

**MATCHINGS AND TOTAL DOMINATION  
SUBDIVISION NUMBER IN GRAPHS  
WITH FEW INDUCED 4-CYCLES**

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**Abstract**

A set  $S$  of vertices of a graph  $G = (V, E)$  without isolated vertex is a *total dominating set* if every vertex of  $V(G)$  is adjacent to some vertex in  $S$ . The *total domination number*  $\gamma_t(G)$  is the minimum cardinality of a total dominating set of  $G$ . The *total domination subdivision number*  $sd_{\gamma_t}(G)$  is the minimum number of edges that must be subdivided (each edge in  $G$  can be subdivided at most once) in order to increase the total domination number. Favaron, Karami, Khoeilar and Sheikholeslami (Journal of Combinatorial Optimization, to appear) conjectured that: For any connected graph  $G$  of order  $n \geq 3$ ,  $sd_{\gamma_t}(G) \leq \gamma_t(G) + 1$ . In this paper we use matchings to prove this conjecture for graphs with at most three induced 4-cycles through each vertex.

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## 1. INTRODUCTION

Let  $G = (V(G), E(G))$  be a simple graph of order  $n$  with minimum degree  $\delta(G) \geq 1$ . The neighborhood of a vertex  $u$  is denoted by  $N_G(u)$  and its degree  $|N_G(u)|$  by  $d_G(u)$  (briefly  $N(u)$  and  $d(u)$  when no ambiguity on the graph is possible). If  $S \subseteq V(G)$ ,  $N(S) = \cup_{x \in S} N(x)$ . We denote by  $N_2(v)$  the set of vertices at distance 2 from the vertex  $v$  and put  $d_2(v) = |N_2(v)|$  and  $\delta_2(G) = \min\{d_2(v); v \in V(G)\}$ . A *matching* is a set of edges with no shared endvertices. A perfect matching  $M$  of  $G$  is a matching with  $V(M) = V(G)$ . If  $n$  is odd, a *near perfect matching* leaves exactly one vertex uncovered, i.e.,  $|V(M)| = n - 1$ . A graph is *factor-critical* if the deletion of any vertex leaves a graph with a perfect matching. Note that factor-critical graphs have odd order. The maximum number of edges of a matching in  $G$  is denoted by  $\alpha'(G)$  ( $\alpha'$  for short). The length of a smallest cycle in a graph  $G$ , that contains cycles, is the *girth* of  $G$  (denoted  $g(G)$ ). We use [15] for terminology and notation which are not defined here.

A set  $S$  of vertices of a graph  $G$  with minimum degree  $\delta(G) > 0$  is a *total dominating set* if  $N(S) = V(G)$ . The minimum cardinality of a total dominating set, denoted by  $\gamma_t(G)$ , is called the *total domination number* of  $G$ . A  $\gamma_t(G)$ -*set* is a total dominating set of  $G$  of cardinality  $\gamma_t(G)$ . The *total domination subdivision number*  $sd_{\gamma_t}(G)$  is the minimum number of edges of  $G$  that must be subdivided once in order to increase the total domination number. This kind of concept was first introduced for the domination number by Velammal in his Ph.D. thesis [14]. The total domination subdivision number was considered by Haynes *et al.* in [8] and since then have been studied by several authors (see for example [2, 4, 5, 3, 6, 7, 10, 11]). Since the total domination number of the graph  $K_2$  does not change when its only edge is subdivided, in the study of total domination subdivision number we must assume that the graph has maximum degree at least two.

It is known that the parameter  $sd_{\gamma_t}$  can take arbitrarily large values [6] and an interesting problem is to find good bounds on  $sd_{\gamma_t}(G)$  in terms of other parameters of  $G$ . For instance it has been proved that for any graph  $G$  of order  $n$ ,  $sd_{\gamma_t}(G) \leq n - \gamma_t(G) + 1$  [4],  $sd_{\gamma_t}(G) \leq 2n/3$  [5] and  $sd_{\gamma_t}(G) \leq n - \Delta + 2$  [2]. Favaron *et al.* in [3] posed the following conjecture

**Conjecture 1.** For any connected graph  $G$  of order  $n \geq 3$ ,

$$sd_{\gamma_t}(G) \leq \gamma_t(G) + 1$$

and proved it for some classes of graphs.

Our purpose in this paper is to prove Conjecture 1 for connected graphs with few induced cycles  $C_4$  through each vertex of  $G$ . We will use the following results on  $\alpha'(G)$ ,  $\gamma_t(G)$  and  $sd_{\gamma_t}(G)$ .

