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VERTEX COLORING THE SQUARE OF OUTERPLANAR GRAPHS OF LOW DEGREE

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Abstract

Vertex colorings of the square of an outerplanar graph have received a lot of attention recently. In this article we prove that the chromatic number of the square of an outerplanar graph of maximum degree $\Delta = 6$ is 7. The optimal upper bound for the chromatic number of the square of an outerplanar graph of maximum degree $\Delta \neq 6$ is known. Hence, this mentioned chromatic number of 7 is the last and only unknown upper bound of the chromatic number in terms of Δ.

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1. Introduction

The square G^2 of a graph G is the graph on the same set of vertices as G, but where each pair of vertices of distance one or two in G is connected with an edge in G^2. The problem of coloring squares of planar graphs has seen much attention mainly for two reasons; firstly in relation to frequency allocation (this models the case when nodes represent both senders and receivers and two senders with a common neighbor will interfere if using the same frequency), and secondly because a conjecture of Wegner [1] dating from 1977 (see [2]) states that the square of every planar graph G of maximum degree $\Delta \geq 8$ has a chromatic number which does not exceed $3\Delta/2+1$. The conjecture matches the maximum clique number of these graphs. Currently the best upper bound known is $1.66\Delta + 78$ by Molloy and Salavatipour [3].

An earlier paper of the current authors [4] gave a bound of $\lceil 1.8\Delta \rceil$ for the chromatic number of squares of planar graph with large maximum degree $\Delta \geq 749$. This is based on bounding the *inductiveness* (or the *degeneracy*) of the graph, which is the maximum over all subgraphs H of the minimum degree of H. It was also shown there that this was the best possible bound on the inductiveness. Borodin *et al.* [5] showed that this bound holds for all $\Delta \geq 48$. Inductiveness has the additional advantage of also bounding the *choosability* or the *list-chromatic number* as well. In [6] the more general $L(p,q)$-labeling of the square of planar graph is studied for positive integers p and q and it is shown that $\lambda(G;p,q) \leq (4q-2)\Delta + 10p + 38q - 24$, where Δ is the maximum degree of G. This implies that $\chi(G^2) \leq 2\Delta + 25$ for any planar graph of maximum degree Δ. This is not an asymptotic improvement of the results mentioned above, but the setting is more general.

Wegner's conjecture is still open, but has been settled for many specific cases of planar graphs. In [7] Wegner's conjecture is proved in the case for K_4-free planar graphs. The case when G is an outerplanar graph has received particularly much attention on this coloring problem of its square. Calamoneri and Petreschi [8] gave a linear time algorithm to color squares of outerplanar graphs, as well as for related problems. They showed that it uses an optimal $\Delta + 1$ colors whenever $\Delta \geq 7$, and at most $\Delta + 2$ colors for $\Delta \geq 3$. This result also appears in the preliminary report [9] from the same year*. There the proof was based on induction and is not as algorithmic as in [8]. This result also appeared recently in [11]. In fact, some specific cases of this result have also appeared in the literature: in [12] the case where the

*A revised and complete version of this report [9] can be found at [10] in the `arXiv`.

outerplanar graph G is chordal is considered, and in [13] the larger upper bound $\chi(G^2) \leq \Delta + 2$ when $G \neq C_5$ is obtained.

That the mentioned bound $\chi(G^2) \leq \Delta + 1$ holds exactly when $\Delta \geq 7$ is not coincidental. The underlying reason for the condition $\Delta \geq 7$ is that in this case one can show that the square G^2 of an outerplanar graph G of maximum degree Δ always contains a vertex of degree at most Δ in G^2. This means that G^2 can be vertex colored in a greedy fashion using the optimal number $\Delta + 1$ colors. Put more precisely, it means that the inductiveness (or the degeneracy) of G^2 is exactly Δ in this case, and consequently the choosability (or the list chromatic number) of G^2 is exactly $\Delta + 1$ for each $\Delta \geq 7$. When $\Delta \leq 6$ this argument fails. That is, it is impossible to obtain the optimal upper bound for $\chi(G^2)$ from the inductiveness of G^2 for $\Delta \leq 6$. Hence, a different line of arguments is needed for this case.

The main purpose of this article is to show that for an outerplanar graph G with $\Delta = 6$ we have $\chi(G^2) = 7$. Note that by the mentioned result above we do have that $\chi(G^2) \leq \Delta + 2 = 8$. This is the last and only case not determined in any of the above mentioned articles. Such an improvement of the upper bound of a chromatic number by a mere one, is many times very difficult and it is ubiquitous when vertex coloring graphs. For example: each graph of maximum degree Δ can trivially be colored by $\Delta + 1$ colors in a greedy fashion, but if we exclude odd cycles and complete graphs then Brook's Theorem states that such a graph can be vertex colored by Δ colors [19, p. 197]. Also (and more notoriously!) that each planar graph is 5-colorable is not too hard to prove (Heawood 1890 [19, p. 257]), whereas that every planar graph is 4-colorable is the well-known Four Color Theorem, the computer-aided proof of which is extremely long and involved.

