Discussiones Mathematicae
Graph Theory 30 (2010) 563–574

PARTITIONING A GRAPH INTO A DOMINATING SET, A TOTAL DOMINATING SET, AND SOMETHING ELSE

Michael A. Henning*

Department of Mathematics
University of Johannesburg
Auckland Park, 2006 South Africa

e-mail: mahenning@uj.ac.za

Christian Löwenstein and Dieter Rautenbach

Institut für Mathematik
TU Ilmenau, Postfach 100565
D–98684 Ilmenau, Germany

e-mails: {christian.loewenstein, dieter.rautenbach}@tu-ilmenau.de

Abstract

A recent result of Henning and Southey (A note on graphs with disjoint dominating and total dominating set, *Ars Comb.* **89** (2008), 159–162) implies that every connected graph of minimum degree at least three has a dominating set D and a total dominating set T which are disjoint. We show that the Petersen graph is the only such graph for which $D \cup T$ necessarily contains all vertices of the graph.

Keywords: domination, total domination, domatic number, vertex partition, Petersen graph.

2010 Mathematics Subject Classification: 05C69.

*Research supported in part by the South African National Research Foundation.

1. Introduction

We consider finite, simple, and undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. For a vertex u in G, the neighbourhood is denoted by $N_G(u)$, the closed neighourhood is denoted by $N_G[u] = N_G(u) \cup \{u\}$, and the degree is denoted by $d_G(u) = |N_G(u)|$. A set D of vertices of G is dominating if every vertex in $V(G) \setminus D$ has a neighbour in D. Similarly, a set T of vertices of G is total dominating if every vertex in $V(G)$ has a neighbour in T [5, 6].

A simple yet fundamental observation made by Ore [13] is that every graph of minimum degree at least one contains two disjoint dominating sets, i.e., the trivial necessary minimum degree condition for the existence of two disjoint dominating sets is also sufficient. In contrast to that, Zelinka [14, 15] observed that no minimum degree condition is sufficient for the existence of three disjoint dominating sets or of two disjoint total dominating sets. In [9] Henning and Southey proved the following result which is somehow located between Ore's positive and Zelinka's negative observation.

Theorem 1 (Henning and Southey [9]). *If G is a graph of minimum degree at least 2 such that no component of G is a chordless cycle of length 5, then $V(G)$ can be partitioned into a dominating set D and a total dominating set T.*

A characterization of graphs with disjoint dominating and total dominating sets is given in [10]. Recently, several authors studied the cardinalities of pairs of disjoint dominating sets in graph [2, 7, 8, 11, 12]. The context of this research motivates the question for which graphs Theorem 1 is best-possible in the sense that the union $D \cup T$ of the two sets necessarily contains all vertices of the graph G. Our following main result gives a partial answer to this question.

Theorem 2. *If G is a graph of minimum degree at least 3 with at least one component different from the Petersen graph, then G contains a dominating set D and a total dominating set T which are disjoint and satisfy $|D| + |T| < |V(G)|$.*

Clearly, if the domatic number [15] of a graph G is at least $2k$, then, by definition, G contains $2k$ disjoint dominating sets and hence also k disjoint total dominating sets. Therefore, the results of Calkin *et al.* [1] and Feige *et al.* [3] imply that a sufficiently large minimum degree and a sufficiently small maximum degree together imply the existence of arbitrarily many disjoint (total) dominating sets.

The rest of the paper is devoted to the proof of Theorem 2.

2. Proof of Theorem 2

A *DT-pair* of a graph G is a pair (D, T) of disjoint sets of vertices of G such that D is a dominating set and T is a total dominating set of G. A DT-pair (D, T) in G is *exhaustive* if $|D| + |T| = |V(G)|$. Thus a DT-pair (D, T) in G is non-exhaustive if $|D| + |T| < |V(G)|$. Note that Theorem 1 implies that every graph with minimum degree at least 2 and with no component that is a chordless 5-cycle, has an exhaustive DT-pair.

