CLIQUE GRAPH REPRESENTATIONS
OF PTOLEMAIC GRAPHS

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Abstract

A graph is ptolemaic if and only if it is both chordal and distance-hereditary. Thus, a ptolemaic graph $G$ has two kinds of intersection graph representations: one from being chordal, and the other from being distance-hereditary. The first of these, called a clique tree representation, is easily generated from the clique graph of $G$ (the intersection graph of the maximal complete subgraphs of $G$). The second intersection graph representation can also be generated from the clique graph, as a very special case of the main result: The maximal $P_n$-free connected induced subgraphs of the $p$-clique graph of a ptolemaic graph $G$ correspond in a natural way to the maximal $P_{n+1}$-free induced subgraphs of $G$ in which every two nonadjacent vertices are connected by at least $p$ internally disjoint paths.

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1. Basic Concepts

For any graph $G$, denote the family of all maxcliques of $G$—meaning the inclusion-maximal complete subgraphs of $G$—as $\mathcal{C}(G)$, and denote the family of all inclusion-maximal induced connected subgraphs of $G$ that are cographs—meaning they contain no induced path of length three—as $\mathcal{CC}(G)$. 
(Of course \( \mathcal{C}(G) \) can be equivalently described as the family of all maximal induced subgraphs of \( G \) that contain no induced path of length two.)

Let \( \Omega(\mathcal{C}(G)) \) [respectively, \( \Omega(\mathcal{CC}(G)) \)] denote the *clique intersection graph* [or the *CC intersection graph*] of \( G \), meaning the intersection graph that has the members of \( \mathcal{C}(G) \) [or \( \mathcal{CC}(G) \)] as nodes, with two nodes adjacent if and only if their vertex sets have nonempty intersection. Let \( \Omega^w(\mathcal{C}(G)) \) and \( \Omega^w(\mathcal{CC}(G)) \) denote their weighted counterparts where, for \( S, S' \) in \( \mathcal{C}(G) \) or in \( \mathcal{CC}(G) \), the weight of the edge \( SS' \) equals \( |V(S) \cap V(S')| \). Figure 1 shows an example.

For each \( p \geq 1 \), the *\( p \)-clique graph* \( K_p(G) \) of \( G \) is the graph that has the maxcliques of \( G \) as nodes, with two nodes \( Q \) and \( Q' \) adjacent in \( K_p(G) \) if and only if \( |V(Q) \cap V(Q')| \geq p \); see [6, section 6.1]. In other words, \( K_p(G) \) is the graph that is formed by the edges of \( \Omega^w(\mathcal{C}(G)) \) that have weight \( p \) or more. The *clique graph* of \( G \) is \( K_1(G) \), typically abbreviated as \( K(G) \); see [9]. For instance, \( K(G) \) for the graph \( G \) in Figure 1 is \( \Omega^w(\mathcal{C}(G)) \) without the edge weights; Figure 2 shows \( K_2(G) \) for the same \( G \).

A graph is *chordal* if every cycle of length four or more has a *chord* (meaning an edge that joins two vertices of the cycle that are not consecutive along the cycle). Among many characterizations in [3, 6], a graph \( G \) is
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Figure 2. The 2-clique graph \( K_2(G) \) of the graph \( G \) in Figure 1.

A set \( S \subset V(G) \) is a vertex separator of a graph \( G \) if there are vertices \( v, w \) that are in a common component of \( G \) but different components of the subgraph induced by \( V(G) - S \); such an \( S \) is also called a \( v, w \)-separator. If \( G \) is chordal with a clique tree \( T \), then the inclusion-minimal vertex separators of \( G \) correspond exactly to \( V(Q) \setminus V(Q') \) where \( QQ' \) is an edge \( T \); see [6, section 2.1] for details.