That $\chi(G^2) = 7$ for each outerplanar graph with $\Delta = 6$ (together with an observation for the case $\Delta = 5$) will then finally yield the tight upper bound for $\chi(G^2)$ where G is outerplanar of maximum degree Δ, for *every* value of $\Delta \in \mathbb{N}$. Therefore, this article will further contribute to the study of vertex colorings of outerplanar graphs of low maximum degree, something that on its own has received considerable attention. We mention some of these related but different coloring results of outerplanar graphs of low degree in this ongoing investigation: in [14] edge colorings are studied and in [15] the vertex-edge-face colorings are studied, both in the cases of $\Delta \leq 4$ respectively. In [16] it is shown that the complete chromatic number (vertex-edge-face chromatic number) of a chordal outerplanar graph with $\Delta = 6$ is

7, and in [17] and [18] the edge-face chromatic number is studied for $\Delta = 6$, in the latter it is shown that the edge-face chromatic number is 6 for $\Delta = 6$.

2. Definitions and Preliminaries

In this section we give some basic definitions and prove results that will be used later on.

Graph notation. The set $\{1, 2, 3, \ldots\}$ of natural numbers will be denoted by $\mathbb{N}$. Unless otherwise stated, a graph G will always be a *simple graph* $G = (V, E)$ where $V = V(G)$ is the finite set of vertices or nodes, and $E = E(G) \subseteq \binom{V(G)}{2}$ the set of edges of G. The edge between the vertices u and v will be denoted by uv (here uv and vu will mean the same undirected edge) rather than the 2-set $\{u, v\}$ that contains both u and v. By *coloring* we will always mean vertex coloring. We denote by $\chi(G)$ the chromatic number of G and by $\omega(G)$ the clique number of G. The degree of a vertex u in a graph G is denoted by $d_G(u)$. We let $\Delta(G)$ denote the maximum degree of a vertex in G. When there is no danger of ambiguity, we simply write Δ instead of $\Delta(G)$. We denote by $N_G(u)$ the open neighborhood of u in G, that is the set of all neighbors of u in G, and by $N_G[u]$ the closed neighborhood of u in G, that additionally includes u.

The square graph G^2 of a graph G is a graph on the same vertex set as G in which additionally to the edges of G, every two vertices with a common neighbor in G are also connected with an edge. Clearly this is the same as the graph on $V(G)$ in which each pair of vertices of distance 2 or less in G are connected by an edge.

Tree terminology. The *diameter* of a tree T is the number of edges in the longest simple path in T and will be denoted by $\mathrm{diam}(T)$. For a tree T with $\mathrm{diam}(T) \geq 1$ we can form the *pruned* tree $\mathrm{pr}(T)$ by removing all the leaves of T. A *center* of T is a vertex of distance at most $\lceil \mathrm{diam}(T)/2 \rceil$ from all other vertices of T. A center of T is either unique or one of two unique adjacent vertices. When T is rooted at $r \in V(T)$, the k-th *ancestor*, if it exists, of a vertex u is the vertex on the unique path from u to r of distance k from u. An *ancestor* of u is a k-th ancestor of u for some $k \geq 0$. Note that u is viewed as an ancestor of itself. The *parent* (*grandparent*) of a vertex is then the 1-st (2-nd) ancestor of the vertex. The *sibling* of a vertex is another child of its parent, and a *cousin* is child of a sibling of its parent.

The *height* of a rooted tree is the length of the longest path from the root to a leaf. The *height* of a vertex u in a rooted tree T is the height of the rooted subtree of T induced by all vertices with u as an ancestor.

Note that in a rooted tree T, vertices of height zero are the leaves (provided that the root is not a leaf). Vertices of height one are the parents of leaves, that is, the leaves of the pruned tree $\mathrm{pr}(T)$ and so on. In general, for $k \geq 0$ let $\mathrm{pr}^k(T)$ be given recursively by $\mathrm{pr}^0(T) = T$ and $\mathrm{pr}^k(T) = \mathrm{pr}(\mathrm{pr}^{k-1}(T))$. Clearly $V(T) \supset V(\mathrm{pr}(T)) \supset \cdots \supset V(\mathrm{pr}^k(T)) \supset \cdots$ is a strict inclusion. With this in mind we have an alternative "root-free" description of the height of vertices in a tree.

Observation 2.1. *Let T be a tree and $0 \leq k \leq \lfloor \mathrm{diam}(T)/2 \rfloor$. The vertices of height k in T are precisely the leaves of $\mathrm{pr}^k(T)$.*

Biconnectivity. The *blocks* of a graph G are the maximal biconnected subgraphs of G. A *cutvertex* is a vertex shared by two or more blocks. A *leaf block* is a block with only one cutvertex (or none, if the graph is already biconnected).

We first show that we can assume, without loss of generality, that G is biconnected when considering the chromatic number of G^2: let G be a graph and $\mathcal{B}$ the set of its blocks. In the same way that $\chi(G) = \max_{B \in \mathcal{B}}\{\chi(B)\}$ we have the following.

