Discussiones Mathematicae
Graph Theory 30 (2010) 425–435

A CHARACTERIZATION OF (γ_t, γ_2)-TREES *

You Lu, Xinmin Hou, Jun-Ming Xu and Ning Li

Department of Mathematics
University of Science and Technology of China
Hefei, Anhui, 230026, China

e-mail: xmhou@ustc.edu.cn

Abstract

Let $\gamma_t(G)$ and $\gamma_2(G)$ be the total domination number and the 2-domination number of a graph G, respectively. It has been shown that: $\gamma_t(T) \leq \gamma_2(T)$ for any tree T. In this paper, we provide a constructive characterization of those trees with equal total domination number and 2-domination number.

Keywords: domination, total domination, 2-domination, (λ, μ)-tree.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, the closed neighborhood and the degree of a vertex $v \in V(G)$ are denoted by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$ and $deg_G(v) = |N_G(v)|$, respectively. For $u \in V(G)$, u is a leaf of G if $deg_G(u) = 1$ and a support vertex of G if u has a leaf as its neighbor in G. For a pair of vertices $u, v \in V(G)$, the distance $d_G(u, v)$ of u and v is the length of a shortest uv-path in G. The diameter of G is $d(G) = \max\{d_G(u, v) : u, v \in V(G)\}$.

For any set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$ and we write $G - S$ for $G[V(G) - S]$. For convenience, we write $G - v$ for

*The work was supported by NNSF of China (No.10701068 and No.10671191).

$G-\{v\}$ for $v \in V(G)$. For any edge $xy \in E(G)$, we use $G - xy$ to denote the subgraph induced by $E(G) - \{xy\}$.

Total domination in graphs was introduced by Cockayne *et al.* [3]. A subset $S \subseteq V(G)$ is a total dominating set (denoted by TDS) if every vertex of $V(G)$ has at least one neighbor in S. The total domination number (denoted by $\gamma_t(G)$) is the minimum cardinality among the total dominating sets of G. The total dominating set of G with cardinality $\gamma_t(G)$ will be called a γ_t-set of G. For a survey on total domination in graphs one can refer to Henning [12].

Let p be a positive integer. In [6], Fink and Jacobson introduced the concept of p-domination. A p-dominating set of G is a subset S of $V(G)$ such that every vertex not in S has at least p neighbors in S. The p-domination number $\gamma_p(G)$ is the minimum cardinality of a p-dominating set of G. The p-dominating set of G with cardinality $\gamma_p(G)$ will be called a γ_p-set of G. Note that p-domination is the classic domination when $p = 1$. For any $S, T \subseteq V(G)$, S p-dominates T in G if every vertex of T not in S has at least p neighbors in S.

An area of research in domination of graphs that has received considerable attention is the characterization of classes of graphs with equal domination parameters. For any two graph parameters λ and μ, G is called a (λ, μ)-graph if $\lambda(G) = \mu(G)$. Characterizing the (λ, μ)-graphs has been investigated in many papers (for example [1, 4, 7, 11, 13]).

In [8], Haynes *et al.* showed that for all trees the total domination number is equal or less than the 2-domination number, and they also gave a necessary condition for all trees with equal total domination number and 2-domination number. In this paper, we give a constructive characterization of trees with equal total domination number and 2-domination number.

2. A Characterization

Let $P_n = u_1 \cdots u_n$ $(n \geq 1)$ be a path with vertex set $\{u_1, \ldots, u_n\}$ and $K(t)$ $(t \geq 2)$ be the tree obtained from a star $K_{1,t}$ with support vertex u by adding a path P_2 to every leaf of $K_{1,t}$. Denote u by $cent(K(t))$. For convenience, we denote a path P_4 by $K(1)$ and let $cent(K(1))$ represent one leaf of P_4.

To state the characterization of (γ_t, γ_2)-trees, we introduce the six types of operations.

Type-1 operation: Attach a path P_1 to each of the two vertices u, w of a tree T, respectively, where u, w locate at a component P_l of $T - xy$ for some edge xy such that either x is in a γ_2-set of T and $P_l = P_4 = uvwx$ or y is in a γ_2-set of T and $P_l = P_5 = uvwxx'$.