**Theorem A** [6]. For any connected graph  $G$  with adjacent vertices  $u$  and  $v$ , each of degree at least two,

$$sd_{\gamma_t}(G) \leq d(u) + d(v) - |N(u) \cap N(v)| - 1 = |N(u) \cup N(v)| - 1.$$

**Theorem B** [3]. For any connected graph  $G$  of order  $n \geq 3$  and  $\gamma_t(G) \leq \alpha'(G)$ ,

$$sd_{\gamma_t}(G) \leq \gamma_t(G) + 1.$$

**Theorem C** [3]. For any connected graph  $G$  of order  $n \geq 3$  with  $\delta = 1$ ,

$$sd_{\gamma_t}(G) \leq \gamma_t(G).$$

**Theorem D** [2]. Let  $G$  be a connected graph of minimum degree at least 2. Then  $sd_{\gamma_t}(G) \leq \delta_2(G) + 3$ .

In the proof of Theorem 1 we use the concept of barrier. If  $S$  is a separator of a graph  $G$ ,  $o(G)$  denotes the number of odd components of  $G - S$ , i.e., components of odd order. A *barrier* of  $G$  is a separator  $S$  such that  $o(G - S) = |S| + t$  where  $t = n - 2\alpha'$  is the number of vertices of  $G$  which are not covered by a maximum matching. By Tutte-Berge's Theorem every connected graph admits barriers. Moreover (see for example exercise 3.3.18 in [12]) if  $S$  is a maximal barrier, then all the components  $G_1, G_2, \dots, G_{|S|+t}$  of  $G - S$  are factor-critical (hence odd) and every maximum matching of  $G$  is formed by a matching pairing  $S$  with  $|S|$  different components of  $G - S$  and a near perfect matching in each component. Therefore, with the notation  $|S| + t = \ell$  and  $|V(G_i)| = n_i$ ,

$$(1) \quad \alpha'(G) = |S| + \sum_{i=1}^{\ell} \frac{n_i - 1}{2}.$$

## 2. MAIN RESULT

**Theorem 1.** *Every connected graph  $G$  of order  $n \geq 3$  such that each vertex belongs to at most three induced  $C_4$  satisfies*

$$\text{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1.$$

**Proof.** By Theorems B and C, we may assume  $\delta(G) \geq 2$  and

$$(2) \quad \alpha'(G) \leq \gamma_t(G) - 1.$$

Let  $S$  be a maximal barrier of  $G$  and  $G_1, G_2, \dots, G_\ell$  the components of  $G - S$ . Let  $S_1$  be the set of the isolated vertices of  $G[S]$ .

If  $S_1 = \emptyset$  and  $G - S$  has only trivial components, then  $S$  is a total dominating set of  $G$  and  $\gamma_t(G) \leq |S| = \alpha'(G)$  by (1), a contradiction with (2). If  $S_1 \neq \emptyset$  and if all the neighbors of a vertex  $x$  of  $S_1$  belong to trivial components of  $G - S$ , then  $N_2(x) \subseteq S - \{x\}$  and by Theorem D, (1) and (2),

$$\text{sd}_{\gamma_t}(G) \leq \delta_2(G) + 3 \leq |N_2(x)| + 3 \leq |S| + 2 \leq \alpha'(G) + 2 \leq \gamma_t(G) + 1.$$

Therefore we can assume that at least one component of  $G - S$  is not trivial and that every isolated vertex of  $S$  has at least one neighbor in a non-trivial component of  $G - S$ .

First suppose that at least two components of  $G - S$ , say  $G_1$  and  $G_2$  with  $n_1 \leq n_2$ , are not trivial. Let  $uv$  be an edge of  $G_1$ . Since  $\delta(G) \geq 2$ , Theorem A, (1) and (2) imply

$$\begin{aligned} \text{sd}_{\gamma_t}(G) &\leq |N(u) \cup N(v)| - 1 \leq n_1 + |S| - 1 \\ &\leq \frac{n_1 + n_2}{2} + |S| - 1 \leq \alpha'(G) \leq \gamma_t(G) - 1. \end{aligned}$$

Now suppose that  $G_1$  is the unique nontrivial component of  $G - S$ . Then  $\alpha'(G) = |S| + \frac{n_1 - 1}{2}$  and every vertex of  $S_1$  has a neighbor in  $G_1$ . Let  $G_i = \{y_i\}$  for  $2 \leq i \leq \ell$ , and  $Y = \{y_i \mid 2 \leq i \leq \ell\}$ . If  $|N(u) \cup N(v)| - 1 \leq \gamma_t(G) + 1$  for some edge  $uv$  of  $G_1$ , then the result follows from Theorem A. Therefore we may assume that for each edge  $uv$  of  $G_1$ ,

$$(3) \quad |N(u) \cup N(v)| \geq \gamma_t(G) + 3.$$

This implies in particular by (2) that

$$|S| + \frac{n_1 - 1}{2} = \alpha'(G) \leq \gamma_t(G) - 1 \leq |N(u) \cup N(v)| - 4 \leq |S| + n_1 - 4$$

and thus  $n_1 \geq 7$ .