Lemma 2.2. *For a graph G with a maximum degree Δ and set $\mathcal{B}$ of biconnected blocks we have that $\chi(G^2) = \max\{\max_{B \in \mathcal{B}}\{\chi(B^2)\}, \Delta + 1\}$.*

Proof. We proceed by induction on $b = |\mathcal{B}|$. The case $b = 1$ is a tautology, so assume G has $b \geq 2$ blocks and that the lemma is true for $b - 1$. Let B be a leaf block and let $G' = \cup_{B' \in \mathcal{B} \setminus \{B\}} B'$, with $w = V(B) \cap V(G')$ as a cutvertex. If Δ' is the maximum degree of G', then by induction hypothesis $\chi(G'^2) = \max\{\max_{B' \in \mathcal{B} \setminus \{B\}}\{\chi(B'^2)\}, \Delta' + 1\}$. Assume we have a $\chi(G'^2)$-coloring of G'^2 and a $\chi(B^2)$-coloring of B^2, the latter given by a map $c_B : V(B) \to \{1, \ldots, \chi(B^2)\}$. Since w is a cutvertex we have a partition $N_G[w] = \{w\} \cup N_B \cup N_{G'}$, where $N_B = N_G(w) \cap V(B)$ and $N_{G'} = N_G(w) \cap V(G')$. In the given coloring c_B all the vertices in N_B have received distinct colors, since they all have w as a common neighbor in B. Since $|N_G[w]| \leq \Delta + 1$ there is a permutation σ of $\{1, \ldots, \max\{\chi(B^2), \Delta + 1\}\}$ such that $\sigma \circ \mathbf{i} \circ c_B$ yields a new $\chi(B^2)$-coloring of B^2 such that all vertices in $N_G[w]$ receive distinct colors (here $\mathbf{i}$ is the inclusion map of $\{1, \ldots, \chi(B^2)\}$ in

$\{1, \ldots, \max\{\chi(B^2), \Delta + 1\}\}$.) This together with the given $\chi(G'^2)$-coloring of G'^2 provides a vertex coloring of G^2 using at most $\max\{\max\{\chi(B^2), \Delta + 1\}, \chi(G'^2)\} \leq \max\{\max_{B \in \mathcal{B}}\{\chi(B^2)\}, \Delta + 1\}$ colors, which completes our proof. ∎

Duals of outerplanar graphs. Recall that a graph G is *outerplanar* if there is an embedding of it in the Euclidean plane such that every vertex bounds the infinite face. Such an explicit embedding is called an *outerplane graph*. For our arguments to come we need a few properties about outerplanar graphs, the first of which is an easy exercise (See [19, p. 240]).

Claim 2.3. Every biconnected outerplanar graph has at least two vertices of degree 2.

By a *k-vertex* we will mean a degree-2 vertex in G with at most k neighbors in G^2.

To study the coloring of the square of an outerplanar graph G, it is useful to consider the *weak dual* of G, denoted by $T^*(G)$ and defined in the following lemma, which is easy to prove.

Lemma 2.4. *Let G be an outerplane graph. Let G^* be its geometrical dual and let $u^*_\infty \in V(G^*)$ be the vertex corresponding to the infinite face of G. The* weak dual *of G is given by $T^*(G) = G^* - u^*_\infty$. The forest $T^*(G)$ is tree if, and only if, G is biconnected.*

Let G be a biconnected outerplane graph. Note that there is a surjective assignment $u \mapsto u^*$ from the degree-2 vertices of G to the leaves u^* of $T^*(G)$ corresponding to the bounded face containing u on its boundary. The vertex u^* of $T^*(G)$ is then the *dual vertex* of the degree-2 vertex u of G. Similarly, for a bounded face f of G the corresponding dual vertex of $T^*(G)$ will be denoted by f^*. In particular, if u is a degree-2 vertex on the boundary of a bounded face f of G, then $f^* = u^*$. We will, however, speak interchangeably of a face f and its corresponding dual vertex f^* from $T^*(G)$ when there is no danger of ambiguity, and we will apply standard forest/tree vocabulary to faces from the tree terminology given previously when each component from $T^*(G)$ is rooted at a center.

A *sibling* of a face f of G is a face g of G with the same parent in $T^*(G)$.

A *k-face* is a face f with k vertices and k edges. This will be denoted by $|f| = k$.

Definition 2.5. A face f of a biconnected outerplane graph G is *i-strongly simplicial*, or *i-ss* for short, if either f is isolated (that is f is the only bounded face of G), or f is a leaf in $T^*(G)$ such that either $i = 0$, or the parent face of f in $T^*(G)$ is $(i-1)$-ss in $\mathrm{pr}(T^*(G))$.

We see from Definition 2.5 that all leafs are 0-ss, while those leafs whose siblings have no children are also 1-ss, and further those leaves whose first cousins have no children are also 2-ss, and so forth.

Convention. For an i-ss face f where $i \geq 2$, then the parent of f is denoted by f' and the grandparent of f (i.e., the parent of f') will be denoted by f''.

Note. (i) If G is a biconnected outerplanar with $\Delta = 6$, then each 1-ss face f has a parent f and a grandparent f''.
(ii) Also, G has at least two 2-ss faces in this case.