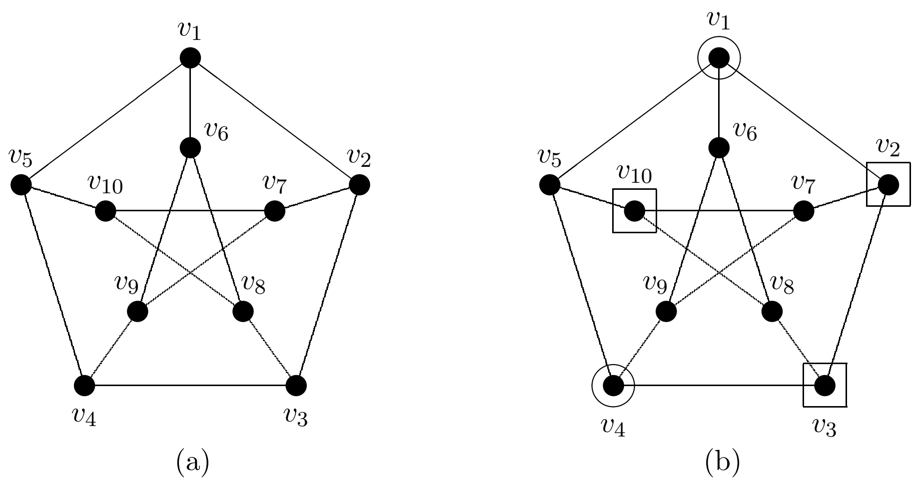


Figure 1. The encircled vertices belong to D and the framed vertices belong to T.

Our first lemma collects some useful observations about the Petersen graph.

Lemma 3. *The following properties hold for the Petersen graph.*

(a) *If G is the union of disjoint Petersen graphs, then every DT-pair in G is exhaustive.*

(b) *If G arises from the Petersen graph by adding an edge between two non-adjacent vertices, then G has a non-exhaustive DT-pair.*

(c) *If G arises from the union of two disjoint Petersen graphs by adding an edge between the two Petersen graphs, then G has a non-exhaustive DT-pair.*

Proof. In order to reduce the number of cases which we have to consider, we will use the known facts that the Petersen graph is 3-*arc transitive*, *distance-transitive*, and *vertex-transitive* (see Sections 4.4 and 4.5 of [4]).

Let P denote the Petersen graph where (see Figure 1(a))

$$\begin{aligned} V(P) &= \{v_1, v_2, \ldots, v_{10}\}, \\ E(P) &= \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\} \\ &\quad \cup \{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\} \\ &\quad \cup \{v_6v_8, v_8v_{10}, v_{10}v_7, v_7v_9, v_9v_6\}. \end{aligned}$$

Let (D, T) be an DT-pair of P. Since P is 3-arc transitive, we may assume, by symmetry, that $v_2, v_3 \in T$ and $v_1, v_4 \in D$. Since $|N_P(v_5) \cap T| \geq 1$, $v_{10} \in T$ (see Figure 1(b)). Suppose no vertex in $\{v_7, v_8\}$ belongs to $D \cup T$. Then, $v_5 \in T$ to totally dominate v_{10}, while $\{v_6, v_9\} \subset D$ to dominate $\{v_7, v_8\}$. But then no vertex of T totally dominates v_6 or v_9. Hence, at least one vertex in $\{v_7, v_8\}$ belongs to $D \cup T$. We may assume, by symmetry, that $v_7 \in D \cup T$.