A graph \( G \) is distance-hereditary if the distance between vertices in a connected induced subgraph of \( G \) always equals their distance in \( G \). Equivalently, \( G \) is distance-hereditary if and only if, for every \( v, w \in V(G) \), all the induced \( v \)-to-\( w \) paths in \( G \) have the same length; see [3]. A graph \( G \) is distance-hereditary if and only if \( G \) has a CC tree \( T \), where \( T \) is a spanning tree of \( (\mathcal{C}(G)) \) such that each subgraph \( T_v \) (defined the same as for clique
trees) is a subtree of $T$. Again, $\mathcal{CC}$ trees are the maximum spanning trees of $\Omega^w(\mathcal{CC}(G))$; see [3, 5, 7] for details of all this. The graph shown in Figure 1 is distance-hereditary, and Figure 3 shows its $\mathcal{CC}$ tree (which is unique in this example).

Let $P_n$ and $C_n$ denote, respectively, a path or cycle with $n$ vertices. Let $v \sim w$ denote that vertices $v$ and $w$ are adjacent, and let $N(v) = \{ x : v \sim x \in V(G) \}$. Define a gem to be a graph that consists of a cycle of length five together with two chords with a common endpoint. For any graph $H$, a graph $G$ is said to be $H$-free if $G$ has no induced subgraph isomorphic to $H$. For any graph $G$ with induced subgraph $H$ and vertex $v \in V(G) - V(H)$, let $H^+v$ denote the subgraph of $G$ induced by $V(H) \cup \{ v \}$.

A graph is ptolemaic if it is both chordal and distance-hereditary; see [3, 4] for history and details. Being ptolemaic is equivalent to being both gem-free and chordal, and also to being both $C_4$-free and distance-hereditary. Ptolemaic graphs therefore have two kinds of tree representations: both a clique tree because of being chordal, and a $\mathcal{CC}$ tree because of being distance-hereditary. Corollary 5 will show how the clique graph of a ptolemaic graph $G$ also determines $\mathcal{CC}(G)$ and thereby the $\mathcal{CC}$ trees of $G$. But first, Theorem 1 will further characterize ptolemaic graphs and Theorem 4 will show how subgraphs of a ptolemaic graph $G$ can be identified in the clique graph of $G$.

**Theorem 1.** Each of the following is equivalent to a chordal graph $G$ being ptolemaic:

1. \( \text{(1.1)} \) Every edge in $K(G)$ is contained in some clique tree for $G$.
2. \( \text{(1.2)} \) For every $p \geq 1$, every induced path in $K_p(G)$ is contained in some clique tree for $G$.

**Proof.** From [4, Theorem 2.4], a graph $G$ is ptolemaic if and only if every nonempty intersection of two maxcliques of $G$ is an inclusion-minimal vertex separator of $G$. Recalling that the inclusion-minimal vertex separators of $G$ correspond exactly to the edges of a clique tree for $G$, and that every maximum spanning tree of $\Omega^w(\mathcal{CC}(G))$ is a clique tree for $G$, it follows that being ptolemaic is equivalent to condition (1.1). Also, the $p = 1$ case of (1.2) implies (1.1).

To finish the proof, suppose $G$ is ptolemaic, condition (1.1) holds, and $p \geq 1$ [toward proving condition (1.2)]. Let $\mathcal{C}(G) = \{ Q_1, \ldots, Q_c \}$ where $\Pi = Q_1, \ldots, Q_n$ is an induced path in $K_p(G)$, and let

$$\mu = \max\{|V(Q_i) \cap V(Q_j)| : 1 \leq i < j \leq n\}.$$
Using (1.1), let $T_2$ be a clique tree that contains the edge $Q_1Q_2$. Each node $Q_i$ of $T_2$—equivalently, each maxclique $Q_i$ of $G$—with $i \notin \{1, 2\}$ will contain a vertex $v_i \notin V(Q_1) \cup V(Q_2)$. Form a new graph $G_2$ from $G$ by creating a set $S_2$ of $\mu - |V(Q_1) \cap V(Q_2)| + 1$ new vertices that are adjacent precisely to each other and to the vertices in $Q_1 \cup Q_2$. For $i \in \{1, 2\}$, let $Q_i^2$ be the subgraph induced by $V(Q_i) \cup S_2$ in $G_2$; for $i \notin \{1, 2\}$, let $Q_i^2 = Q_i$. The maxcliques of $G_2$ will be precisely $Q_1^2, Q_2^2$ (since each $N(v_i) \cap S_2 = \emptyset$).