**Claim.**  $\gamma_t(G_1) \leq \frac{n_1 - 1}{2}$ .

**Proof of the Claim.** Let  $M = \{u_1v_1, \dots, u_{\frac{n_1-1}{2}}v_{\frac{n_1-1}{2}}\}$  be a near perfect matching of  $G_1$  and let  $\{x\} = V(G_1) - V(M)$ . Without loss of generality we assume  $x$  is adjacent to  $u_{\frac{n_1-1}{2}}$ . Let  $X$  be a subset of  $V(G_1)$  satisfying the following properties:

- (a)  $u_{\frac{n_1-1}{2}} \in X$  and  $x \notin X$ ,
- (b)  $|X \cap \{u_i, v_i\}| \leq 1$  for  $1 \leq i \leq \frac{n_1-1}{2}$ ,
- (c)  $G[X]$  has no isolated vertex if  $|X| > 1$ .

Choose  $X$  to be maximum among all such sets. By Property (b),  $|X| \leq \frac{n_1-1}{2}$ . Suppose  $|X| < \frac{n_1-1}{2}$ . Without loss of generality we may assume  $X \cap \{u_i, v_i\} = \emptyset$  for  $1 \leq i \leq r < \frac{n_1-1}{2}$ . Let  $R = \{u_i v_i \mid 1 \leq i \leq r\}$ ,  $R' = \{u_i, v_i \mid 1 \leq i \leq r\}$  and  $G' = G[R'] - R$ . If  $uv \in E(G')$ , then  $X' = X \cup \{u, v\}$  satisfies Properties (a) to (c), a contradiction with the choice of  $X$ . Similarly if  $G$  contains an edge  $uv$  with  $u \in X$  and  $v \in R'$ , then  $X' = X \cup \{v\}$  contradicts the choice of  $X$ . Hence  $G'$  is empty and no edge exists between  $X$  and  $R'$ . Therefore  $N(u_1) \cup N(v_1) \subseteq S \cup \{u_1, v_1\} \cup (V(G_1) \setminus (X \cup R'))$ . Since  $|V(G_1)| \leq 2|X| + |R'| + 1$ , we have by (2),

$$|N(u_1) \cup N(v_1)| \leq |S| + |X| + 3 < \alpha'(G) + 3 \leq \gamma_t(G) + 2,$$

in contradiction with (3). Therefore  $|X| = \frac{n_1-1}{2} \geq 3$  and from its construction, it is clear that  $X$  is a total dominating set of  $G_1$ .  $\square$

Let  $X$  be a total dominating set of  $G_1$  of order  $\frac{n_1-1}{2}$  as in the claim. If  $S_1 = \emptyset$  or if every vertex of  $S_1$  has a neighbor in  $X$ , then  $S \cup X$  is a total dominating set of  $G$  and thus  $\gamma_t(G) \leq |S| + |X| = \alpha'(G)$ , a contradiction with (2). Hence the set  $S_2$  of the isolated vertices of  $G[S]$  with no neighbor in  $X$  is not empty.

If  $N(y_i) \not\subseteq S_2$  for each  $i$  with  $2 \leq i \leq \ell$ , we associate to each vertex  $x$  of  $S_2$  one of its neighbors  $f(x)$  in  $V(G_1) - X$  (recall that each vertex of  $S_1$  has at least one neighbor in  $G_1$ ) and we let  $S'_2 = \{f(x) \mid x \in S_2\}$ . Clearly  $|S'_2| \leq |S_2|$ ,  $S'_2$  dominates  $S_2$ , and  $X \cup S'_2$  is a total dominating set of  $V(G_1) \cup S_1$ . Therefore  $(S - S_2) \cup X \cup S'_2$  is a total dominating set of  $G$  and  $\gamma_t(G) \leq |S| + |X| = \alpha'(G)$ , a contradiction with (2).