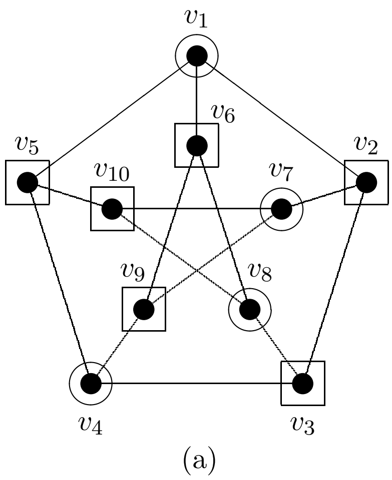
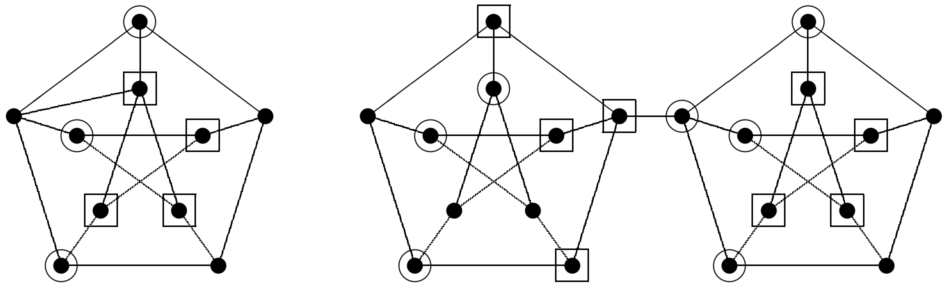


Figure 2. The encircled vertices belong to D and the framed vertices belong to T.

First, we assume $v_7 \in D$. Since $|N_P(v_9) \cap T| \geq 1$, $v_6 \in T$. Since $|N_P[v_8] \cap D| \geq 1$, $v_8 \in D$. Since $|N_P(v_6) \cap T| \geq 1$, $v_9 \in T$. Since $|N_P(v_{10}) \cap T| \geq 1$, $v_5 \in T$ (see Figure 2(a)). Now, $|D| + |T| = |V(P)|$.

Next, we assume $v_7 \in T$. Since $|N_P[v_7] \cap D| \geq 1$, $v_9 \in D$. Since $|N_P(v_6) \cap T| \geq 1$, $v_8 \in T$. Since $|N_P[v_8] \cap D| \geq 1$, $v_6 \in D$. Since $|N_P[v_{10}] \cap D| \geq 1$, $v_5 \in D$ (see Figure 2(a)). Again, $|D| + |T| = |V(P)|$.

Since in both cases (D, T) is exhaustive, the proof of (a) is complete. Since the Petersen graph is distance-transitive, Figure 3(a) proves (b).



(a) (b)

Figure 3. The encircled vertices constitute a dominating set and the framed vertices constitute a total dominating set.

Finally, since the Petersen graph is vertex-transitive, Figure 3(b) proves (c). ∎

The next lemma contains the core of our argument.

Lemma 4. *If G is a graph such that*

(i) *the minimum degree of G is at least* 3,

(ii) *G is not the union of disjoint Petersen graphs, and*

(iii) *the set of vertices of degree at least* 4 *is independent,*

then G has a non-exhaustive DT-pair.

Proof. For sake of contradiction, we assume that G is a counterexample of minimum order. Hence G satisfies condition (i), (ii) and (iii), but does not have a non-exhaustive DT-pair.

By (i) and Theorem 1, G has a non-exhaustive DT-pair if and only if some component of G has a non-exhaustive DT-pair. Hence, by the minimality of G, the graph G is connected.

We establish a series of claims concerning G.

Claim A. For $u \in V(G)$, the subgraph $G - \{u\}$ of G induced by $V(G) \setminus \{u\}$ has no C_5-component.

***Proof of Claim* A.** For contradiction, we assume that for some vertex u of G, the graph $G' = G - \{u\}$ has at least one C_5-component. Let V_5 denote the set of vertices of all C_5-components of G'. By the minimum degree condition (i) in G, we note that u is adjacent to every vertex of V_5 in G. If $V_5 \cup \{u\} = V(G)$, then letting $v \in V_5$, we have that $(D, T) = (\{u\}, V_5 \setminus \{v\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $V_5 \cup \{u\} \neq V(G)$. Let $G'' = G - (\{u\} \cup V_5)$. Then, G'' has no C_5-component and has minimum degree at least 2. Thus, by Theorem 1, G'' has an exhaustive DT-pair (D'', T''). If $v \in V_5$, then $(D, T) = (D'' \cup \{u\}, T'' \cup (V_5 \setminus \{v\}))$ is a non-exhaustive DT-pair of G, a contradiction. □

Claim B. For a 5-cycle C in G, the graph $G - V(C)$ either has a C_5-component or is of minimum degree less than 2.