To show that $G_2$ is chordal, suppose $C$ were a chordless cycle of $G_2$ with length four or more such that $C$ contained a vertex $s \in S_2$ [arguing by contradiction]. Then $C$ would consist of edges $sq_1$ and $sq_2$ with $q_1 \in V(Q_1) - V(Q_2)$ and $q_2 \in V(Q_2) - V(Q_1)$, together with an induced $q_1$-to-$q_2$ path $\pi$ within $G$. Because $Q_1Q_2$ is an edge of the clique tree $T_2$, the set $V(Q_1) \cap V(Q_2)$ will be a $q_1,q_2$-separator, and so the path $\pi$ must contain an internal vertex $w \in V(Q_1) \cap V(Q_2)$, making $w \sim s$ [contradicting that $C$ was chordless].

To show that $G_2$ is gem-free, suppose $\{a, b, c, d, e, a\}$ induced a gem in $G_2$ [arguing by contradiction], where $a, b, c, d, e, a$ is a cycle that has exactly the two chords $be$ and $ce$. If $a \in S_2$, then $a, b, e \in V(Q_1^2)$ where $i \in \{1, 2\}$ and $c, d \notin V(Q_i^2)$; then there would exist a $v \in V(Q_i)$ with $c \not\sim v \not\sim d$, which would make $\{v, b, c, d, e\}$ induce a gem in $G$ [contradicting that $G$ is ptolemaic]. The case $d \in S_2$ is similar. If $b \in S_2$, then (without loss of generality) vertex $a$ is in $Q_1 - Q_2$, vertex $c$ is in $Q_2 - Q_1$, vertex $d$ is not in $Q_1 \cup Q_2$, and vertices $b$ and $e$ are in $Q_1 \cap Q_2$; then there would exist a $v \in V(Q_1) \cap V(Q_2) - \{e\}$, which would make $\{a, v, c, d, e\}$ induce a gem in $G$ [contradicting that $G$ is ptolemaic]. The case $c \in S_2$ is similar. Note that $e \notin S_2$, since $e$ is in at least three maxcliques of $G$.

Therefore, $G_2$ is ptolemaic.

Repeat the $G_2$ construction to form new ptolemaic graphs $G_i$—from $G_{i-1}$ using $\mu - |V(Q_{i-1}) \cap V(Q_i)| + 1$ new vertices adjacent precisely to each other and to the vertices in $V(Q_{i-1}) \cup V(Q_i)$—whenever $3 \le i \le n$. The final ptolemaic graph $G_n$ will have maxcliques $Q_1^n, \ldots, Q_c^n$ that contain $Q_1, \ldots, Q_c$ respectively, where $Q_1^n, \ldots, Q_c^n$ forms an induced path $\Pi_n$ of maximum-weight edges of $K_{\mu+1}(G_n)$. Let $T_n$ be a maximum spanning tree of $\Omega^\gamma(C(G_n))$ that contains $\Pi_n$. This $T_n$ will be a clique tree for $G_n$ and, by suppressing all the vertices in $V(G_n) - V(G)$, this $T_n$ will correspond to a clique tree of $G$ that contains the edges of $\Pi$.

The following consequence of Theorem 1 will be used several times in Section 2.
Lemma 2. If $G$ is ptolemaic with $p \geq 1$ and $n \geq 2$ and if $Q_1, \ldots, Q_n$ is an induced path in $K_p(G)$, then there exist $v_0, \ldots, v_n \in V(G)$ such that $v_0 \in V(Q_j)$ exactly when $j = 1$, each $1 \leq i \leq n - 1$ has $e_i \in V(Q_j)$ exactly when $j \in \{i, i + 1\}$, and $v_n \in V(Q_j)$ exactly when $j = n$.

