Discussiones Mathematicae
Graph Theory 30 (2010) 407–423

STRUCTURE OF THE SET OF ALL MINIMAL TOTAL DOMINATING FUNCTIONS OF SOME CLASSES OF GRAPHS

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Abstract

In this paper we study some of the structural properties of the set of all minimal total dominating functions ($\mathfrak{F}_T$) of cycles and paths and introduce the idea of function reducible graphs and function separable graphs. It is proved that a function reducible graph is a function separable graph. We shall also see how the idea of function reducibility is used to study the structure of $\mathfrak{F}_T(G)$ for some classes of graphs.

Keywords: minimal total dominating functions (MTDFs), convex combination of MTDFs, basic minimal total dominating functions (BMTDFs), simplex, polytope, simplicial complex, function separable graphs, function reducible graphs.

2010 Mathematics Subject Classification: Primary 05C69; Secondary 05C35.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected graph which does not contain loops and multiple edges. In this paper, unless specified otherwise, we follow the terminology of D.B. West [9]. Total domination, as an analogue to domination, is well studied by many graph theorists. For the terms and definitions related to domination, which are not given in this paper, readers may refer the books Fundamentals of Domination and Domination in Graphs — Advanced Topics [5, 6]. A *total dominating set* of $G = (V, E)$ is a subset S of V such that every vertex of V is adjacent to at least one vertex in S. Smallest such set is called a *minimal total dominating set*. The characteristic functions of the dominating set is a $0-1$ valued function such that, the sum of the function values over the open neighborhood of each vertex is at least one.

Fractional analog of the total dominating set is a *total dominating function* (TDF) defined as the real valued function $f : V \to [0, 1]$ such that

$$\sum_{x \in N(v)} f(x) \geq 1$$

for all $v \in V$, where $N(v)$ is the open neighborhood of v. This definition was first given by Hedetniemi and Wimer [3] in 1994. A *minimal total dominating function* (MTDF) is a TDF such that f is not a TDF if for any $v \in V$, the value of $f(v)$ is decreased.

For an MTDF f of G, denote $\sum_{x \in N(v)} f(x)$ by $f(N(v))$. The *boundary* of f or B_f is $\{v \in V : \sum_{x \in N(v)} f(x) = 1\}$ and the *positive set* of f or P_f is $\{v \in V : f(v) > 0\}$. For two subsets A and B of V, we write $A \to_t B$ if every vertex in B is adjacent to some vertex in A. Identifying BMTDFs from a collection of MTDFs is not a difficult task, if we use the following theorem.

Theorem 1.1 [3]. *A total dominating function f of the graph G is a minimal total dominating function if and only if $B_f \to_t P_f$.*

An interpolation problem motivated to define the convex combination of minimal dominating functions. This problem can be stated as follows. “Given two minimal dominating functions f and g of the graph G and for any real number x, such that $ag(f) < x < ag(g)$, where $ag(f) = \sum_{v \in V} f(v)$, does there exist a minimal dominating function h of G such that $ag(h) = x$?”.

A similar question raised about total domination, motivated to define the convex combination of two MTDFs. Let f and g be two MTDFs of G, a convex combination of f and g is $h_\lambda = \lambda f + (1-\lambda)g$ where $0 < \lambda < 1$. This function is clearly a TDF. Hence the set of all TDFs forms a convex set. However it is evident from the following theorem that the convex combination of two MTDFs need not always be an MTDF.

Theorem 1.2 [3]. *A convex combination of two MTDFs f and g is minimal if and only if $B_f \cap B_g \rightarrow_t P_f \cup P_g$.*

An MTDF f of G is called a *universal minimal total dominating function* if and only if every convex combination of f and any other MTDF is minimal. Theorem 1.2 is true for any finite number of MTDFs.