***Proof of Claim* B.** For contradiction, we assume that $C\colon v_1v_2v_3v_4v_5v_1$ is a 5-cycle in G such that $G' = G - V(C)$ has minimum degree at least 2 and no C_5-component. By Theorem 1, G' has an exhaustive DT-pair (D', T'). If a vertex in T' is adjacent to a vertex of C, say to v_1, then $(D, T) = (D' \cup \{v_2, v_5\}, T' \cup \{v_3, v_4\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, by condition (i), every vertex of C has a neighbour in D'. But then $(D, T) = (D', T' \cup \{v_1, v_2, v_3\})$ is a non-exhaustive DT-pair of G, once again producing a contradiction. □

Claim C. G contains no 3-cycle.

***Proof of Claim* C.** For contradiction, we assume that $C\colon v_1v_2v_3v_1$ is a 3-cycle in G. First, we assume that there is a vertex $v_4 \in V(G) \setminus V(C)$ which is adjacent to at least two vertices of C, say to v_1 and to v_2. By (iii), at least one of the vertices v_1 and v_2 has degree exactly 3, say v_2. Now the graph $G' = G - \{v_1\}$ has minimum degree at least 2 and, by Claim A, has no C_5-component. Thus, by Theorem 1, G' has an exhaustive DT-pair (D', T'). Since $d_{G'}(v_2) = 2$, $|D' \cup \{v_2, v_3, v_4\}| > 0$ and $|T' \cup \{v_3, v_4\}| > 0$. Thus $(D, T) = (D', T')$ is a non-exhaustive DT-pair of G, a contradiction. Hence, every vertex in $V(G) \setminus V(C)$ is adjacent to at most one vertex of C. Thus the graph $G' = G - V(C)$ has minimum degree at least 2. If G' has a C_5-component G_5, then $G - V(G_5)$ has no C_5-component and is of minimum degree at least 2 which contradicts Claim B. Hence, G' has no C_5-component. Applying Theorem 1 to G', the graph G' has an exhaustive DT-pair (D', T'). If a vertex in T' is adjacent to a vertex of C, say to v_1, then $(D, T) = (D' \cup \{v_3\}, T' \cup \{v_1\})$ is a non-exhaustive DT-pair of G, a contradition. Hence, every vertex of C has a neighbour in D'. But then $(D, T) = (D', T' \cup \{v_1, v_2\})$ is a non-exhaustive DT-pair of G, once again producing a contradiction. $\square$

Claim D. G contains no $K_{3,3}$ as a subgraph.

***Proof of Claim* D.** For contradiction, we assume that G contains a $K_{3,3}$-subgraph with partite sets $V_v = \{v_1, v_2, v_3\}$ and $V_w = \{w_1, w_2, w_3\}$. Note that, by Claim C, every $K_{3,3}$-subgraph of G is induced. By (iii), we may assume that all vertices in V_v have degree exactly 3. Since $K_{3,3}$ has a non-exhaustive DT-pair, we may assume that w_1 has degree more than 3. Now the graph $G' = G - \{w_1\}$ is of minimum degree at least 2 and, by Claim A, has no C_5-component. By Theorem 1, G' has an exhaustive DT-pair (D', T'). Since $|N_{G'}(v_1) \cap T'| \geq 1$, $|D' \cap \{w_2, w_3\}|$ is either 0 or 1. If $|D' \cap \{w_2, w_3\}| = 0$, then $\{v_1, v_2, v_3\} \subseteq D'$, $\{w_2, w_3\} \subset T'$, and $(D, T) = ((D' \setminus \{v_1, v_2\}) \cup \{w_1\}, T' \cup \{v_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $|D' \cap \{w_2, w_3\}| = 1$. But then $(D, T) = ((D' \setminus V_v) \cup \{v_1\}, (T' \setminus V_v) \cup \{v_2\})$ is a non-exhaustive DT-pair of G, once again producing a contradiction. $\square$

Claim E. G contains no $K_{3,3} - e$ as a subgraph.