Proof. Suppose $G$ is ptolemaic with $p \geq 1$ and $n \geq 2$, suppose $\Pi = Q_1, \ldots, Q_n$ is an induced path in $K_p(G)$ and, within this proof, identify each $Q_i$ with $V(Q_i)$. Therefore $|i - j| = 1$ implies $|Q_i \cap Q_j| \geq p$ and $|i - j| > 1$ implies $|Q_i \cap Q_j| < p$. The existence of the desired $v_0 \in Q_1$ and $v_n \in Q_n$ follows from $Q_1 \not\subseteq Q_2$ and $Q_n \not\subseteq Q_{n-1}$ (since maxcliques of any graph have incomparable vertex sets). The existence of the desired $v_1 \in Q_1 \cap Q_2 - Q_3$ follows from $Q_1 \cap Q_2 \not\subseteq Q_2 \cap Q_3$ (since $|Q_1 \cap Q_3| < p$); the existence of $v_{n-1} \in Q_n \cap Q_{n-1} - Q_{n-2}$ follows similarly.

Suppose $1 < i < n - 1$ [toward showing the existence of $v_i \in (Q_i \cap Q_{i+1}) - (Q_{i-1} \cup Q_{i+2})$]. Suppose instead that $Q_i \cap Q_{i+1} \subseteq Q_{i-1} \cup Q_{i+2}$ [arguing by contradiction]. By Theorem 1, $\Pi$ is a path in some clique tree $T$ for $G$. Because $\Pi$ is an induced path, the three cardinality-$p$ sets $Q_{i-1} \cap Q_i$, $Q_i \cap Q_{i+1}$, and $Q_{i+1} \cap Q_{i+2}$ are pairwise unequal, and so there exist $v \in Q_{i-1} \cap Q_i + Q_{i+1} - Q_{i-1}$ and $w \in Q_i \cap Q_{i+1} - Q_{i+2}$ (and so $w \in Q_{i-1}$, since $Q_i \cap Q_{i+1} \subseteq Q_{i-1} \cup Q_{i+2}$). There would also exist $t \in Q_{i-1} - Q_i$, $u \in Q_{i-1} \cap Q_i - Q_{i+1}$, and $x \in Q_{i+1} \cap Q_{i+2} - Q_i$ (just as for the $i = 0, 1, n - 1$ cases, respectively, but now for the path $Q_{i-1}, Q_i, Q_{i+1}, Q_{i+2}$). So $\{t, u, w\}$, $\{u, v, w\}$, and $\{v, w, x\}$ would induce triangles (inside $Q_{i-1}$, $Q_i$ and $Q_{i+1}$ respectively), and $u \not\sim x \not\sim t \not\sim v$ (for instance, $u \not\sim x$ since $u$ and $x$ are not in a common maxclique, using that $T$ is a clique tree for $G$). But then $\{t, u, v, w, x\}$ would induce a gem in $G$ [contradicting that $G$ is ptolemaic].

2. Representing Subgraphs of $G$ within $K(G)$

For each $p \geq 1$ and $n \geq 2$, let $\langle G, p, n \rangle$ denote the family of all induced subgraphs of $G$ that are maximal with respect to both being $P_n$-free and having every two nonadjacent vertices connected by at least $p$ internally-disjoint paths (such paths form what is sometimes called a $p$-skein). That second condition is equivalent to the subgraph being either $p$-connected or complete. For example, $\langle G, 1, 2 \rangle = V(G)$, $\langle G, 1, 3 \rangle = C(G)$, and $\langle G, 1, 4 \rangle = CC(G)$, while $\langle G, 2, 4 \rangle$ consists of the 2-connected members of $CC(G)$ together with any bridges (edges that are not in cycles) and isolated vertices.
If $H$ is a connected induced subgraph of $K(G)$ and $H$ is a connected induced subgraph of $G$, then say that $H$ represents $H$ in $G$ if $H$ is induced by the vertices that are in the union of the maxcliques of $G$ that correspond to the nodes of $H$. In Figure 1 for instance, the path $H$ of $K(G)$ induced by the nodes $ghjk$, $jkm$, and $mn$ represents the subgraph $H$ of $G$ that is induced by $\{g, h, j, k, m, n\}$. Every connected induced subgraph $H$ of $K(G)$ clearly represents a connected induced subgraph $H$ of $G$ with $H \cong K(H)$, but not conversely: for instance, $V(H) = \{e, h, k\}$ is not even a union of maxcliques of $G$.