Type-2 operation: Attach a path P_2 to a vertex v of a tree T by joining one leaf of P_2 to v, where v is a vertex such that $T - v$ has a component P_2.

Type-3 operation: Attach t (≥ 1) paths P_3 to a vertex v of a tree T by joining one leaf of each P_3 to v, where v is a vertex such that either $T - v$ has a component P_2 or $T - v$ has two components P_1 and P_3 that a leaf of P_3 is adjacent to v in T.

Type-4 operation: Attach a path P_3 to a vertex v of a tree T by joining its support vertex to v, where v is a vertex such that v is not contained in any γ_t-set of T and $T - v$ has a component P_3 that one of its leaves is adjacent to v in T.

Type-5 operation: Attach a tree $K(t)$ $(t \geq 1)$ to a vertex v of a tree T by joining $cent(K(t))$ to v, where v is in a γ_2-set of T if $t = 1$.

Type-6 operation: Attach a path P_5 to a vertex v of a tree T by joining one of its support vertices to v, where v is a vertex such that $T - v$ has a component $H \in \{P_2, P_3, P_5\}$ and v is adjacent to a support vertex of H if $H = P_5$.

From the survey on total domination in graphs [12], it is hard to recognize whether a vertex v is in no γ_t-set or no γ_2-set.

Let $\mathscr{A}$ be the family of trees with equal total domination number and 2-domination number, that is

$$\mathscr{A} = \{T : T \text{ is a tree satisfying } \gamma_t(T) = \gamma_2(T)\}.$$

We also define the family $\mathscr{B}$ as:

$$\mathscr{B} = \{T : T \text{ is obtained from } P_3 \text{ by a finite sequence of operations of Type-}i, \text{ where } 1 \leq i \leq 6\}.$$

We shall show that

Theorem 1. $\mathscr{A} = \mathscr{B} \cup \{P_2\}$.

3. The Proof of Theorem 1

We need some known results.

Lemma 2 ([8])**.** *Let T be a tree without isolated vertices, then $\gamma_t(T) \leq \gamma_2(T)$.*

Lemma 3 ([2])**.** *Every 2-dominating set of a graph G contains all leaves of G.*

Lemma 4 ([8])**.** *If T is a tree satisfying $\gamma_t(T) = \gamma_2(T)$, then every support vertex of T is adjacent to at most two leaves.*

Let T be a rooted tree. For every $v \in V(T)$, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively, and $D[v] = D(v) \cup \{v\}$. Define

$$C'(v) = \{u \in C(v) : \text{every vertex of } D[u] \text{ has distance at most two from } v \text{ in } T\}.$$

By Lemma 4, each vertex of $C'(v)$ has degree at most three. Hence we can partition $C'(v)$ into $C'_1(v), C'_2(v), C'_3(v)$ such that every vertex of $C'_i(v)$ has degree i in T, $i = 1, 2$ or 3.

Lemma 5. *Let T be a rooted tree satisfying $\gamma_t(T) = \gamma_2(T)$ and $w \in V(T)$. We have*

(1) *If $C'_2(w) \neq \emptyset$, then $C'_1(w) = C'_3(w) = \emptyset$.*

(2) *If $C'_3(w) \neq \emptyset$, then $C'_1(w) = C'_2(w) = \emptyset$ and $|C'_3(w)| = 1$.*

(3) *If $C(w) = C'(w) \neq C'_1(w)$, then $C'_1(w) = C'_3(w) = \emptyset$.*

Proof. Let $C'_1(w) = \{x_1, \ldots, x_r\}$, $C'_2(w) = \{y_1, \ldots, y_s\}$ and $C'_3(w) = \{z_1, \ldots, z_t\}$. Then $|C'_1(w)| = r$, $|C'_2(w)| = s$ and $|C'_3(w)| = t$. For each $i = 1, \ldots, t$, let u_i be a leaf adjacent with z_i in T. Let $T' = T - \{x_1, \ldots, x_r, u_1, \ldots, u_t\}$.

