three cardinality-p sets $Q_{i-1} \cap Q_i$, $Q_i \cap Q_{i+1}$, and $Q_{i+1} \cap Q_{i+2}$ are pairwise unequal, and so there exist $v \in Q_i \cap Q_{i+1} - Q_{i-1}$ and $w \in Q_i \cap Q_{i+1} - Q_{i+2}$ (and so $w \in Q_{i-1}$, since $Q_i \cap Q_{i+1} \subset Q_{i-1} \cup Q_{i+2}$). There would also exist $t \in Q_{i-1} - Q_i$, $u \in Q_{i-1} \cap Q_i - Q_{i+1}$, and $x \in Q_{i+1} \cap Q_{i+2} - Q_i$ (just as for the $i = 0, 1, n-1$ cases, respectively, but now for the path $Q_{i-1}, Q_i, Q_{i+1}, Q_{i+2}$). So $\{t, u, w\}$, $\{u, v, w\}$, and $\{v, w, x\}$ would induce triangles (inside Q_{i-1}, Q_i and Q_{i+1} respectively), and $u \not\sim x \not\sim t \not\sim v$ (for instance, $u \not\sim x$ since u and x are not in a common maxclique, using that T is a clique tree for G). But then $\{t, u, v, w, x\}$ would induce a gem in G [contradicting that G is ptolemaic]. ■

2. Representing Subgraphs of G within $K(G)$

For each $p \geq 1$ and $n \geq 2$, let $\langle G, p, n \rangle$ denote the family of all induced subgraphs of G that are maximal with respect to both being P_n-free and having every two nonadjacent vertices connected by at least p internally-disjoint paths (such paths form what is sometimes called a *p-skein*). That second condition is equivalent to the subgraph being either p-connected or complete. For example, $\langle G, 1, 2 \rangle = V(G)$, $\langle G, 1, 3 \rangle = \mathcal{C}(G)$, and $\langle G, 1, 4 \rangle = \mathcal{CC}(G)$, while $\langle G, 2, 4 \rangle$ consists of the 2-connected members of $\mathcal{CC}(G)$ together with any bridges (edges that are not in cycles) and isolated vertices.

If $\mathcal{H}$ is a connected induced subgraph of $K(G)$ and H is a connected induced subgraph of G, then say that $\mathcal{H}$ *represents* H in G if H is induced by the vertices that are in the union of the maxcliques of G that correspond to the nodes of $\mathcal{H}$. In Figure 1 for instance, the path $\mathcal{H}$ of $K(G)$ induced by the nodes $ghjk$, jkm, and mn represents the subgraph H of G that is induced by $\{g, h, j, k, m, n\}$. Every connected induced subgraph $\mathcal{H}$ of $K(G)$ clearly represents a connected induced subgraph H of G with $\mathcal{H} \cong K(H)$, but not conversely: for instance, $V(H) = \{e, h, k\}$ is not even a union of maxcliques of G.

Given a family $\text{FAM}_{K(G)}$ of connected induced subgraphs of $K(G)$ and a family FAM_G of connected induced subgraphs of G, say that the members of $\text{FAM}_{K(G)}$ *represent precisely* the members of FAM_G if every $\mathcal{H} \in \text{FAM}_{K(G)}$ represents an $H \in \text{FAM}_G$ and every $H \in \text{FAM}_G$ is represented by some $\mathcal{H} \in \text{FAM}_{K(G)}$. For instance, the nodes of $K(G)$ always represent precisely the maxcliques of G.

Theorem 4 will look at certain kinds of subgraphs of the clique graph of a ptolemaic graph G and at the kinds of subgraphs of G that they represent. For instance, Corollary 5 will show that the maxcliques of the clique graph of a ptolemaic graph G represent precisely the members of $\mathcal{CC}(G)$. Theorem 4 will use the following lemma.