3. The Chromatic Number when $\Delta \leq 6$

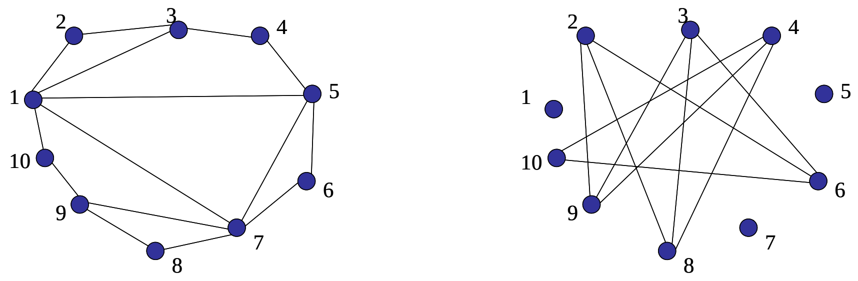
The following theorem appears in [8, 9, 10] and [11].

Theorem 3.1. *If G is an outerplanar graph with maximum degree Δ, then $\chi(G^2) \leq \Delta + 2$ for $\Delta \geq 3$ and $\chi(G^2) \leq \Delta + 1$ for $\Delta \geq 7$.*

By considering the five-cycle with one chord and the six-cycle with three chords forming and internal triangle in it, we see that the upper bound for $\Delta = 3, 4$ respectively in the above Theorem 3.1 is optimal. The bound for $\Delta \geq 7$ is clearly optimal since $\chi(G^2)$ matches the clique number $\omega(G^2)$ of G^2 in that case.

Consider now the case $\Delta = 5$. From Theorem 3.1 we have $\chi(G^2) \leq \Delta + 2 = 7$. We now briefly argue that this upper bound is indeed optimal (something, that to the best of the authors knowledge, is not discussed in the literature.) Let G_{10} be the graph on ten vertices given in Figure 1A. To see that G_{10}^2 requires 7 colors, consider its complement graph $\overline{G_{10}^2}$ shown in Figure 1B, where we connect every pair of vertices that are not connected in G_{10}^2. Clearly, $\chi(G_{10}^2) = 7$ if, and only if, the least number of cliques to cover $\overline{G_{10}^2}$ is seven. Each of the vertices u_1, u_5 and u_7 in $\overline{G_{10}^2}$ require their own clique, while for the remaining 7 vertices, there is no 3-clique. So at least

$\lceil 7/2 \rceil = 4$ cliques are needed to cover these remaining 7 vertices. Hence, 7 cliques are required to cover $\overline{G_{10}^2}$. That is, 7 colors are required to color G_{10}^2.



(A) G_{10}, $\chi(G_{10}^2) = 7 = \Delta + 2$

(B) $\overline{G_{10}^2}$, the complement of the square of G_{10}

Figure 1. A biconnected outerplane graph G with $\Delta = 5$ and $\chi(G^2) = 7 = \Delta + 2$.

We note that G_{10} has four edges with endvertices of degree 2 and 3 respectively. By fusing together two copies of G_{10} along these edges in such a way that a degree-2 vertex in one copy is identified with a degree-3 vertex in another copy, we can make an infinite family of outerplanar graphs with $\Delta = 5$, such that their square has chromatic number of 7. We summarize in the following.

Proposition 3.2. *There are infinitely many biconnected outerplanar graphs G with maximum degree $\Delta = 5$ such that $\chi(G^2) = 7$.*

We now delve into the case where G is an outerplanar graph with $\Delta = 6$. By Theorem 3.1 we have in this case that $7 \leq \omega(G^2) \leq \chi(G^2) \leq 8$, so $\chi(G^2)$ is either 7 or 8. We will show that $\chi(G^2) = 7$ always holds here. As discussed in the introduction, unlike the cases where $\Delta \geq 7$, we cannot prove that G always contains a vertex of degree Δ or less in G^2. A different approach is needed. However, there are some "local arguments" one can use. The main idea here will be based on the *method of infinite decent*[†] where we assume there is a counter example to our assertion with the smallest number of vertices. We then show that this example must have certain local

[†]an offshoot of the well-ordering principle devised by Fermat

properties which, together with its minimality, we can use to extend a 7-coloring of certain smaller graphs to a 7-coloring of this counter example, thereby obtaining a contradiction to the existence of such a counter example. When considering coloring the square of an outerplanar graph G, we can by Lemma 2.2 assume G to be biconnected and hence, by Lemma 2.4, its weak dual $T^*(G)$ to be a connected tree.

Definition 3.3. A *minimal criminal* is a biconnected outerplanar graph G with maximum degree $\Delta = 6$ and a minimum number of vertices satisfying $\chi(G^2) = 8$.

First we note the following.