***Proof of Claim* E.** For contradiction, we assume that G contains a $(K_{3,3} - e)$-subgraph, i.e., there is a subset $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ of vertices in G such that $\{v_1w_1, v_1w_2, v_1w_3, v_2w_1, v_2w_2, v_2w_3, v_3w_1, v_3w_2\} \subseteq E(G)$ and $v_3w_3 \notin E(G)$. By Claim C, $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are independent sets.

If $d_G(v_3) > 3$ and $d_G(w_3) > 3$, then, by (iii), $d_G(v_1) = d_G(w_1) = d_G(v_2) = d_G(w_2) = 3$. The graph $G' = G - \{v_1, v_2, w_1, w_2\}$ has minimum degree at least 2. Since $d_{G'}(u) \geq 3$ for all $u \in V(G') \setminus \{v_3, w_3\}$, G' contains no C_5-component. Therefore, by Theorem 1, G' has an exhaustive DT-pair (D', T'). If $v_3 \in D'$, let $(D, T) = (D' \cup \{w_1\}, T' \cup \{v_2, w_2\})$. If $v_3 \in T'$, let $(D, T) = (D' \cup \{v_1, w_1\}, T' \cup \{w_2\})$. In both cases, (D, T) is a non-exhaustive DT-pair of G, a contradiction. Hence, $d_G(v_3) = 3$ or $d_G(w_3) = 3$. By symmetry and (iii), we may assume that $d_G(v_1) = d_G(v_2) = d_G(v_3) = 3$.

Suppose that $d_G(w_3) > 3$. If at least one vertex in $\{w_1, w_2\}$ is of degree more than 3, say w_2, then $G' = G - \{v_1, v_2, w_1\}$ has minimum degree at least 2. By Claim C, at most two neighbours of w_1 can belong to a possible C_5-component of G'. Since w_2, w_3, and the three neighbours of w_1 are the only vertices which can have degree exactly 2 in G', G' contains no C_5-component. Thus, by Theorem 1, G' has an exhaustive DT-pair (D', T'). If $\{v_3, w_2\} \subset D'$, let $(D, T) = (D', T' \cup \{v_1, w_1\})$. If $\{v_3, w_2\} \subset T'$, let $(D, T) = (D' \cup \{v_1, w_1\}, T')$. If $v_3 \in D'$ and $w_2 \in T'$, let $(D, T) = (D' \cup \{w_1\}, T' \cup \{v_1\})$. If $v_3 \in T'$ and $w_2 \in D'$, let $(D, T) = (D' \cup \{v_1\}, T' \cup \{w_1\})$. In all cases, (D, T) is a non-exhaustive DT-pair of G, a contradiction. Hence, $d_G(w_1) = d_G(w_2) = 3$. Thus, $G' = G - \{v_1, v_2, v_3, w_1, w_2\}$ has minimum degree at least 2. Let $N(v_3) = \{w_1, w_2, v_3'\}$. Since $d_{G'}(u) \geq 3$ for all $u \in V(G') \setminus \{w_3, v_3'\}$, G' contains no C_5-component. Thus, by Theorem 1, G' has an exhaustive DT-pair (D', T'). Now, $(D, T) = (D' \cup \{v_1, w_1\}, T' \cup \{v_2, w_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $d_G(w_3) = 3$.

Suppose that at least one vertex in $\{w_1, w_2\}$ is of degree more than 3, say w_2. Then, $G' = G - \{v_2, v_3, w_1\}$ has minimum degree at least 2. Let $N(v_3) = \{w_1, w_2, v_3'\}$ and let $w_2' \in V(G) \setminus \{v_1, v_2, v_3\}$ be a neighbour of w_2. By Claim C, $v_3' \neq w_2'$.