Given a family $\text{FAM}_{K(G)}$ of connected induced subgraphs of $K(G)$ and a family $\text{FAM}_G$ of connected induced subgraphs of $G$, say that the members of $\text{FAM}_{K(G)}$ represent precisely the members of $\text{FAM}_G$ if every $H \in \text{FAM}_{K(G)}$ represents an $H \in \text{FAM}_G$ and every $H \in \text{FAM}_G$ is represented by some $H \in \text{FAM}_{K(G)}$. For instance, the nodes of $K(G)$ always represent precisely the maxcliques of $G$.

Theorem 4 will look at certain kinds of subgraphs of the clique graph of a ptolemaic graph $G$ and at the kinds of subgraphs of $G$ that they represent. For instance, Corollary 5 will show that the maxcliques of the clique graph of a ptolemaic graph $G$ represent precisely the members of $\mathcal{C}(G)$. Theorem 4 will use the following lemma.

**Lemma 3.** If $G$ is ptolemaic with $p \geq 1$ and $n \geq 2$ and if $H \in \langle G, p, n \rangle$, then $\mathcal{C}(H) \subseteq \mathcal{C}(G)$.

**Proof.** Suppose $G$ is ptolemaic (and so chordal and distance-hereditary) with $p \geq 1$ and $n \geq 2$, and suppose $H \in \langle G, p, n \rangle$ and $Q \in \mathcal{C}(H) - \mathcal{C}(G)$ [arguing by contradiction]: so there exists $v \in V(G) - V(H)$ with $Q \subseteq N(v)$. The maximality of $H$ from being in $\langle G, p, n \rangle$ implies that $H^+v \notin \langle G, p, n \rangle$, and so $H$ must be $p$-connected (as opposed to $H$ being complete with $|V(H)| = |V(Q)| \leq p$). Also, $H$ must be chordal (since $G$ is), and so $H$ will have a clique tree $T$. Since $H$ is $p$-connected, every edge $Q_iQ_j$ of $T$ will have $|V(Q_i) \cap V(Q_j)| \geq p$ (since $V(Q_i) \cap V(Q_j)$ will be a minimal vertex separator in $G$), and so every node $Q_1$ of $T$ will have $|V(Q_i)| \geq p$. In particular, $|V(Q)| \geq p$, which makes $H^+v$ also $p$-connected. Hence, $H^+v \notin \langle G, p, n \rangle$ implies that there must exist an induced path $\pi = v_1, \ldots, v_n$ in $G$ that has $v \in V(\pi) \subseteq V(H) \cup \{v\}$.

Vertex $v$ cannot be an interior vertex of $\pi$—otherwise $\{v_1, v_n\} \subseteq V(H)$ and $H \in \langle G, p, n \rangle$ would imply there is an induced $v_1$-to-$v_n$ path within $H$ shorter than $\pi$ [contradicting that $G$ is distance-hereditary].
Without loss of generality, say $v = v_1$ and suppose for the moment that $v_2 \notin V(Q)$. Note that $v_1 \notin V(Q)$ for $i \geq 3$ (since $V(Q) \subset N(v)$ and $\pi$ induced implies such $v_1$ not adjacent to $v_1$). Because $v_2 \notin V(Q)$ and $Q$ is a maxclique of $G$, there is a $q \in V(Q)$ such that $v_2 \sim q \sim v_1$. Note that $v_i \sim q$ for $i \geq 3$ [otherwise some $q, v_1, \ldots, v_i, q$ would be an induced cycle in $G$ with length $i + 1 \geq 4$, contradicting that $G$ is chordal]. So $q, v_1, \ldots, v_n$ is an induced $q$-to-$v_n$-path of length $n$ in $G$. But $H \in \langle G, p, n \rangle$ would imply there is an induced $q$-to-$v_n$-path within $H$ of length less than $n$ [again contradicting that $G$ is distance-hereditary].