Lemma 3.4. *A minimal criminal has no* 6*-vertex.*

Proof. Assume that a minimal criminal G has a 6-vertex u. Then u has degree 2 in G and at most 6 neighbors in G^2. If v is one of the two neighbors of u in G, let G/uv be the simple contraction of G with respect to the edge uv. Here, we imagine that v has "swallowed" the vertex u and the edge uv so that v is still a vertex in the contraction G/uv. Since G/uv has one less vertex than G, is biconnected and outerplanar with maximum degree at most 6, then by minimality of G the square of G/uv can be colored by at most 7 colors. Since the distance between a pair of vertices in G/uv of two or more will remain at least that in G, we can use the coloring of $(G/uv)^2$ to color all the vertices of G except u. Since u has at most 6 neighbors in G^2, there is at least one available color for u to complete the 7-coloring of G^2. This shows that G cannot be a minimal criminal, which contradicts our assumption. ∎

We will now reduce our considerations to some key cases regarding the weak dual $T^*(G)$ of our minimal criminal G. We will assume, unless otherwise stated, that G is outerplane, i.e., with a fixed embedding in the plane.

Firstly, if f is a 0-ss face (so f^* is a leaf in $T^*(G)$) with $|f| \geq 5$, then any degree-2 vertex with both its neighbors of degree 2 in G is a 4-vertex in G^2 which cannot be by Lemma 3.4. Secondly, let f be a 1-ss face of G. If $|f| = 4$, then one of the two degree-2 vertices of f, say u, has two neighbors in G, one of degree 2 and the other of degree at most 4 in G (if f has a sibling that shares a vertex with f.) This means that u is a 5-vertex in G^2, contradicting Lemma 3.4. Thirdly, if f is a 1-ss with $|f| = 3$ and has two siblings, then the unique degree-2 vertex of f is a 6-vertex, again contradicting Lemma 3.4.

We summarize in the following.

Lemma 3.5. *For a minimal criminal G we have that*

1. *each* 0*-ss face f has $|f| \in \{3, 4\}$,*
2. *each* 1*-ss face f has $|f| = 3$,*
3. *each* 1*-ss face f has at most one sibling in $T^*(G)$.*

In general, we say that a biconnected outerplanar G is 3-restricted *if it satisfies the above three conditions.*

If G is 3-restricted, then we have in general the following for the parent f' of a 1-ss face f in $T^*(G)$.

Lemma 3.6. *If f is a* 1*-ss face of a* 3*-restricted G with $\Delta = 6$ and $|f'| \geq 5$, then either f or its parent f' contains a* 6*-vertex.*

Proof. Note that f' (that is to say f'^*) is a leaf in the pruned tree $\mathrm{pr}(T^*(G))$. Let f' be bounded by the vertices $v_0, \ldots, v_\alpha$ with $\alpha \geq 4$ and where $v_0 v_\alpha$ is the edge in G that bounds the grandparent f'' (that is to say, is dual to the edge in $T^*(G)$ incident to f'^*). If v_i and v_{i+1} are on the boundary of either f or its unique sibling g (in the case that f has a sibling g) for some $i \in \{1, \ldots, \alpha - 2\}$, then the degree-2 vertex on either f or g has degree at most five in G^2. Otherwise, all the $\alpha - 2 \geq 2$ edges $v_1 v_2, \ldots, v_{\alpha-2} v_{\alpha-1}$ bound the infinite face of G, in which case the $\alpha - 3 \geq 1$ vertices, $v_2, \ldots, v_{\alpha-2}$, are all degree-2 vertices with at most six neighbors in G^2 (exactly six only if $\alpha = 5$, $v_0 v_1$ bounds f and $v_{\alpha-1} v_\alpha$ bounds its sibling g.) This completes the proof. ■

As a result, for every 1-ss face f in a minimal criminal, we have $|f'| \in \{3, 4\}$ for its parent f'.

Consider now a 3-restricted G and a 1-ss face f of G where f' is bounded by four vertices v_0, v_1, v_2, v_3 and the edge $v_0 v_3$ bounds f'' as well. Assume f is bounded by u, v_1, v_2. Whether or not f has a unique sibling in $T^*(G)$, the degree-2 vertex u has at most 5 neighbors in G^2. Hence, if G is a minimal criminal, then $v_1 v_2$ must bound the infinite face of G.

To make further restrictions, assume that G is 3-restricted and is induced by the cycle C_n on the vertices $\{u_1, \ldots, u_n\}$ in clockwise order. If

$d_G(u_3) = 4$ and u_3 is adjacent to both u_1 and u_5 in G, then for any coloring of the square G^2 the vertices $u_1, \dots, u_5$ must all receive distinct colors, say $1, \dots, 5$ respectively, since $N_G[u_3] = \{u_1, \dots, u_5\}$ induces a clique in G^2. Consider the outerplanar graph G' obtained by first removing both the edges u_3u_4 and u_3u_5 and then connecting a new vertex u_3' to each of the vertices u_3, u_4 and u_5. In this way G becomes the contraction of G', namely $G = G'/u_3u_3'$. Note that if G has a maximum degree of $\Delta = 6$, then so does G'. In addition, given the mentioned coloring of G^2 where u_i has color i for $1 \le i \le 5$, then we can obtain a coloring of G'^2 by retaining the colors of u_i from G^2 for all $i \notin \{2, 3, 4\}$, and then assigning colors $3, 2, 4, 3$ to the vertices u_2, u_3, u_3', u_4 respectively. In particular, the graph G' obtained from G in the mentioned fashion cannot be a minimal criminal. We summarize our findings in the following.