(1). We prove that if $s \geq 1$ then $r + t = 0$. Assume $r + t \geq 1$. Since $s \geq 1$, we can choose a γ_2-set D of T such that $w \in D$, and a γ_t-set S' of T' such that $w \in S'$. It is not difficult to check that $D - \{x_1, \ldots, x_r, u_1, \ldots, u_t\}$ is a 2-dominating set of T' and S' is a TDS of T. Hence,

$$\begin{aligned}\gamma_t(T') &= |S'| \geq \gamma_t(T) = \gamma_2(T) \\ &= |D| > |D - \{x_1, \ldots, x_r, u_1, \ldots, u_t\}| \geq \gamma_2(T'),\end{aligned}$$

a contradiction with Lemma 2.

(2) and (3). Suppose either $C'_3(w) \neq \emptyset$ or $C(w) = C'(w) \neq C'_1(w)$. Then $s + t \geq 1$. Choose a γ_t-set S' of T' such that $w \in S'$. Then S' is also a TDS of T. Hence $\gamma_t(T') = |S'| \geq \gamma_t(T)$. By the definition of γ_2-set and Lemma 3, there is a γ_2-set, denoted by D, of T satisfying $D \cap \{y_1, \ldots, y_s, z_1, \ldots, z_t\} = \emptyset$. Then $(D \cap V(T')) \cup \{w\}$ is a 2-dominating set of T'. Hence

$$\begin{aligned}\gamma_2(T') &\leq |(D \cap V(T')) \cup \{w\}| \\ &\leq |D| - (r + t) + 1 \\ &= \gamma_2(T) - (r + t) + 1 \\ &= \gamma_t(T) - (r + t) + 1.\end{aligned}$$

If $t \geq 1$, then $\gamma_2(T') \leq \gamma_t(T) \leq \gamma_t(T') \leq \gamma_2(T')$, the last inequality is by Lemma 2, which implies that $r + t = 1$ and $w \notin D$. So $r = 0$ and $t = 1$. By (1), we have $s = 0$. Hence (2) is valid.

If $C(w) = C'(w) \neq C'_1(w)$, then $s + t \geq 1$. By (1) and (2), $r = 0$. We show that $t = 0$. If not, similar to the proof of (2), we have $w \notin D$, $t = 1$ and $s = 0$. Since $C(w) = C'(w)$, we know that $deg_T(w) = 2$. To 2-dominate w, $z_1 \in D$, which contradicts with the choice of D. ■

Lemma 6. *If $T' \in \mathscr{A}$ with order at least three and T is obtained from T' by an operation of Type-i, $1 \leq i \leq 6$, then $T \in \mathscr{A}$.*

Proof. Since $T' \in \mathscr{A}$, we have $\gamma_t(T') = \gamma_2(T')$. By Lemma 2, we only need to prove that $\gamma_t(T) \geq \gamma_2(T)$.

Case 1. $i = 1$. Assume that T is obtained from T' by attaching u' and w' to u and w, respectively, where u and w satisfy the conditions of Type-1 operation. Then there is an edge xy in T' such that either x is in a γ_2-set of T' and $T' - xy$ has a component $P_4 = uvwx$, or y is in a γ_2-set of T' and $T' - xy$ has a component $P_5 = uvwxx'$. Clearly, $\gamma_t(T') = \gamma_t(T) - 1$.

If $T' - xy$ contains a path $P_4 = uvwx$, then let D' be a γ_2-set of T' containing x. From Lemma 3 and the definition of γ_2-set, we have $D' \cap$

$\{u, v, w, x\} = \{u, w\}$ or $\{u, v\}$. Thus $D = (D' - \{u, v, w\}) \cup \{u', v, w'\}$ is a 2-dominating set of T with $|D| = |D'| + 1 = \gamma_2(T') + 1$. So, $\gamma_t(T) = \gamma_t(T') + 1 = \gamma_2(T') + 1 = |D| \geq \gamma_2(T)$.