Thus $v_2 \in V(Q)$. As before, $v_i \notin V(Q)$ for $i \geq 3$. Because $v_3 \notin V(Q)$, there is a $q \in V(Q)$ such that $v_3 \sim q \sim v_2$. Note that $v_i \sim q$ when $i \geq 4$ (otherwise some $q, v_2, \ldots, v_i, q$ would be an induced cycle in $G$ with length $n = i \geq 4$). But then $q, v_2, \ldots, v_n$ would form an induced $P_n$ in $H$ [contradicting $H \in \langle G, p, n \rangle$].

**Theorem 4.** If $G$ is ptolemaic with $p \geq 1$ and $n \geq 2$, then the subgraphs of $K(G)$ in $\langle K_p(G), 1, n \rangle$ represent precisely the subgraphs of $G$ in $\langle G, p, n+1 \rangle$.

Before proving Theorem 4, it will be helpful to illustrate it using Figure 1 and Figure 2: When $p = 2$ and $n = 3$, the six $\mathcal{H} \in \langle K_2(G), 1, 3 \rangle$—these are the six maxcliques of $K_2(G)$—represent the six subgraphs of $G$ that are induced by \{a, b, c, d, e, g, h\}, \{g, h, j, k, l, m\}, \{d, e, g, h, j, k\}, \{f, g\}, \{i, j\}, and \{m, n\}, and these are precisely the six subgraphs $H \in \langle G, 2, 4 \rangle$. When $p = 2$ and $n = 4$, the subgraph $\mathcal{H} \in \langle K_2(G), 1, 4 \rangle$ that is induced by the four nodes $\text{degh}$, $\text{ghjk}$, $\text{jkl}$, and $\text{jkm}$ represents the subgraph $H \in \langle G, 2, 5 \rangle$ that is induced by $\{d, e, g, h, j, k, l, m\}$. When $p = 3$ and $n = 3$, the maxclique $\mathcal{H} \in \langle K_3(G), 1, 3 \rangle$ that is formed by the edge between $\text{abde}$ and $\text{acde}$ represents the subgraph $H \in \langle G, 3, 4 \rangle$ that is induced by $\{a, b, c, d, e\}$.

**Proof.** Suppose $G$ is ptolemaic (and so chordal and distance-hereditary) with $p \geq 1$ and $n \geq 2$.

First suppose $\mathcal{H} \in \langle K_p(G), 1, n \rangle$ and $\mathcal{H}$ represents a subgraph $H$ of $G$. To show $H \in \langle G, p, n+1 \rangle$ requires showing three things: (i) that $H$ is $P_{n+1}$-free, (ii) that every two nonadjacent vertices of $H$ are connected by at least $p$ internally-disjoint paths of $H$, and (iii) the maximality of $H$ with respect to (i) and (ii). Within this proof, identify each maxclique $Q$ with $V(Q)$.

To show (i), suppose instead that $\pi = v_1, v_2, \ldots, v_{n+1}$ is an induced path in $H$ [arguing by contradiction]. Observe that $\mathcal{H} \in \langle K_p(G), 1, n \rangle$ is a subgraph of an induced subgraph $\mathcal{H}^* \in \langle K(G), 1, n \rangle$ on the same node-set as
For each $i \in \{1, 2, \ldots, n-1\}$, let $Q_i$ be a maxclique of $G$ that is a node of $\mathcal{H}^*$ such that $Q_i \cap V(\pi) = \{v_i, v_{i+1}\}$. Note that $|i-j| = 1$ implies $Q_i \cap Q_j \neq \emptyset$ (because $v_i \in Q_i \cap Q_{i+1}$). If $|i-j| > 1$ and $x \in Q_i \cap Q_j$, then $x$ will be adjacent to $v_i$ and $v_{i+1}$ (because $x \in Q_i$), to $v_j$ and $v_{j+1}$ (because $x \in Q_j$), and to every $v_{i'}$ with $i + 1 < i' < j$ (since $\pi$ being induced implies that such $x_{i,j}$ would be the only possible chords in the cycle formed by edges from $E(\pi) \cup \{x_{i+1,j}, x_{j+1,i}\}$ in the chordal graph $G$). Therefore $|i-j| > 1$ implies $Q_i \cap Q_j = \emptyset$ (because if $x \in Q_i \cap Q_j$, then $\{x, v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ would induce a gem in $G$ [contradicting that $G$ is distance-hereditary]). Thus $Q_1, \ldots, Q_n$ would be an induced path in $\mathcal{H}^*$ [contradicting $\mathcal{H}^* \not\in \langle K(G), 1, n \rangle$].

