A NOTE ON HAMILTONIAN CYCLES IN GENERALIZED HALIN GRAPHS

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Abstract
We show that every 2-connected (2)-Halin graph is Hamiltonian.

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1. Introduction
We generalize the well-known notion of a Halin graph in the following way. An \((n)\)-Halin graph is a planar simple graph having the property that its edge set \(E\) can be partitioned as \(E = T \cup C_1 \cup C_2 \cup \cdots \cup C_n\) where \(T\) is a tree with no vertices of degree two and \(C_1, C_2, \ldots, C_n\) are pairwise disjoint cycles such that \(V(C_1) \cup \cdots \cup V(C_n)\) is the set of all leaves of \(T\) (see Figure 1). Thus, \((1)\)-Halin graphs are the usual Halin graphs. It is well known that Halin graphs are Hamiltonian, even Hamiltonian connected (see, for instance, Barefoot [1]). In this note we show that each 2-connected \((2)\)-Halin graph is Hamiltonian.

Figure 1. An example of a \((2)\)-Halin graph.
Theorem 1. A (2)-Halin graph is Hamiltonian if and only if it is 2-connected.

Our proof relies on Lemma 2 below. By a rooted Halin graph we mean a planar graph $F$ which is the union of a rooted tree $T$, where the root of $T$ is a vertex of degree at least two and all other vertices, except the leaves, are of degree at least three, and a path $P = \ell_1 \ell_2 \cdots \ell_m$ whose vertices are precisely the leaves of $T$. The endvertices $\ell_1$ and $\ell_m$ of $P$ are called left and rights corners of $H$ respectively.

Lemma 2 [2]. Let $F$ be a rooted Halin graph and let $x$, $y$ be two different vertices from the set which consists of the root of $F$ and its two corners. Then $F$ contains a Hamiltonian path joining $x$ and $y$.

Proof of Theorem 1. Clearly, since each Hamiltonian graph is 2-connected, we need only to prove that 2-connectivity is a sufficient condition for a (2)-Halin graph to be Hamiltonian. Let $G$ be a (2)-Halin graph which decomposes into a tree $T$ and two cycles $C_1$ and $C_2$, and let $\hat{G}$ be an embedding of $G$ into the plane. Without loss of generality we may assume that in the embedding $\hat{G}$ the faces corresponding to $C_1$ and $C_2$ are both bounded. Let $x_1y_1 [x_2y_2]$ denote an edge of $C_1 [C_2]$ which belongs to the unbounded face, and let $P_x$ and $P_y$ denote the disjoint paths contained in $T$ which join vertices $x_1, x_2$ and $y_1, y_2$ respectively. Note that because $H$ is 2-connected $P_x$ and $P_y$ have to exist. Finally, let $P = v_1 v_2 \cdots v_n$, $n \geq 2$, be the unique path which joins the paths $P_x$ and $P_y$ in $T$.

Observe that if we remove $P_m$ from the tree $T$, it decomposes into a number of ‘rooted Halin trees’, attached to vertices of the cycles $C_1$ and $C_2$. Moreover, since $v_{n-1}$ has degree at least three, it must have a neighbor which does not lie on $P$; thus, without loss of generality, we may assume that it has a neighbor which is the root of a Halin tree attached to $C_1$. Now, using Lemma 2, we can define a Hamiltonian cycle $H$ in $G$ in the following way (see Figure 2). Start at the vertex $y_1$ and move to $v_1$, going through all vertices of the rooted Halin tree which contains $y_1$. Then go through $y_2$ and collect the vertices of all Halin rooted trees attached to $C_2$ up to $v_1$. Next, pass through the first $n-1$ vertices of $P$ and then visit all vertices of the remaining Halin rooted trees attached to $C_1$ up to $x_1$ and finally, go back to $y_1$. ■
Let us conclude the note with a few remarks. It is tempting to generalize the above result to \((n)\)-Halin graphs and conjecture that, say, a \((3)\)-Halin graph is Hamiltonian whenever it is 1-tough. Unfortunately, it is not the case; Figure 3 shows a 1-tough \((3)\)-Halin graph which, as one can easily check, contains no Hamiltonian cycle. Moreover, unlike \((1)\)-Halin graphs (which are always 3-connected), 3-connected \((2)\)-Halin graphs are not always Hamiltonian connected (see Figure 4). Finally, we remark that from the proof of Theorem 1 it follows that a Hamiltonian cycle in \((2)\)-Halin graph, if exists, can be found in polynomial time. It is not clear whether the same holds for \((3)\)-Halin graph, and more generally, if there exists \(k\) such that the problem of deciding hamiltonicity of \((k)\)-Halin graph is \(NP\)-complete.

Figure 2. A construction of a Hamiltonian cycle in \((2)\)-Halin graph.

Figure 3. An example of a non-Hamiltonian 1-tough \((3)\)-Halin graph.
Figure 4. An example of a 3-connected (2)-Halin graph, which is not Hamiltonian connected (there are no Hamiltonian path between $v$ and $w$).

References


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