If $T' - xy$ contains a path $P_5 = uvwxx'$, then let D' be a γ_2-set of T' containing y. By Lemma 3 and the definition of γ_2-set, we have $D' \cap \{u, v, w, x, x'\} = \{u, w, x'\}$. Thus $D = (D' \setminus \{u, w\}) \cup \{u', v, w'\}$ is a 2-dominating set of T with $|D| = |D'|+1 = \gamma_2(T')+1$. So, $\gamma_t(T) = \gamma_t(T')+1 = \gamma_2(T') + 1 = |D| \geq \gamma_2(T)$.

Case 2. $i = 2$. Assume that T is obtained from T' by attaching a path $P_2 = uu'$ to a vertex v of T' such that $uv \in E(T)$, where $T' - v$ has a component $P_2 = wx$ satisfying $vw \in E(T')$. It is easy to show that $\gamma_t(T) = \gamma_t(T') + 1$. By the definition of γ_2-set, there exists a γ_2-set D' of T' containing the vertex v. Then $D' \cup \{u'\}$ is a 2-dominating set of T. Hence, $\gamma_t(T) = \gamma_t(T') + 1 = \gamma_2(T') + 1 = |D' \cup \{u'\}| \geq \gamma_2(T)$.

Case 3. $i = 3$. Assume that T is obtained from T' by attaching t (≥ 1) paths P_3, denoted by $\{x_iy_iz_i : 1 \leq i \leq t\}$, to a vertex v of T' such that $x_iv \in E(T)$ for $1 \leq i \leq t$, where either $T' - v$ has a component $P_2 = uu'$ satisfying $uv \in E(T')$, or $T' - v$ has a component $P_1 = u_0$ and a component $P_3 = uu'u''$ satisfying $uv \in E(T')$. By the definitions of γ_t-set and γ_2-set, we can easily prove that $\gamma_t(T) \geq \gamma_t(T') + 2t$ and $\gamma_2(T') + 2t \geq \gamma_2(T)$. Since $\gamma_t(T') = \gamma_2(T')$, we have $\gamma_t(T) \geq \gamma_t(T') + 2t = \gamma_2(T') + 2t \geq \gamma_2(T)$.

Case 4. $i = 4$. Assume that T is obtained from T' by attaching a path $P_2 = xyz$ to a vertex v of T' such that $yv \in E(T)$, where v is not in any γ_t-set of T' and $T' - v$ has a component $P_3 = uu'u''$ satisfying $uv \in E(T')$. For any γ_2-set D' of T', $D' \cup \{x, z\}$ is a 2-dominating set of T. So $\gamma_2(T') + 2 \geq \gamma_2(T)$. Let S be a γ_t-set of T containing the vertex u, then $y \in S$ and $|S \cap \{v, x, z\}| = 1$.

If $v \notin S$, then $|S \cap V(T')| = |S| - 2 = \gamma_t(T) - 2 \geq \gamma_t(T')$ since $S \cap V(T')$ is a TDS of T'. By $\gamma_t(T') = \gamma_2(T')$, $\gamma_t(T) \geq \gamma_t(T')+2 = \gamma_2(T')+2 \geq \gamma_2(T)$.

If $v \in S$, then $S \cap \{v, x, z\} = \{v\}$ and $|S \cap V(T')| = |S| - 1 = \gamma_t(T) - 1 \geq \gamma_t(T')$ since $S \cap V(T')$ is a TDS of T'. Suppose that $\gamma_t(T) \leq \gamma_2(T) - 1$, then, by $\gamma_t(T') = \gamma_2(T')$, $\gamma_2(T) \geq \gamma_t(T) + 1 \geq \gamma_t(T') + 2 = \gamma_2(T') + 2 \geq \gamma_2(T)$. So $|S \cap V(T')| = \gamma_t(T) - 1 = \gamma_t(T')$, and $S \cap V(T')$ is a γ_t-set of T' containing v, which contradicts with v is not in any γ_t-set of T'. Hence $\gamma_t(T) \geq \gamma_2(T)$.

Case 5. $i = 5$. Assume that T is obtained from T' by attaching a $K(t)$ $(t \geq 1)$ to a vertex v of T' by joining $u = cent(K(t))$ to v, where v satisfies the condition of Type-5 operation. Clearly, $\gamma_t(T) \geq \gamma_t(T') + 2t$.