To show (ii), suppose $v$ and $w$ are nonadjacent vertices of $H$ and suppose $\Pi = Q_1, \ldots, Q_t$ is an induced path in $\mathcal{H}$ (so $|i-j| = 1$ implies $|Q_i \cap Q_j| \geq p$ and $|i-j| > 1$ implies $|Q_i \cap Q_j| < p$) with $v \in Q_1 - Q_2$ and $w \in Q_t - Q_{t-1}$. By Theorem 1, $\Pi$ is a subpath of some clique tree for $G$. Therefore if $i < j < k$ and $x \in Q_i \cap Q_k$, then $x \in Q_j$. Hence, for each $1 \leq i \leq t - 1$, it is possible to pick distinct vertices $x_{(i,1), \ldots, x_{(i,p)}} \in Q_i \cap Q_{i+1}$ such that $x_{(i,j)} \neq x_{(i',j')}$ whenever $j \neq j'$. (It is possible that $x_{(i,j)} = x_{(i',j')}$. Thus for $1 \leq j \leq p$, each set $\{v, x_{(1,j), \ldots, x_{(t,j), w}}\}$ will then contain the vertices of a $v$-to-$w$ path $\pi_j$ in $H$ with $2 \leq |E(\pi_j)| \leq l$ such that $\pi_j$ and $\pi_{j'}$ are internally disjoint whenever $j \neq j'$.)

To show (iii), suppose that $H$ is a proper induced subgraph of $H' \in \langle G, p, n + 1 \rangle$ [arguing by contradiction]. Specifically, suppose there exists a $v \in V(H') - V(H)$. Note that $H \not\subseteq N(v)$ in $H'$, by the maximality of $H'$ from being in $\langle G, p, n + 1 \rangle$. Thus there exists a $w \in V(H)$ with $v \not\sim w$. Let $\pi_1, \ldots, \pi_p$ be internally-disjoint induced $v$-to-$w$ paths in $H'$, and let $u_i$ be the neighbor of $v$ along each $\pi_i$. Whenever $u_i \neq u_j$, the edge $u_i u_j$ must be a chord of the cycle $E(\pi_i) \cup E(\pi_j)$ (because $G$ is chordal and $\pi_i$ and $\pi_j$ are induced paths). Thus $\{u_1, \ldots, u_p\}$ will induce a complete subgraph of $G$. Let $Q'$ be a maxclique of $H'$ and so a node of $\mathcal{H}'$—that contains $\{u_1, \ldots, u_p, v\}$. Note that $v \not\in V(H)$ implies that $Q'$ is not a node of $\mathcal{H}$. Since there also exists a maxclique of $H$ and so a node of $\mathcal{H}$—that contains $\{u_1, \ldots, u_p\}$ (but not $v$), $\mathcal{H}^+Q'$ will also be a connected subgraph of $K_p(G)$. The maximality of $\mathcal{H}$ from being in $\langle K_p(G), 1, n \rangle$ implies that $\mathcal{H}^+Q'$ is not $P_p$-free, and so there must be an induced path $Q_1, Q_2, \ldots, Q_n$ of $K_p(G)$ in $\mathcal{H}^+Q'$ with $Q' = Q_i$ where $1 \leq i \leq n$. By Lemma 2, there would then exist an induced path $v_0, v_1, v_2, \ldots, v_n$ in $H'$ [contradicting $H' \not\in \langle G, p, n + 1 \rangle$].
of maxcliques of $G$. Let $\mathcal{H}^+$ be the connected subgraph of $K(G)$ that is induced by the nodes that correspond to those maxcliques of $G$ whose union is $H$—so $\mathcal{H}^+$ represents $H$. Let $\mathcal{H}$ be the subgraph of $K_p(G)$ induced by the nodes of $\mathcal{H}^+$. Then $\mathcal{H}$ also represents $H$ and is connected in $K_p(G)$ (since every two nonadjacent vertices of $H$ are connected by $p$ internally-disjoint paths in $H$). To show that $\mathcal{H}$ is $P_n$-free, suppose instead that $Q_1, \ldots, Q_n$ is an induced path in $\mathcal{H}$ (arguing by contradiction). By Lemma 2, there would exist an induced path $v_0, v_1, \ldots, v_n$ in $H$ [contradicting $H \notin \langle \langle G, 1, n + 1 \rangle \rangle$]. The maximality of $H$ from being in $\langle \langle G, p, n + 1 \rangle \rangle$ implies the maximality of $\mathcal{H}$ that ensures $\mathcal{H} \in \langle \langle K_p(G), 1, n \rangle \rangle$.