Claim 3.7. Let G be 3-restricted and f a 1-ss face of G where its parent f' is bounded by the vertices v_0, v_1, v_2, v_3 and v_0v_3 bounds f'' as well.

1. If G is a minimal criminal, then v_1v_2 must bound the infinite face.
2. If v_0v_1 bounds f and v_2v_3 bounds its sibling g, then G is not a minimal criminal.

This yields further restrictions on the 1-ss face f and its parent f' of a minimal criminal G.

Theorem 3.8. *Let G be a minimal criminal and f a 1-ss face of G with a parent f' and grandparent f''.*

1. *If f has no sibling, then $|f'| = 4$ and all the faces f, f' and f'' have exactly one vertex in common on their boundaries.*
2. *If f has one sibling g, then $|f'| = 3$ and hence all the faces f, g and f' are bounded by exactly three vertices and edges.*

Proof. Since G is a minimal criminal then G is 3-restricted and $|f'| \in \{3, 4\}$. If f has no sibling and $|f'| = 3$ then the degree-2 vertex bounding f has ≤ 6 neighbors in G^2, a contradiction.

If f has no sibling and $|f'| = 4$, assume f' is bounded by four vertices v_0, v_1, v_2, v_3 where v_0v_3 bounds f'' as well. By Claim 3.7 the edge v_1v_2 must bound the infinite face. Therefore, either v_0v_1 or v_2v_3 bounds f and hence the faces f, f' and f'' all share a common vertex, namely v_0 or v_3.

If f has one sibling g, and $|f'| = 4$, assume again f' is bounded by four vertices v_0, v_1, v_2, v_3 where v_0v_3 bounds f'' as well. Since v_1v_2 bounds the infinite face, then it bounds neither f nor g. By symmetry we may assume g to be to the right of f in the plane embedding of G. In that case we have that v_0v_1 bounds f, the edge v_2v_3 bounds g and v_1v_2 bounds the infinite face of G, contradicting Claim 3.7. ■

What Theorem 3.8 implies, in particular, is that in a minimal criminal G, each configuration $C(f, f')$ of a 1-ss face f and its parent f' is itself induced by a 5-cycle on the vertices v_1, v_2, v_3, v_4, v_5 in a clockwise order, and is of one of the following three types (if f has a sibling g, then it is unique and we may assume g to be to the right of f in the planar embedding of $T^*(G)$ when viewed from f'):

(a) $C(f, f')$ is the 5-cycle on $\{v_1, v_2, v_3, v_4, v_5\}$ in which v_3 is connected to v_1. Here f is bounded by $\{v_1, v_2, v_3\}$ and f' is bounded by $\{v_1, v_3, v_4, v_5\}$.

(b) $C(f, f')$ is the 5-cycle on $\{v_1, v_2, v_3, v_4, v_5\}$ in which v_3 is connected to v_5. Here f is bounded by $\{v_3, v_4, v_5\}$ and f' is bounded by $\{v_1, v_2, v_3, v_5\}$.

(c) $C(f, f')$ is the 5-cycle on $\{v_1, v_2, v_3, v_4, v_5\}$ in which v_3 is connected to both v_1 and v_5. Here f is bounded by $\{v_1, v_2, v_3\}$, the face g is bounded by $\{v_3, v_4, v_5\}$ and f' is bounded by $\{v_1, v_3, v_5\}$.

Here, for all the three types of configurations, it is assumed that the edge v_1v_5 bounds the faces f' and f''.

Remarks. Note that plane configurations (a) and (b) are mirror images of each other. Also, note that in all configurations, all the edges v_iv_{i+1} where $1 \le i \le 4$ of the 5-cycle that induces $C(f, f')$, except one edge v_1v_5, bound the infinite face of G.

Definition 3.9. Let G be 3-restricted with $\Delta = 6$. If for each 1-ss face f of G the configuration cycle $C(f, f')$ of f and its parent f' is of type (a), (b) or (c) from above, then G is *fully restricted*. Hence, a minimal criminal is always fully restricted.

Let G be a biconnected outerplanar graph induced by the cycle C_n on the vertices $\{u_1, \ldots, u_n\}$ in clockwise order. Assume that $d_G(u_4) = 6$ and that u_4 is adjacent to u_1, u_2, u_3, u_5, u_6 and u_7.

From G we construct four other outerplanar graphs $\tilde{G}$, G', G'' and G''' in the following way:

1. Let $\tilde{G}$ be obtained by replacing the edge u_1u_2 by the 2-path (u_1, x, u_2).
2. Let G' be obtained from G by replacing the edges u_1u_2 and u_6u_7 by the 2-paths (u_1, x, u_2) and (u_6, y, u_7) respectively.
3. Let G'' be obtained from G by replacing the edge u_1u_2 by the 2-path (u_1, x, u_2) and connecting the additional vertex y to both u_6 and u_7.
4. Let G''' be obtained from G by connecting the additional vertex x to both u_1 and u_2 and the additional vertex y to u_6 and u_7.