First, we assume that G' contains a C_5-component C. By Claim C, at most two neighbours of w_1 can belong to C. Since w_2 and w_3 are the only neighbours of v_1 in G', either $|V(C) \cap \{w_2, v_1, w_3\}| = 0$ or $|V(C) \cap \{w_2, v_1, w_3\}| = 3$. Since w_2, w_3, v_3', and the neighbours of w_1 are the only vertices which can have degree exactly 2 in G', we have that $V(C) = \{v_1, v_3', w_2, w_2', w_3\}$ implying that $d_G(v_3') = d_G(w_2') = 3$, $d_G(w_2) = 4$, and $\{w_1 w_2', v_3' w_3, v_3' w_2'\} \subset E(G)$. Thus the graph F shown in Figure 4 is a subgraph of G. We note that the degree of every vertex in the subgraph F, except possibly for the vertex w_1, is the same as its degree in the graph G; that is, $d_F(v) = d_G(v)$ for all $v \in V(F) \setminus \{w_1\}$.

If $G = F$, then $(D, T) = (\{v_1, w_1, w_2'\}, \{v_2, v_3', w_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, $G \neq F$. We now consider the graph $G'' = G - V(F)$. Every vertex in G'' has degree at least 3, except possibly for vertices in $N_G(w_1) \setminus V(F)$ which have degree at least 2 in G''. By Claim A, the graph G'' has no C_5-component. Thus, by Theorem 1, G'' has an exhaustive DT-pair (D'', T''). Now, $(D, T) = (D'' \cup \{v_2, w_2, w_2'\}, T'' \cup \{v_3, v_3', w_3\})$ is a non-exhaustive DT-pair of G, a contradiction. We deduce, therefore, that G' has no C_5-component.

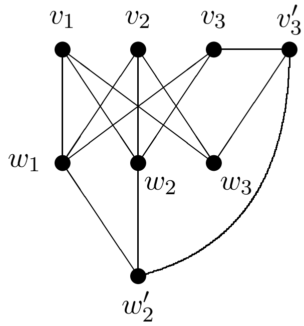


Figure 4. Configuration in the proof of Claim E.

By Theorem 1, G' has an exhaustive DT-pair (D', T'). If $w_2 \in T'$, let $(D,T) = (D' \cup \{w_1\}, T' \cup \{v_2\})$. If $\{v_1, w_2\} \subset D'$, let $(D,T) = (D', T' \cup \{v_2, w_1\})$. If $w_2 \in D'$ and $v_1 \in T'$, let $(D,T) = (D' \cup \{v_2\}, T' \cup \{w_1\})$. In all cases, (D,T) is a non-exhaustive DT-pair of G, a contradiction. We deduce, therefore, that the vertices $v_1, v_2, v_3, w_1, w_2, w_3$ are all of degree 3 in G.

Let $N(v_3) = \{w_1, w_2, v_3'\}$. We now consider the graph G' obtained from $G - \{v_2, v_3, w_1\}$ by adding the edge w_2v_3'. Then, G' has minimum degree at least 2. Since $d_{G'}(u) \geq 3$ for all $u \in V(G') \setminus \{v_1, w_2, w_3\}$, the graph G' contains no C_5-component. Thus, by Theorem 1, G' has an exhaustive DT-pair (D', T').

If $\{v_1, w_2\} \subseteq D'$, then $\{w_3, v_3'\} \subseteq T'$, and let $(D,T) = (D' \cup \{v_3\}, T' \cup \{v_2\})$. If $v_1 \in D'$ and $w_2 \in T'$, then $v_3' \in T'$ and let $(D,T) = (D' \cup \{w_1\}, T' \cup \{v_3\})$. If $v_1 \in T'$ and $w_2 \in D'$, then $w_3 \in T'$ and let $(D,T) = (D' \cup \{v_3\}, T' \cup \{w_1\})$. Finally, if $\{v_1, w_2\} \subseteq T'$, then $\{w_3, v_3'\} \subseteq D'$, and let $(D,T) = (D' \cup \{v_2\}, T' \cup \{v_3\})$. In all cases, (D,T) is a non-exhaustive DT-pair of G, a contradiction which completes the proof of the claim. $\square$

Claim F. G contains no $K_{2,3}$ as a subgraph.