**Corollary 5.** If $G$ is ptolemaic, then the maxcliques of $K(G)$ represent precisely the subgraphs of $G$ that are in $\mathcal{CC}(G)$.

**Proof.** This is the $p = 1, n = 3$ case of Theorem 4. For the graph $G$ in Figure 1 for instance, $\mathcal{CC}(G)$ has exactly four members, induced by the vertex sets $\{a, b, c, d, e, g, h\}$ and $\{d, e, f, g, h, j, k\}$ (represented by the two $K_3$ maxcliques of $K(G)$), $\{g, h, i, j, k, l, m\}$ (represented by the $K_4$ maxclique of $K(G)$), and $\{j, k, m, n\}$ (represented by the $K_2$ maxclique of $K(G)$).

Ptolemaic graphs are not characterized by Corollary 5, as shown by taking $G$ to be the non-ptolemaic graph formed by the union of the length-10 cycle $v_1, v_2, \ldots, v_{10}, v_1$ and the length-5 cycle $v_1, v_3, v_5, v_7, v_9, v_1$. We leave as an open question how this might be modified into an actual characterization.

As another consequence of the $p = 1$ case of Theorem 4, the clique graph $K(G)$ of a connected ptolemaic graph $G$ is complete—equivalently, $K(G)$ is $P_4$-free—if and only if $G$ is $P_4$-free. Such $P_4$-free ptolemaic (equivalently, $P_4$-free chordal) graphs have been well-studied under various names in the literature, including ‘trivially perfect,’ ‘nested interval,’ ‘hereditary upper bound,’ and ‘quasi-threshold’ graphs; see [6, section 7.9].

For any graph $G$, the **diameter** of $G$, denoted $\text{diam } G$, is the maximum distance between vertices in $G$. If $G$ is distance-hereditary, then $\text{diam } G \leq k$ if and only if $G$ is $P_{k+2}$-free. (The equivalence fails for graphs that are not distance-hereditary; for instance, $\text{diam } C_5 = 2$ and yet $C_5$ contains induced $P_4$ subgraphs.) Reference [1] shows that $G$ is ptolemaic if and only if $K(G)$ is ptolemaic. Using that, the following would be another consequence of the $p = 1$ case of Theorem 4: A ptolemaic graph $G$ always satisfies $\text{diam } K(G) = \text{diam } G - 1$. (This is also a special case of the following much more general
result from [1, 2, 8], in which \( K^1(G) = K(G) \) and \( K^i(G) = K(K^{i-1}(G)) \) when \( i \geq 2 \): A chordal graph \( G \) always satisfies \( \text{diam} K^i(G) = \text{diam} G - i \) whenever \( i \leq \text{diam} G \).

**References**


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