Lemma 3. *If G is ptolemaic with $p \geq 1$ and $n \geq 2$ and if $H \in \langle G, p, n\rangle$, then $\mathcal{C}(H) \subseteq \mathcal{C}(G)$.*

Proof. Suppose G is ptolemaic (and so chordal and distance-hereditary) with $p \geq 1$ and $n \geq 2$, and suppose $H \in \langle G, p, n\rangle$ and $Q \in \mathcal{C}(H) - \mathcal{C}(G)$ [arguing by contradiction]; so there exists $v \in V(G) - V(H)$ with $Q \subseteq N(v)$. The maximality of H from being in $\langle G, p, n\rangle$ implies that $H^{+v} \notin \langle G, p, n\rangle$, and so H must be p-connected (as opposed to H being complete with $|V(H)| = |V(Q)| \leq p$). Also, H must be chordal (since G is), and so H will have a clique tree T. Since H is p-connected, every edge Q_iQ_j of T will have $|V(Q_i) \cap V(Q_j)| \geq p$ (since $V(Q_i) \cap V(Q_j)$ will be a minimal vertex separator in G), and so every node Q_i of T will have $|V(Q_i)| \geq p$. In particular, $|V(Q)| \geq p$, which makes H^{+v} also p-connected. Hence, $H^{+v} \notin \langle G, p, n\rangle$ implies that there must exist an induced path $\pi = v_1, \ldots, v_n$ in G that has $v \in V(\pi) \subseteq V(H) \cup \{v\}$.

Vertex v cannot be an interior vertex of π—otherwise $\{v_1, v_n\} \subset V(H)$ and $H \in \langle G, p, n\rangle$ would imply there is an induced v_1-to-v_n path within H shorter than π [contradicting that G is distance-hereditary].

Without loss of generality, say $v = v_1$ and suppose for the moment that $v_2 \notin V(Q)$. Note that $v_i \notin V(Q)$ for $i \geq 3$ (since $V(Q) \subset N(v)$ and π induced implies such v_i not adjacent to v_1). Because $v_2 \notin V(Q)$ and Q is a maxclique of G, there is a $q \in V(Q)$ such that $v_2 \not\sim q \sim v_1$. Note that $v_i \not\sim q$ for $i \geq 3$ [otherwise some $q, v_1, \ldots, v_i, q$ would be an induced cycle in G with length $i + 1 \geq 4$, contradicting that G is chordal]. So $q, v_1, \ldots, v_n$ is an induced q-to-v_n-path of length n in G. But $H \in \langle G, p, n\rangle$ would imply there is an induced q-to-v_n-path within H of length less than n [again contradicting that G is distance-hereditary].

Thus $v_2 \in V(Q)$. As before, $v_i \notin V(Q)$ for $i \geq 3$. Because $v_3 \notin V(Q)$, there is a $q \in V(Q)$ such that $v_3 \not\sim q \sim v_2$. Note that $v_i \not\sim q$ when $i \geq 4$ (otherwise some $q, v_2, \ldots, v_i, q$ would be an induced cycle in G with length $n = i \geq 4$). But then $q, v_2, \ldots, v_n$ would form an induced P_n in H [contradicting $H \in \langle G, p, n\rangle$]. ■

Theorem 4. *If G is ptolemaic with $p \geq 1$ and $n \geq 2$, then the subgraphs of $K(G)$ in $\langle K_p(G), 1, n\rangle$ represent precisely the subgraphs of G in $\langle G, p, n+1\rangle$.*

Before proving Theorem 4, it will be helpful to illustrate it using Figure 1 and Figure 2: When $p = 2$ and $n = 3$, the six $\mathcal{H} \in \langle K_2(G), 1, 3\rangle$—these are the six maxcliques of $K_2(G)$—represent the six subgraphs of G that are induced by $\{a, b, c, d, e, g, h\}$, $\{g, h, j, k, l, m\}$, $\{d, e, g, h, j, k\}$, $\{f, g\}$, $\{i, j\}$, and $\{m, n\}$, and these are precisely the six subgraphs $H \in \langle G, 2, 4\rangle$. When $p = 2$ and $n = 4$, the subgraph $\mathcal{H} \in \langle K_2(G), 1, 4\rangle$ that is induced by the four nodes *degh*, *ghjk*, *jkl*, and *jkm* represents the subgraph $H \in \langle G, 2, 5\rangle$ that is induced by $\{d, e, g, h, j, k, l, m\}$. When $p = 3$ and $n = 3$, the maxclique $\mathcal{H} \in \langle K_3(G), 1, 3\rangle$ that is formed by the edge between *abde* and *acde* represents the subgraph $H \in \langle G, 3, 4\rangle$ that is induced by $\{a, b, c, d, e\}$.

Proof. Suppose G is ptolemaic (and so chordal and distance-hereditary) with $p \geq 1$ and $n \geq 2$.

First suppose $\mathcal{H} \in \langle K_p(G), 1, n\rangle$ and $\mathcal{H}$ represents a subgraph H of G. To show $H \in \langle G, p, n+1\rangle$ requires showing three things: (i) that H is P_{n+1}-free, (ii) that every two nonadjacent vertices of H are connected by at least p internally-disjoint paths of H, and (iii) the maximality of H with respect to (i) and (ii). Within this proof, identify each maxclique Q with $V(Q)$.