Note that G is a contraction of each of the graphs $\tilde{G}$, G', G'' and G''', namely

$$\tilde{G}/u_1x = (G'/u_1x)/u_6y = (G''/u_1x)/u_6y = (G'''/u_1x)/u_6y = G.$$

Lemma 3.10. *A 7-coloring of G^2 can be extended to a 7-coloring of $\tilde{G}^2$.*

Proof. Since $N_G[u_4] = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ induces a clique in G^2, we may assume u_i to have color i for $1 \leq i \leq 7$ in the 7-coloring of G^2. To obtain a 7-coloring of $\tilde{G}^2$ we retain the colors of u_i from G^2 for all $i \notin \{2, 3, 5\}$ and then assign colors $2, 3, 5, 2$ to vertices x, u_2, u_3, u_5 respectively (note that we do not need to know the colors of all the neighbors of neither u_1 nor u_7 in the given 7-coloring of G^2). ∎

Definition 3.11. Let G be a biconnected outerplanar graph with $\Delta = 6$. For a legitimate partial vertex coloring of G^2 we call a vertex u *c-simplicial* if all the colored neighbors of u in G^2 have combined at most 6 colors.

Clearly every 6-vertex of G^2 is c-simplicial with respect to any partial coloring of G^2. Also, note that $\tilde{G}$ from above cannot be a minimal criminal, since u_5 is a 6-vertex and hence always c-simplicial; if we have a 7-coloring of $(\tilde{G} - u_5)^2$ then we can extend it to a 7-coloring of $\tilde{G}^2$.

Our next theorem will provide our main tool for this section.

Theorem 3.12. *If G and the constructed graphs G', G'' and G''' are as defined above, then none of the graphs G', G'' or G''' are minimal criminals.*

Proof. If G' is a minimal criminal, then by definition G^2 has a legitimate 7-coloring. Again, we may assume u_i to have color i for $1 \leq i \leq 7$. By retaining the colors of u_i from G^2 for $i \notin \{2, 3, 5, 6\}$ and then assigning colors $2, 3, 6, 2, 5, 6$ to the vertices x, u_2, u_3, u_5, u_6, y respectively, we obtain a legitimate 7-coloring of G'^2. Hence, G' cannot be a minimal criminal.

If G'' is a minimal criminal, then G^2 has a legitimate 7-coloring. We can assume u_i to have color i for $1 \leq i \leq 7$. By Lemma 3.10 we obtain a 7-coloring of $\tilde{G}^2$, as given in its proof. If y is c-simplicial (with respect to this mentioned 7-coloring of G^2) then we can obtain a 7-coloring of G''^2. Therefore y cannot be c-simplicial in this case. This means the neighbors of u_7 among $V(G) \setminus \{u_2, \ldots, u_6\}$ have the colors $1, 3$ and 5 precisely, since $d_G(u_7) = 5$ and $d_{G''}(u_7) = 6$. In this case assign the colors $2, 5, 6, 3, 2, 6$ to the vertices x, u_2, u_3, u_5, u_6, y. This is a legitimate 7-coloring of G''^2 and hence G'' cannot be a minimal criminal.

If G''' is a minimal criminal, then $G^2 = (G''' - \{x, y\})^2$ has a legitimate 7-coloring. We can assume u_i to have color i for $1 \leq i \leq 7$. If both x and y are c-simplicial, then we can extend the given coloring of G^2 to that of G'''^2, since x and y are of distance 3 or more from each other in G'''. If neither x nor y are c-simplicial, then we must have that the neighbors of u_1 among $V(G) \setminus \{u_2, \ldots, u_6\}$ have the colors $5, 6$ and 7 precisely, and the neighbors of u_7 among $V(G) \setminus \{u_2, \ldots, u_6\}$ have the colors $1, 2$ and 3 precisely. In this case we assign the colors $2, 3, 6, 2, 5, 6$ to the vertices x, u_2, u_3, u_5, u_6, y respectively (as in the case with G') and obtain a legitimate 7-coloring of G'''^2. We consider lastly the case where one of x and y is c-simplicial and the other is not. By symmetry, it suffices to consider the case where x is c-simplicial and y is not. The fact that x is c-simplicial means that it can be assigned a color that must be from $\{5, 6, 7\}$ and thereby obtain a 7-coloring of $(G''' - y)^2$. Since y is not c-simplicial means that the neighbors of u_7 among $V(G) \setminus \{u_2, \ldots, u_6\}$ have the colors $1, 2$ and 3 precisely. We now consider the following three cases:

x has color 5: In this case assign the colors $2, 5, 3, 2, 6, 5$ to the vertices x, u_2, u_3, u_5, u_6, y, thereby obtaining a legitimate 7-coloring of G'''^2.

x has color 6: In this case assign the colors $2, 6, 3, 2, 5, 6$ to the vertices x, u_2, u_3, u_5, u_6, y, thereby obtaining a legitimate 7-coloring of G'''^2.

x has color 7: Here u_1 and u_7 cannot be connected since both x and u_7 have color 7. In this case assign the colors $7, 2, 5, 3, 6, 5$ to the vertices x, u_2, u_3, u_5, u_6, y, thereby obtaining a legitimate 7-coloring of G'''^2.