***Proof of Claim* F.** For contradiction, we assume that G contains a $K_{2,3}$-subgraph, i.e., there are two vertices v_1 and v_2 that have $\ell \geq 3$ common neighbours $w_1, w_2, \ldots, w_\ell$. By Claim C, $\{v_1, v_2\}$ and $\{w_1, w_2, \ldots, w_l\}$ are independent sets. We now consider the graph $G' = G - \{v_1, v_2, w_1, w_2, \ldots, w_\ell\}$. By Claims C, D and E, every vertex in $V(G')$ is adjacent in G to at most one vertex in $V(G) \setminus V(G')$. Hence, G' has minimum degree at least 2. By Claim B, G' therefore has no C_5-component. Hence, by Theorem 1, G' has an exhaustive DT-pair (D',T'). Now, $(D,T) = (D' \cup \{v_1, w_1\}, T' \cup \{v_2, w_2\})$ is a non-exhaustive DT-pair of G, a contradiction. $\square$

Claim G. G contains no 4-cycle.

***Proof of Claim* G.** For contradiction, we assume that $C\colon v_1v_2v_3v_4v_1$ is a 4-cycle in G. Let $G' = G - V(C)$. By Claim C and F, every vertex in $V(G')$ is adjacent in G to at most one vertex in $V(G)\setminus V(G')$. Hence, G' has minimum degree at least 2. By Claim B, G' therefore has no C_5-component. Hence, by Theorem 1, G' has an exhaustive DT-pair (D',T'). If a vertex in D' is adjacent to a vertex of C, say to v_1, then $(D,T) = (D' \cup \{v_3\}, T' \cup \{v_1, v_2\})$ is a non-exhaustive DT-pair of G, a contradiction. Hence, no vertex in D' is adjacent to a vertex of C'. Thus, every vertex of C has a neighbour in T'. But then $(D,T) = (D' \cup \{v_1, v_2\}, T')$ is a non-exhaustive DT-pair of G, a contradiction. $\square$

Claim H. G contains no 5-cycle.

***Proof of Claim* H.** For contradiction, we assume that $C\colon v_1v_2v_3v_4v_5v_1$ is a 5-cycle in G. Let $G' = G - V(C)$. By Claim C and G, every vertex in $V(G')$ is adjacent in G to at most one vertex in $V(G)\setminus V(G')$. Hence, G' has minimum degree at least 2. By Claim B, G' therefore has a C_5-component $C'\colon v_6v_8v_{10}v_7v_9v_6$ and, again by Claim B, $V(G) = V(C) \cup V(C')$. We may assume that $v_1v_6 \in E(G)$. By (i), symmetry, and Claims C and G, we may assume that $v_2v_7 \in E(G)$ and $v_3v_8 \in E(G)$. Now Claims C and G imply $v_5v_{10} \in E(G)$, $v_2v_7 \in E(G)$, and $v_4v_9 \in E(G)$, i.e., G is the Petersen graph, a contradiction. $\square$

We now return to our proof of Lemma 4. By Claims C, G, and H, the graph G contains no 3-cycle, 4-cycle, or 5-cycle. Let $P\colon v_1v_2v_3v_4$ be a path in G and let $v_1' \in V(G)\setminus\{v_1, v_3\}$ be a neighbour of v_2. Let $G' = G - \{v_1, v_2, v_3, v_4, v_1'\}$. Since G has girth at least 6, the graph G' has minimum degree at least 2 and contains no C_5-component. Hence, by Theorem 1, G' has an exhaustive DT-pair (D',T').