If $t \geq 2$, then, by $\gamma_t(T') = \gamma_2(T')$, it is obvious that $\gamma_t(T) \geq \gamma_t(T') + 2t = \gamma_2(T') + 2t \geq \gamma_2(T)$.

If $t = 1$, then let $K(1) = uxyz$ and D' be a γ_2-set of T' containing v. Thus $D' \cup \{z, x\}$ is a 2-dominating set of T. Hence $\gamma_t(T) \geq \gamma_t(T') + 2 = \gamma_2(T') + 2 = |D' \cup \{z, x\}| \geq \gamma_2(T)$.

Case 6. $i = 6$. Assume that T is obtained from T' by attaching a path $P_5 = x_1x_2x_3x_4x_5$ to a vertex v of a tree T such that $x_2v \in E(T)$, where T' and v satisfy the condition of Type-6 operation. Then we can choose a subset S of $V(T)$ as a γ_t-set of T such that $S \cap N_{T'}(v) \neq \emptyset$. Thus $S \cap V(T')$ is a TDS of T' and then $|S \cap V(T')| \geq \gamma_t(T')$. By the definition of γ_2-set, we have $\gamma_2(T') + 3 \geq \gamma_2(T)$. Hence $\gamma_t(T) = |S| = |S \cap V(P_5)| + |S \cap V(T')| \geq 3 + \gamma_t(T') = 3 + \gamma_2(T') \geq \gamma_2(T)$. ■

Lemma 7. *If $T \in \mathscr{A}$ with order at least three, then $T \in \mathscr{B}$.*

Proof. Let $n = |V(T)|$. Since $T \in \mathscr{A}$, we have $\gamma_t(T) = \gamma_2(T)$. If $d(T) = 2$, then T is a star $K_{1,n-1}$. Since $2 = \gamma_t(T) = \gamma_2(T) = n - 1$, $n = 3$. So $T = P_3 \in \mathscr{B}$. If $d(T) = 3$, then T contains exactly $n - 2$ leaves. Since $2 = \gamma_t(T) = \gamma_2(T) \geq n - 2$, $n = 4$. So $T = P_4$. However, $\gamma_2(P_4) = 3 \neq \gamma_t(P_4)$, a contradiction. If $d(T) = 4$, then there is a vertex w of T with distance at most two from the other vertices in T. Hence $C(w) = C'(w) \neq C'_1(w)$ if we root T at w. By (3) of Lemma 5, T is a tree obtained from a star $K_{1,t}$ by attaching a vertex to every leaf of $K_{1,t}$, where $2t + 1 = n$. Clearly, T can be obtained from P_3 by $t - 1$ operations of Type-2. By Lemma 6, $T \in \mathscr{B}$. In the following, we will assume that $d(T) \geq 5$ and prove $T \in \mathscr{B}$ by induction on the order of $n = |V(T)|$.

If $n < 6$, then $d(G) \leq 4$. The result is true from the above proof. If $n = 6$, then $T = P_6 \in \mathscr{B}$. This establishes the base cases. Assume that $n > 6$ and the result is true for all the trees T' with order $|V(T')| < n$, that is, if $T' \in \mathscr{A}$ with order $|V(T')| < n$ then $T' \in \mathscr{B}$.

Claim 1. If there is a vertex $a \in V(T)$ such that $T - a$ contains at least two components P_2, then $T \in \mathscr{B}$.

Proof. Assume that $P_2 = bb'$ and $P_2 = cc'$ are two components of $T - a$ such that $ab, ac \in E(T)$. Let $T' = T - \{b, b'\}$, then we use S' and D to denote a γ_t-set of T' containing a and a γ_2-set of T, respectively. Since $a \in S'$, $S' \cup \{b\}$ is a TDS of T, and so $\gamma_t(T') \geq \gamma_t(T) - 1$. Since D is a γ_2-set of T, $D \cap \{a, b, b'\} = \{a, b'\}$ by the definition of γ_2-set. So $D \cap V(T')$ is a 2-dominating set of T'. Hence $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D \cap V(T')| \geq \gamma_2(T')$. By Lemma 2, $\gamma_t(T') = \gamma_2(T')$, and so $T' \in \mathscr{A}$. By the induction on T', $T' \in \mathscr{B}$. Since T can be obtained from T' by Type-2 operation. So $T \in \mathscr{B}$. The claim holds.