Hence some vertex  $y_i$  of  $Y$ , say  $y_2$ , has all its neighbors in  $S_2$ . Since  $\delta(G) \geq 2$ ,  $|S_2| \geq 2$ . Let  $uv$  be an edge of  $G[X]$  (such an edge exists since  $n_1 \geq 7$ ). Then  $N(u) \cup N(v) \subseteq (S - S_2) \cup V(G_1)$ . By (2) and (3),

$$|S| + \frac{n_1 - 1}{2} = \alpha'(G) \leq \gamma_t(G) - 1 \leq |N(u) \cup N(v)| - 4 \leq |S| - |S_2| + n_1 - 4.$$

Therefore

$$n_1 \geq 2|S_2| + 7 \geq 11.$$

Let  $z_1$  and  $z_2$  be two neighbors of  $y_2$ . The neighborhoods  $N_{G_1}(z_1)$  and  $N_{G_1}(z_2)$  are contained in  $V(G_1) - X$ . Let  $|N_{G_1}(z_1) \cap N_{G_1}(z_2)| = p$  and suppose without loss of generality  $d_{G_1}(z_1) \leq d_{G_1}(z_2)$ . Then

$$2|N_{G_1}(z_1)| \leq |N_{G_1}(z_1)| + |N_{G_1}(z_2)| \leq \frac{n_1 + 1}{2} + p$$

and since  $p \geq 0$  and  $n_1 \geq 11$ ,

$$(4) \quad |N_{G_1}(z_1)| \leq \frac{n_1 + 1}{4} + \frac{p}{2} \leq \frac{n_1 - 1}{2} - 2 + p.$$

Note that each of the  $p$  vertices of  $N_{G_1}(z_1) \cap N_{G_1}(z_2)$  induces with  $y_2, z_1, z_2$  a cycle  $C_4$  containing one vertex in  $Y$  and one vertex in  $G_1$ .

Let  $A = N_Y(z_1) - \{y_2\}$  and  $B = N(y_2) - \{z_1\}$  ( $\subseteq S_2$ ). For each  $a \in A$ , let  $a'$  be one of its neighbors in  $S - \{z_1\}$  ( $a'$  exists since  $\delta(G) \geq 2$ ) and let  $A' = \{a' \mid a \in A\}$ . Then  $|A'| \leq |A|$  and  $|A| - |A'|$  is at most the number of pairs  $a_i, a_j$  of vertices of  $A$  such that  $a'_i = a'_j$ . Note that if  $a'_i = a'_j$ , then  $a'_i, a_i, a_j, z_1$  induce a  $C_4$  not containing  $y_2$ . Since the set  $B \cup A'$  is contained in  $S - \{z_1\}$ ,  $|B \cup A'| \leq |S| - 1$ . Each vertex  $a'$  of  $B \cap A'$  corresponds to at least one induced  $C_4$  of the form  $z_1 y_2 a' a z_1$  (possibly more if  $a'$  is associated to several vertices of  $A$ ). Hence if we denote by  $q$  the number of induced cycles  $C_4$  containing  $z_1$  and two vertices of  $Y$ , we get  $|A| - |A'| + |B \cap A'| \leq q$ .

Therefore

$$\begin{aligned}
 |N_Y(z_1)| + |N(y_2)| &= |A| + 1 + |B| + 1 \\
 &= |A - A'| + |A'| + |B| + 2 \\
 &= |A - A'| + |A' \cup B| + |A' \cap B| + 2 \\
 &\leq |A' \cup B| + q + 2 \\
 &\leq |S| + q + 1.
 \end{aligned}$$

Since  $N(z_1) \cap N(y_2) = \emptyset$  and by Theorem A, (1), (2) and (4), we get

$$\begin{aligned}
 \text{sd}_{\gamma_t}(G) &\leq |N(z_1)| + |N(y_2)| - 1 \\
 &\leq |N_{G_1}(z_1)| + |N_Y(z_1)| + |N(y_2)| - 1 \\
 &\leq \left(\frac{n_1-1}{2} - 2 + p\right) + (|S| + q + 1) - 1 \\
 &\leq \alpha'(G) + p + q - 2 \\
 &\leq \gamma_t(G) - 3 + p + q.
 \end{aligned}$$

By hypothesis,  $z_1$  is contained in at most three induced cycles  $C_4$ . Hence  $p + q \leq 3$  and  $\text{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$ , which completes the proof of Theorem 1.  $\blacksquare$

**Corollary 2.** *For any connected graph  $G$  of order  $n \geq 3$  with girth greater than 4,  $\text{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$ .*

**Corollary 3.** *For any connected chordal graph  $G$  of order  $n \geq 3$ ,  $\text{sd}_{\gamma_t}(G) \leq \gamma_t(G) + 1$ .*

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