If a vertex in D' is adjacent to a vertex in $\{v_1, v_1'\}$, say to v_1', let $(D,T) = (D' \cup \{v_1, v_4\}, T' \cup \{v_2, v_3\})$. If every vertex in $\{v_1, v_4, v_1'\}$ has a neighbour in T', let $(D,T) = (D' \cup \{v_2, v_3\}, T' \cup \{v_1, v_4\})$. If every vertex of $\{v_1, v_1'\}$ has a neighbour in T' and v_4 has a neighbour in D', then $(D,T) = (D' \cup \{v_2\}, T' \cup \{v_3, v_4\})$. In all cases, (D,T) is a non-exhaustive DT-pair of G, a contradiction which completes the proof of the lemma. $\blacksquare$

With the help of Lemma 4, the proof of Theorem 2 follows readily. Recall the statement of Theorem 2: *If G is a graph of minimum degree at least* 3 *with at least one component different from the Petersen graph, then G contains a dominating set D and a total dominating set T which are disjoint and satisfy $|D| + |T| < |V(G)|$.*

***Proof of Theorem* 2.** We prove the result by induction on the number of edges between vertices of degree at least 4. If there is no such edge, then the result follows immediately from Lemma 4. Hence, we assume that $e \in E(G)$ is such an edge. If e is a bridge, then deleting e results in two components G_1 and G_2. If both of G_1 and G_2 are the Petersen graph, then the result follows from Lemma 3(c). If at least one of G_1 or G_2 is not the Petersen graph, then the result follows by induction. Hence, we may assume that e is no bridge. If $G' = G - e$ is the Petersen graph, then the result follows from Lemma 3(b). If G' is not the Petersen graph, then the result follows by induction. This completes the proof of the theorem. ∎

References

[1] N.J. Calkin and P. Dankelmann, *The domatic number of regular graphs*, Ars Combin. **73** (2004) 247–255.

[2] G.S. Domke, J.E. Dunbar and L.R. Markus, *The inverse domination number of a graph*, Ars Combin. **72** (2004) 149–160.

[3] U. Feige, M.M. Halldórsson, G. Kortsarz and A. Srinivasan, *Approximating the domatic number*, SIAM J. Comput. **32** (2002) 172–195.

[4] C. Godsil and G. Royle, Algebraic Graph Theory (Springer, 2001).

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (eds), Domination in graphs: Advanced topics (Marcel Dekker, New York, 1998).

[7] S.M. Hedetniemi, S.T. Hedetniemi, R.C. Laskar, L. Markus and P.J. Slater, *Disjoint dominating sets in graphs*, in: Proc. Internat. Conf. Discrete Math., ICDM 2006, 87–100, Ramanujan Math. Soc., Lecture Notes Series in Mathematics, 2008.

[8] M.A. Henning, C. Löwenstein and D. Rautenbach, *Remarks about disjoint dominating sets*, Discrete Math. **309** (2009) 6451–6458.

[9] M.A. Henning and J. Southey, *A note on graphs with disjoint dominating and total dominating sets*, Ars Combin. **89** (2008) 159–162.

[10] M.A. Henning and J. Southey, *A characterization of graphs with disjoint dominating and total dominating sets*, Quaestiones Mathematicae **32** (2009) 119–129.

[11] V.R. Kulli and S.C. Sigarkanti, *Inverse domination in graphs*, Nat. Acad. Sci. Lett. **14** (1991) 473–475.

[12] C. Löwenstein and D. Rautenbach, *Pairs of disjoint dominating sets and the minimum degree of graphs*, Graphs Combin. **26** (2010) 407–424.

[13] O. Ore, Theory of Graphs, Amer. Math. Soc. Transl. **38** (Amer. Math. Soc., Providence, RI, 1962) 206–212.

[14] B. Zelinka, *Total domatic number and degrees of vertices of a graph*, Math. Slovaca **39** (1989) 7–11.

[15] B. Zelinka, *Domatic numbers of graphs and their variants: A survey*, in: Domination in graphs: Advanced topics, T.W. Haynes *et al.* eds (Marcel Dekker, New York, 1998), 351–377.

Received 18 July 2009
Revised 9 November 2009
Accepted 9 November 2009