By Claim 1, we only need consider the case that, for every vertex a, $T - a$ has at most one component P_2. Let $P = uvwxyz \cdots r$ be a longest path in T and we root T at r.

Clearly, $C(w) = C'(w) \neq C'_1(w)$. By (3) of Lemma 5, $C'_1(w) = C'_3(w) = \emptyset$. Hence $P_3 = uvw$ is a component of $T - x$. Let t be the number of components P_3 of $T[D(x)]$ such that a leaf of every P_3 is adjacent to x. Note that $T[D(x)]$ possible has other components. We suppose $T[D(x)]$ has s components P_3 with its support vertex is adjacent to x, k components P_2 and h components P_1. By Lemmas 4 and 5, $s, k \in \{0, 1\}$ and $h \in \{0, 1, 2\}$. Denote the t components P_3 of $T[D(x)]$ with one of its leaves is adjacent to x in T by $P_3 = u_i v_i w_i$ $(1 \leq i \leq t)$, where $xw_i \in E(T)$ for $1 \leq i \leq t$. We prove the result according to the values of $\{s, k, h\}$.

Case 1. $s = k = h = 0$.
Then $T[D[x]] = K(t), t \geq 1$. Let $T' = T - D[x]$. Then $3 \leq |V(T')| < n$. Clearly, $\gamma_t(T') \geq \gamma_t(T) - 2t$. Let D be a γ_2-set of T such that D contains as few vertices of $T[D[x]]$ as possible. Then, $x \notin D$ and $|D \cap D[x]| = 2t$ by the definition of γ_2-set. So $D \cap V(T')$ is a 2-dominating set of T'. Thus $\gamma_t(T') \geq \gamma_t(T) - 2t = \gamma_t(T) - 2t = |D \cap V(T')| \geq \gamma_2(T)$. By Lemma 2, $\gamma_t(T') = \gamma_2(T')$ and $D \cap V(T')$ is a γ_2-set of T'. So $T' \in \mathscr{A}$. Applying the inductive hypothesis on T', $T' \in \mathscr{B}$.

If $t \geq 2$, then it is obvious that T is obtained from T' by Type-5 operation, and so $T \in \mathscr{B}$.

If $t = 1$, then $T[D[x]] = K(1) = P_4 = uvwx$, and so $D \cap \{u, v, w, x\} = \{u, w\}$. To 2-dominate x, $y \in D$, and so $y \in D \cap V(T')$, which implies that y is in some γ_2-set of T'. Hence T can be obtained from T' by Type-5 operation, and $T \in \mathscr{B}$, too.

Case 2. $s \neq 0$. By the proof procedure of Lemma 5, $s = 1$ and $k = h = 0$. Denote the component P_3 of $T[D[x]]$ whose support vertex is adjacent to x in T by $P_3 = abc$ and let $T' = T - \{a, b, c\}$. Clearly, $3 \leq |V(T')| < n$. Let D be a γ_2-set of T which does not contain b.

We claim that x is not in any γ_t-set of T'. Suppose that T' has a γ_t-set containing x, denoted by S', then $S' \cup \{b\}$ is a TDS of T. So $\gamma_t(T') \geq \gamma_t(T) - 1$. Since $b \notin D$, then $D \cap V(T')$ is a 2-dominating set of T'. Hence $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D \cap V(T')| + 1 \geq \gamma_2(T') + 1$, which contradicts $\gamma_t(T') \leq \gamma_2(T')$. The claim holds. Therefore, T can be obtained from T' by Type-4 operation.

Now we prove that $T' \in \mathscr{B}$. Let S' be a γ_t-set of T'. By the above claim, $x \notin S'$. Since $S' \cup \{x, b\}$ is a TDS of T, $\gamma_t(T') \geq \gamma_t(T) - 2$. Since $b \notin D$, $D \cap V(T')$ is a 2-dominating set of T'. Hence $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2(T) - 2 = |D \cap V(T')| \geq \gamma_2(T')$. By Lemma 2, $\gamma_t(T') = \gamma_2(T')$, which implies $T' \in \mathscr{A}$. Applying the inductive hypothesis on T', $T' \in \mathscr{B}$, and so $T \in \mathscr{B}$.