Theorem 1.3 [7]. *A convex combination of n MTDFs $f_1, f_2, \ldots f_n$ is minimal if and only if $B_{f_1} \cap B_{f_2} \cap \cdots \cap B_{f_n} \rightarrow_t P_{f_1} \cup P_{f_2} \cup \cdots \cup P_{f_n}$.*

Fractional version of total domination, convexity of two MTDFs and the existence of universal MTDFs have been studied by many authors [2, 3, 4]. Since the set of TDFs is convex, some TDFs cannot be expressed as a convex combination two or more TDFs. Motivated by this, in 2000 K. Reji Kumar introduced basic total dominating functions (BTDFs) and basic minimal total dominating functions (BMTDFs) [7]. An MTDF is called a *basic minimal total dominating function* or BMTDF, if it cannot be expressed as a proper convex combination of two distinct MTDFs. A necessary and sufficient condition for an MTDF to be a basic MTDF is known and based on this we have developed an algorithm to decide whether a given MTDF is basic.

Theorem 1.4 [7]. *Let f be an MTDF. Then f is a BMTDF if and only if there does not exist an MTDF g such that $B_f = B_g$ and $P_f = P_g$.*

Theorem 1.5 [7]. *Let f be an MTDF of a graph $G = (V, E)$ with $B_f = \{v_1, v_2, \ldots, v_m\}$ and $P'_f = \{u \in V : 0 < f(u) < 1\} = \{u_1, u_2, \ldots, u_n\}$. Let $A = (a_{ij})$ be an $m \times n$ matrix defined by*

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } u_j, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the system of linear equations given by

$$\sum_j a_{ij}x_j = 0, \text{ where } 1 \le i \le m. \tag{1.1}$$

Then f is a BMTDF if and only if (1.1) *does not have a non-trivial solution.*

Corollary 1.6 [7]**.** *If $f(v) \in \{0, 1\}$ for all $v \in V$, then the MTDF f of G is a BMTDF.*

Let G be a graph. We define

$$C_0(G) = \{v \in V : f(v) = 0 \text{ for any MTDF } f \text{ of } G\}, \text{ and}$$
$$C_1(G) = \{v \in V : f(v) = 1 \text{ for any MTDF } f \text{ of } G\}.$$

The set of *leaves* of G is, $L = \{v \in V : d(v) = 1\}$ and the set of *remote vertices* is defined by $R = \{v \in V : v \in N(u) \text{ for } u \in L\}$. Here $d(v)$ is the number of vertices adjacent to a vertex $v \in V$. Nice characterizations of the sets $C_0(G)$ and $C_1(G)$ of a graph G are given by Cockayne *et al.* in [3].

Proposition 1.7 [3]**.** *For any graph G and vertex v,*

1. *$v \in C_0(G)$ if and only if v is in no MTDS of G;*
2. *$v \in C_1(G)$ if and only if v is in every MTDS of G.*

Theorem 1.8 [3]**.** *A graph G has either a unique MTDF or infinitely many MTDFs.*

Theorem 1.9 [3]**.** *For any graph G, $C_1(G) = R$.*

Theorem 1.10 [3]**.** *The vertex $v \in C_0(G)$, if and only if for any $u \in N(v)$ there exists a vertex w such that $N(w) \subseteq N(u) - v$.*

Let K be a convex subset of $\mathbf{R}^n$. A point $x \in K$ is an *extreme point* of K if $y, z \in K$, $0 < \lambda < 1$, and $x = \lambda y + (1 - \lambda)z$ imply $x = y = z$. The set of all extreme points of K is denoted by $ext(K)$. A set $F \subseteq K$ is a face of K if either $F = \emptyset$ or $F = K$ or there exists a supporting hyperplane H of K such that $F = K \cap H$. An *n-simplex* in the Euclidean space is the convex hull of $n + 1$ affinely independent points. A *convex polytope* is the convex hull of a finite set. A finite family $\mathfrak{B}$ of convex polytopes in $\mathbf{R}^n$ is called a *simplicial complex* if it satisfies the following conditions

1. Every face of a member of $\mathfrak{B}$ is itself a member of $\mathfrak{B}$;
2. The intersection of any two members of $\mathfrak{B}$ is a face of each of them.

For further study of simplices, polytopes and complexes, the reader is referred to [1]. We use the notations $\mathfrak{F}_T(G)$ and $\mathfrak{F}_{BT}(G)$ to denote the set of all MTDFs and the set of all BMTDFs of a graph G, respectively.