To show (i), suppose instead that $\pi = v_1, v_2, \ldots, v_{n+1}$ is an induced path in H [arguing by contradiction]. Observe that $\mathcal{H} \in \langle K_p(G), 1, n\rangle$ is a subgraph of an induced subgraph $\mathcal{H}^* \in \langle K(G), 1, n\rangle$ on the same node-set as

$\mathcal{H}$. For each $i \in \{1, 2, \ldots, n-1\}$, let Q_i be a maxclique of G that is a node of $\mathcal{H}^*$ such that $Q_i \cap V(\pi) = \{v_i, v_{i+1}\}$. Note that $|i-j| = 1$ implies $Q_i \cap Q_j \neq \emptyset$ (because $v_i \in Q_i \cap Q_{i+1}$). If $|i-j| > 1$ and $x \in Q_i \cap Q_j$, then x will be adjacent to v_i and v_{i+1} (because $x \in Q_i$), to v_j and v_{j+1} (because $x \in Q_j$), and to every $v_{i'}$ with $i+1 < i' < j$ (since π being induced implies that such $xv_{i'}$ would be the only possible chords in the cycle formed by edges from $E(\pi) \cup \{xv_{i+1}, xv_j\}$ in the chordal graph G). Therefore $|i-j| > 1$ implies $Q_i \cap Q_j = \emptyset$ (because if $x \in Q_i \cap Q_j$, then $\{x, v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ would induce a gem in G [contradicting that G is distance-hereditary]). Thus $Q_1, \ldots, Q_n$ would be an induced path in $\mathcal{H}^*$ [contradicting $\mathcal{H}^* \in \langle K(G), 1, n\rangle$].

To show (ii), suppose v and w are nonadjacent vertices of H and suppose $\Pi = Q_1, \ldots, Q_l$ is an induced path in $\mathcal{H}$ (so $|i-j| = 1$ implies $|Q_i \cap Q_j| \geq p$ and $|i-j| > 1$ implies $|Q_i \cap Q_j| < p$) with $v \in Q_1 - Q_2$ and $w \in Q_l - Q_{l-1}$. By Theorem 1, Π is a subpath of some clique tree for G. Therefore if $i < j < k$ and $x \in Q_i \cap Q_k$, then $x \in Q_j$. Hence, for each $1 \leq i \leq l-1$, it is possible to pick distinct vertices $x_{(i,1)}, \ldots, x_{(i,p)} \in Q_i \cap Q_{i+1}$ such that $x_{(i,j)} \neq x_{(i',j')}$ whenever $j \neq j'$. (It is possible that $x_{(i,j)} = x_{(i',j)}$ with $i \neq i'$). For each $1 \leq j \leq p$, each set $\{v, x_{(1,j)}, \ldots, x_{(l,j)}, w\}$ will then contain the vertices of a v-to-w path π_j in H with $2 \leq |E(\pi_j)| \leq l$ such that π_j and $\pi_{j'}$ are internally disjoint whenever $j \neq j'$.

To show (iii), suppose that H is a proper induced subgraph of $H' \in \langle G, p, n+1\rangle$ [arguing by contradiction]. Specifically, suppose there exists a $v \in V(H') - V(H)$. Note that $H \not\subseteq N(v)$ in H', by the maximality of H' from being in $\langle G, p, n+1\rangle$. Thus there exists a $w \in V(H)$ with $v \not\sim w$. Let $\pi_1, \ldots, \pi_p$ be internally-disjoint induced v-to-w paths in H', and let u_i be the neighbor of v along each π_i. Whenever $u_i \neq u_j$, the edge u_iu_j must be a chord of the cycle $E(\pi_i) \cup E(\pi_j)$ (because G is chordal and π_i and π_j are induced paths). Thus $\{u_1, \ldots, u_p, v\}$ will induce a complete subgraph of G. Let Q' be a maxclique of H'—and so a node of $\mathcal{H}'$—that contains $\{u_1, \ldots, u_p, v\}$. Note that $v \notin V(H)$ implies that Q' is not a node of $\mathcal{H}$. Since there also exists a maxclique of H—and so a node of $\mathcal{H}$—that contains $\{u_1, \ldots, u_p\}$ (but not v), $\mathcal{H}^{+Q'}$ will also be a connected subgraph of $K_p(G)$). The maximality of $\mathcal{H}$ from being in $\langle K_p(G), 1, n\rangle$ implies that $\mathcal{H}^{+Q'}$ is not P_n-free, and so there must be an induced path $Q_1, Q_2, \ldots, Q_n$ of $K_p(G)$ in $\mathcal{H}^{+Q_1}$ with $Q' = Q_i$ where $1 \leq i \leq n$. By Lemma 2, there would then exist an induced path $v_0, v_1, v_2, \ldots, v_n$ in H' [contradicting $H' \in \langle G, p, n+1\rangle$].