Case 3. $k \neq 0$. By the proof procedure of Lemma 5, $s = h = 0$.
Let $T' = T - \cup_{i=1}^{t}\{u_i, v_i, w_i\}$. It is clearly that $3 \leq |V(T')| < n$ and T is obtained from T' by Type-3 operation.

We only need to prove that $T' \in \mathscr{B}$. Let $S' \subseteq V(T')$ be a γ_t-set of T', then $S' \cup (\cup_{i=1}^{t}\{v_i, w_i\})$ is a TDS of T. So $\gamma_t(T') \geq \gamma_t(T) - 2t$. Since $T - x$ has a component $P_2 = ab$, we can choose $D \subseteq V(T)$ as a γ_2-set of T containing x. Then $D \cap V(T')$ is a 2-dominating set of T', and so $\gamma_2(T) = |D| = 2t + |D \cap V(T')| \geq 2t + \gamma_2(T')$. By $\gamma_t(T) = \gamma_2(T)$, we have $\gamma_t(T') = \gamma_2(T')$, and so $T' \in \mathscr{A}$. Applying the inductive hypothesis on T', $T' \in \mathscr{B}$.

Case 4. $h \neq 0$. By Lemmas 4 and 5, $h \in \{1, 2\}$ and $s = k = 0$.
We claim that $h = 1$. If not, then $h = 2$. We denote the two components P_1 of $T[D(x)]$ by x' and x''. Let $T' = T - x''$. Clearly, $\gamma_t(T') = \gamma_t(T)$. Let D be a γ_2-set of T containing $\{w_1, \ldots, w_t\}$. By Lemma 3, $\{x', x''\} \subseteq D$. Since $D \cap V(T')$ is 2-dominating set of T' with $|D \cap V(T')| = \gamma_2(T) - 1$, we have $\gamma_t(T') = \gamma_t(T) = \gamma_2(T) > \gamma_2(T) - 1 \geq \gamma_2(T')$, which contradicts $\gamma_t(T') \leq \gamma_2(T')$.

Case 4.1. $t \geq 2$.
Let $T' = T - \cup_{i=2}^{t}\{u_i, v_i, w_i\}$, then T is obtained from T' by Type-3 operation. By the definitions of γ_t-set and γ_2-set, it is easy to see that $\gamma_t(T') + 2(t-1) = \gamma_t(T)$ and $\gamma_2(T') + 2(t-1) = \gamma_2(T)$. Hence $\gamma_t(T') = \gamma_2(T')$ and $T' \in \mathscr{A}$. Applying the inductive hypothesis on T', $T' \in \mathscr{B}$, and so $T \in \mathscr{B}$.

Case 4.2. $t = 1$. Denote the component P_1 of $T[D(x)]$ by $P_1 = x'$.

Case 4.2.1. If $T[D(y) \setminus D[x]]$ has a component $H \in \{P_2, P_3, P_5\}$, then let $T' = T - D[x]$. We can easily check that T is obtained from T' by Type-6 operation. By the definition of γ_2-set, $\gamma_2(T') + 3 = \gamma_2(T)$. For any γ_t-set S' of T', $S' \cup \{v, w, x\}$ is a TDS of T. So $\gamma_t(T') \geq \gamma_t(T) - 3 = \gamma_2(T) - 3 = \gamma_2(T')$. By Lemma 2, $\gamma_t(T') = \gamma_2(T')$ and $T' \in \mathscr{A}$. Applying the inductive hypothesis on T', $T' \in \mathscr{B}$, and so $T \in \mathscr{B}$.