This shows that G''' cannot be a minimal criminal. This completes our proof. ■

Remark. To test the legitimacy of the extended colorings we note first of all that the vertices u_1, u_4 and u_7 always keep their color from the one provided by G^2. In addition, the colors of the neighbors of u_1 among $\{u_1, \ldots, u_7\}$ are the same, unless x is not c-simplicial, which gives concrete information about the colors of the other neighbors of u_1. Similarly the colors of the neighbors of u_7 among $\{u_1, \ldots, u_7\}$ are the same, unless (as for x) y is not c-simplicial, which again gives concrete information about the colors of the remaining neighbors of u_7.

We are now ready for the proof of the following main result of this section.

Theorem 3.13. *There is no minimal criminal; every biconnected outerplanar graph G with $\Delta = 6$ has $\chi(G^2) = 7$.*

Proof. We will show that a minimal criminal must have the form of one of the graphs G', G'' or G''', thereby obtaining a contradiction by Theorem 3.12.

Assume G is a minimal criminal, which must therefore be fully restricted. Sinc e $\Delta = 6$, each 1-ss face f of G has a parent f' and a grandparent f'' in $T^*(G)$. Assume now f is a 2-ss face of G. In this case f'' (that is f''^*) is a leaf in $\mathrm{pr}^2(T^*(G))$ or a single vertex. Since G is fully restricted, the configuration $C(f, f')$ in G is induced by a 5-cycle on vertices $\{v_1, v_2, v_3, v_4, v_5\}$ in clockwise order and is of type (a), (b) or (c) mentioned earlier. Assume f'' is bounded by $u_1, \ldots, u_m$ where $m \geq 3$. If f'' is not a single vertex but a leaf in $\mathrm{pr}^2(T^*(G))$, then let the edge $u_m u_1$ of G be the dual edge of the unique edge with f''^* as an endvertex in $\mathrm{pr}^2(T^*(G))$. In any case (whether f''^* is a leaf or a single vertex in $\mathrm{pr}^2(T^*(G))$) *at least one* of the edges $u_1u_2, \ldots, u_{m-1}u_m$ must be identified with an edge v_1v_5 of a configuration $C(f, f')$ of type (a), (b) or (c).

We now argue that *every* edge $u_1u_2, \ldots, u_{m-1}u_m$ bounds a configuration $C(f, f')$ of the type (a), (b) or (c). That is, for each $i \in \{1, \ldots, m-1\}$ the edge u_iu_{i+1} is identified with the edge v_1v_5 that defines a $C(f, f')$ configuration (i.e., $v_1 = u_i$ and $v_5 = u_{i+1}$). If there is an edge u_iu_{i+1} bounding either the infinite face or a leaf face of G, then there is such an edge adjacent to another such edge that bounds a $C(f, f')$ configuration. In that case one of the degree-2 vertices v_2 or v_4 of this $C(f, f')$ configuration is a 6-vertex in G, contradicting Lemma 3.4. Hence, each edge $u_1u_2, \ldots, u_{m-1}u_m$ bounds a parent f' with a child f, which must be a $C(f, f')$ configuration since G is fully restricted.

Consider now the vertex u_2. Since both the edges u_1u_2 and u_2u_3 are identified with edges v_1v_5 of configurations $C(f, f')$ each of type (a), (b) or (c), then we must have $d_G(u_2) \in \{4, 5, 6\}$. We now discuss each of these cases. In what follows "*Case* (x,y)" will mean that configuration $C(f, f')$ of type (x) is to the left of u_2 and configuration $C(f, f')$ of type (y) is to the right of u_2.

$d_G(u_2) = 4$: Here we must have *Case* (a,b), in which case both degree-2 vertices adjacent to u_2 are 6-vertices in G, contradicting Lemma 3.4.

$d_G(u_2) = 5$: Here we must have one of the cases *Case* (a,a), *Case* (a,c), *Case* (b,b) or *Case* (c,b). In each of these four cases the unique degree-2 neighbor of u_2 that bounds a triangular face f is indeed a 6-vertex of G, contradicting Lemma 3.4.

$d_G(u_2) = 6$: Here we must have one of the cases *Case* (b,a), *Case* (b,c), *Case* (c,a) or *Case* (c,c). We dispatch each of them as follows.

Case (b,a). Here G is of type G' as stated in Theorem 3.12 (with u_2 here in the role of u_4 mentioned there), and therefore cannot be a minimal criminal.

Case (b,c). Here G is of type G'' as stated in Theorem 3.12, and therefore cannot be a minimal criminal.

Case (c,a). Here G is a mirror image of a type G'' (previous case) as stated in Theorem 3.12, and therefore cannot be a minimal criminal.

Case (c,c). Here G is of type G''' as stated in Theorem 3.12, and therefore cannot be a minimal criminal.

This concludes the proof, that there is no minimal criminal. Hence the square of each biconnected outerplanar graph with $\Delta = 6$ is 7-colorable. ■

By Theorem 3.13 and Lemma 2.2 we have the following corollary.

Corollary 3.14. *For every outerplanar graph G with $\Delta = 6$ we have $\chi(G^2) = 7$.*

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