Theorem 1.11 [8]. *Let $A \subseteq \mathfrak{F}_{BT}(G)$ such that the convex combination f_{A_i} of all BMTDFs in $A_i \subseteq A$ is an MTDF for any subset A_i and $B_{A_1} \neq B_{A_2}$ or $P_{A_1} \neq P_{A_2}$ for any two nonempty subsets A_1 and A_2. Then the convex combination f_A is a simplex with dimension $|A| - 1$.*

Theorem 1.12 [8]. *Let G be a graph with $|V| = n$. Then the Euclidian dimension of $\mathfrak{F}_T(G)$ is at most n.*

Theorem 1.13 [8]. *Let G be a graph having order n such that $|\mathfrak{F}_{BT}(G)| = r$, and $\mathfrak{F}_T(G)$ is convex.*

1. *If $r \leq (n+1)$ and for all different subsets A_1 and A_2 of $\mathfrak{F}_{BT}(G)$, $B_{f_{A_1}} \neq B_{f_{A_2}}$ or $P_{f_{A_1}} \neq P_{f_{A_2}}$ then $\mathfrak{F}_T(G)$ is an r-1 simplex. Otherwise $\mathfrak{F}_T(G)$ is a convex polytope having dimension at most $n - 1$.*
2. *If $r > (n+1)$, $\mathfrak{F}_T(G)$ is a convex polytope having dimension at most n and there exists two subsets A_1 and A_2 of $\mathfrak{F}_{BT}(G)$, such that $B_{f_{A_1}} = B_{f_{A_2}}$ and $P_{f_{A_1}} = P_{f_{A_2}}$.*

Theorem 1.14 [8]. *If $\mathfrak{F}_T(G)$ is not convex, then it is a simplicial complex.*

Theorem 1.15 [8]. *For the complete bipartite graph $G = K_{m,n}$, the set $\mathfrak{F}_T(K_{m,n})$ is isomorphic to*

1. *the n-1-simplex if $m = 1$ and $n \geq 2$.*
2. *a convex polytope otherwise.*

2. Structure of the Set of all MTDFs of Some Classes of Graphs

In this section, our focus is on the study of the structure of the set of all MTDFs of cycles and paths. We shall show that, $\mathfrak{F}_T(C_n)$ is convex only if $n = 4$ or 8 and $\mathfrak{F}_T(P_n)$ is convex only if $n \leq 8$. There exists a bijection from the set of all functions $f : V \to [0,1]$ of the graph $G = (V, E)$ to the n dimensional cube (I^n) in $\mathbb{R}^n$. So the set of all TDFs is isomorphic to a subset of I^n.

Theorem 2.1. *Let G be a vertex transitive graph. The set $\mathfrak{F}_T(G)$ is convex if and only if $B_f = V$ for all MTDF f of G.*

Proof. Suppose that $\mathfrak{F}_T(G)$ is convex. Assume that $B_f \neq V$ for some MTDF f of G. Let f_v be an MTDF of G such that, $v \notin B_f$. Since the graph is vertex transitive, there exists a function f_u such that, $u \notin B_{f_u}$ for each $u \in V$. Now, by the convexity of $\mathfrak{F}_T(G)$, there must exist an MTDF g such that, $B_g = \emptyset$. This is a contradiction. Conversely, if $B_f = V$ for all MTDF f of G, then it directly follows that, $\mathfrak{F}_T(G)$ is a convex set. ■

Lemma 2.2. *If f is a BMTDF of an even cycle or a path, then f is a $0-1$ BMTDF. The odd cycle has exactly one BMTDF which is not a $0-1$ BMTDF.*

Proof. Suppose that f is not a 0-1 BMTDF. Then there exist vertices $u_1, u_2, \ldots, u_r \in P'_f$. Now consider the corresponding system of equations (1.1), (Theorem 1.5). In this system, each equation should contain exactly two u_i's and each should appear in at most two equations. Rank of all possible system of equations obeying this rule, is less than the number of equations used in it, except when the graph is a cycle and $P'_f = V$. Using row echelon form, we can prove that, the system has trivial solution only if the graph is an odd cycle and $P'_f = V$. If the graph is a path or an even cycle, the system has a non-trivial solution. Then f is not basic, which is a contradiction. Hence the function f of the odd cycle defined by, $f(v) = \frac{1}{n}$ for all $v \in V$ is a BMTDF. ■

Theorem 2.3. *The set $\mathfrak{F}_T(C_n)$ is convex only if $n = 4$ or 8.*

Proof. First we shall show that, C_n except when $n = 4$ or 8 have one MTDF f with $B_f \neq V$. Since cycles are vertex transitive, by Theorem 2.1 it follows that $\mathfrak{F}_T(C_n)$ is not convex when $n \neq 4$ or 8. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$.