Case 4.2.2. If $T[D(y) \setminus D[x]]$ has no component P_2, P_3 or P_5, we consider the structure of $T[D(y)]$. By the above discussion, $T[D(y)]$ consists of a component $P_5 = uvwxx'$ and ℓ components P_1, denoted by $\{y_1, \ldots, y_\ell\}$. By Lemma 4, $\ell \leq 2$. However, if $\ell = 2$, then let $T' = T - D[y]$. It can be easily checked that $\gamma_t(T') + 4 \geq \gamma_t(T) = \gamma_2(T) = \gamma_2(T') + 5$, which contradicts $\gamma_t(T') \leq \gamma_2(T')$. Hence $\ell \leq 1$.

Let $T' = T - \{u, x'\}$. Then we can easily check that $\gamma_t(T') + 1 = \gamma_t(T)$. Let D be a γ_2-set of T such that D contains as few vertices of $D[y]$ as possible and $D \cap D[x] = \{u, w, x'\}$. Then $D' = (D \setminus \{u, w, x'\}) \cup \{v, x\}$ is a 2-dominating set of T'. So $\gamma_t(T') = \gamma_t(T) - 1 = \gamma_2(T) - 1 = |D'| \geq \gamma_2(T')$, which implies that $\gamma_t(T') = \gamma_2(T')$ and D' is a γ_2-set of T'. By $\gamma_t(T') = \gamma_2(T')$, $T' \in \mathscr{A}$. Applying the inductive hypothesis to T', $T' \in \mathscr{B}$.

If $\ell = 0$, then $deg_T(y) = 2$. Since $x \notin D$, to 2-dominate y, $y \in D$. Thus y is in the γ_2-set D' of T'. Hence T is obtained from T' by Type-1 operation. Thus $T \in \mathscr{B}$.

If $\ell = 1$, then $deg_T(y) = 3$. Since $x \notin D$, to 2-dominate y, we have $y \notin D$ and $z \in D$ by the choice of D. Thus z is in the γ_2-set D' of T'. Hence T is obtained from T' by Type-1 operation. Thus $T \in \mathscr{B}$.

This completes the proof of Lemma 7. ■

Note that $\{P_2, P_3\} \subseteq \mathscr{A}$. Lemma 6 implies that $\mathscr{B} \subseteq \mathscr{A}$ and Lemma 7 implies that $\mathscr{A} \subseteq \mathscr{B} \cup \{P_2\}$. Hence Theorem 1 is true.

References

[1] M. Blidia, M. Chellalia and T.W. Haynes, *Characterizations of trees with equal paired and double domination numbers*, Discrete Math. **306** (2006) 1840–1845.

[2] M. Blidia, M. Chellali and L. Volkmann, *Some bounds on the p-domination number in trees*, Discrete Math. **306** (2006) 2031–2037.

[3] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980) 211–219.

[4] E.J. Cockayne, O. Favaron, C.M. Mynhardt and J. Puech, *A characterization of* (γ, i)*-trees*, J. Graph Theory **34** (2000) 277–292.

[5] G. Chartrant and L. Lesniak, Graphs & Digraphs, third ed. (Chapman & Hall, London, 1996).

[6] J.F. Fink and M.S. Jacobson, *n-Domination in graphs*, in: Y. Alavi, A.J. Schwenk (eds.), Graph Theory with Applications to Algorithms and Computer Science (Wiley, New York, 1985) 283–300.

[7] F. Harary and M. Livingston, *Characterization of trees with equal domination and independent domination numbers*, Congr. Numer. **55** (1986) 121–150.

[8] T.W. Haynes, S.T. Hedetniemi, M.A. Henning and P.J. Slater, *H-forming sets in graphs*, Discrete Math. **262** (2003) 159–169.

[9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (New York, Marcel Deliker, 1998).

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (New York, Marcel Deliker, 1998).

[11] T.W. Haynes, M.A. Henning and P.J. Slater, *Strong quality of domination parameters in trees*, Discrete Math. **260** (2003) 77–87.

[12] M.A. Henning, *A survey of selected recently results on total domination in graphs*, Discrete Math. **309** (2009) 32–63.

[13] X. Hou, *A characterization of* $(2\gamma, \gamma_p)$*-trees*, Discrete Math. **308** (2008) 3420–3426.

Received 27 February 2009
Revised 28 July 2009
Accepted 1 September 2009