To finish the proof, suppose now that $H \in \langle G, p, n+1\rangle$ [toward showing H is represented by some $\mathcal{H} \in \langle K_p(G), 1, n\rangle$]. By Lemma 3, H is the union of maxcliques of G. Let $\mathcal{H}^+$ be the connected subgraph of $K(G)$ that is induced by the nodes that correspond to those maxcliques of G whose union is H—so $\mathcal{H}^+$ represents H. Let $\mathcal{H}$ be the subgraph of $K_p(G)$ induced by the nodes of $\mathcal{H}^+$. Then $\mathcal{H}$ also represents H and is connected in $K_p(G)$ (since every two nonadjacent vertices of H are connected by p internally-disjoint paths in H). To show that $\mathcal{H}$ is P_n-free, suppose instead that $Q_1, \ldots, Q_n$ is an induced path in $\mathcal{H}$ [arguing by contradiction]. By Lemma 2, there would exist an induced path $v_0, v_1, \ldots, v_n$ in H [contradicting $H \in \langle G, 1, n+1\rangle$]. The maximality of H from being in $\langle G, p, n+1\rangle$ implies the maximality of $\mathcal{H}$ that ensures $\mathcal{H} \in \langle K_p(G), 1, n\rangle$. ■

Corollary 5. *If G is ptolemaic, then the maxcliques of $K(G)$ represent precisely the subgraphs of G that are in $\mathcal{CC}(G)$.*

Proof. This is the $p = 1, n = 3$ case of Theorem 4. ■

For the graph G in Figure 1 for instance, $\mathcal{CC}(G)$ has exactly four members, induced by the vertex sets $\{a, b, c, d, e, g, h\}$ and $\{d, e, f, g, h, j, k\}$ (represented by the two K_3 maxcliques of $K(G)$), $\{g, h, i, j, k, l, m\}$ (represented by the K_4 maxclique of $K(G)$), and $\{j, k, m, n\}$ (represented by the K_2 maxclique of $K(G)$).

Ptolemaic graphs are not characterized by Corollary 5, as shown by taking G to be the non-ptolemaic graph formed by the union of the length-10 cycle $v_1, v_2, \ldots, v_{10}, v_1$ and the length-5 cycle $v_1, v_3, v_5, v_7, v_9, v_1$. We leave as an open question how this might be modified into an actual characterization.

As another consequence of the $p = 1$ case of Theorem 4, *the clique graph $K(G)$ of a connected ptolemaic graph G is complete*—equivalently, $K(G)$ is P_3-free—*if and only if G is P_4-free.* Such P_4-free ptolemaic (equivalently, P_4-free chordal) graphs have been well-studied under various names in the literature, including 'trivially perfect,' 'nested interval,' 'hereditary upper bound,' and 'quasi-threshold' graphs; see [6, section 7.9].

For any graph G, the *diameter* of G, denoted $\operatorname{diam} G$, is the maximum distance between vertices in G. If G is distance-hereditary, then $\operatorname{diam} G \leq k$ if and only if G is P_{k+2}-free. (The equivalence fails for graphs that are not distance-hereditary; for instance, $\operatorname{diam} C_5 = 2$ and yet C_5 contains induced P_4 subgraphs.) Reference [1] shows that G is ptolemaic if and only if $K(G)$ is ptolemaic. Using that, the following would be another consequence of the $p = 1$ case of Theorem 4: *A ptolemaic graph G always satisfies* $\operatorname{diam} K(G) = \operatorname{diam} G - 1$. (This is also a special case of the following much more general result from [1, 2, 8], in which $K^1(G) = K(G)$ and $K^i(G) = K(K^{i-1}(G))$ when $i \geq 2$: *A chordal graph G always satisfies* $\operatorname{diam} K^i(G) = \operatorname{diam} G - i$ *whenever* $i \leq \operatorname{diam} G$.)

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