Case 1. When $n = 3$, the required function has the following values $f(v_1) = 1$, $f(v_2) = 1$ and $f(v_3) = 0$.

Case 2. When $n = 5$, f is defined as $f(v_1) = 1$, $f(v_2) = 1$, $f(v_3) = 1$, $f(v_4) = 0$ and $f(v_5) = 0$.

Case 3. When $n = 6$, f has the values $f(v_1) = 1$, $f(v_2) = 1$, $f(v_3) = 0$, $f(v_4) = 1$, $f(v_5) = 1$ and $f(v_6) = 0$.

Case 4. When $n = 7$, the function is defined as $f(v_1) = 1$, $f(v_2) = 1$, $f(v_3) = 0$, $f(v_4) = 1$, $f(v_5) = 1$, $f(v_6) = 0$ and $f(v_7) = 0$.

Case 5. When $n = 9$, $f(v_1) = 1$, $f(v_2) = 1$, $f(v_3) = 0$, $f(v_4) = 0$, $f(v_5) = 1$, $f(v_6) = 1$, $f(v_7) = 1$, $f(v_8) = 0$ and $f(v_9) = 0$ are the function values.

Case 6. When $n = 10$, $f(v_1) = 1$, $f(v_2) = 1$, $f(v_3) = 0$, $f(v_4) = 1$, $f(v_5) = 1$, $f(v_6) = 0$, $f(v_7) = 1$, $f(v_8) = 1$, $f(v_9) = 0$ and $f(v_{10}) = 0$ are the function values.

Case 7. When $n = 11$, f takes the values, $f(v_1) = 1$, $f(v_2) = 1$, $f(v_3) = 0$, $f(v_4) = 1$, $f(v_5) = 1$, $f(v_6) = 0$, $f(v_7) = 0$, $f(v_8) = 1$, $f(v_9) = 1$, $f(v_{10}) = 0$, and $f(v_{11}) = 0$.

Case 8. When $n = 12$, $f(v_1) = 1$, $f(v_2) = 1$, $f(v_3) = 0$, $f(v_4) = 1$, $f(v_5) = 1$, $f(v_6) = 0$, $f(v_7) = 1$, $f(v_8) = 1$, $f(v_9) = 0$, $f(v_{10}) = 1$, $f(v_{11}) = 1$ and $f(v_{12}) = 0$ are the values at different vertices.

Next we shall show that, if C_n has an MTDF f such that, $f(v_i) = 0$, $f(v_{i+1}) = 1$, $f(v_{i+2}) = 1$, $f(v_{i+3}) = 0$ and $B_f \neq V(C_n)$, then f can be extended to an MTDF f' of C_{n+4} satisfying $B_{f'} \neq V(C_{n+4})$. We can make C_{n+4} from C_n by joining the path u_1, u_2, u_3 and u_4 between the vertices v_{i+1} and v_{i+2}. Consider the function $f' : V(C_{n+4}) \to [0, 1]$, defined by $f'(v_i) = f(v_i)$ for all i, $f'(u_1) = 1$, $f'(u_2) = 0$, $f'(u_3) = 0$ and $f'(u_4) = 1$. Now the function f' has the property, $B_{f'} \neq V(C_{n+4})$.

Since C_n, when $n = 9, 10, 11$, and 12 has an MTDF f such that $B_f \neq V(C_n)$, we can apply the above procedure repeatedly over these cycles and get similar kind of MTDFs for higher order cycles. Hence for C_n where $n \geq 9$ has at least one MTDF f with $B_f \neq V(C_n)$.

When $n = 4$, let us denote the four 0-1 MTDFs of C_4 by $f_1 = (1, 1, 0, 0)$, $f_2 = (0, 1, 1, 0)$, $f_3 = (0, 0, 1, 1)$ and $f_4 = (1, 0, 0, 1)$. Let g be any MTDF of C_4. We shall show that, g can be expressed as a convex combination of these four MTDFs. Clearly $B_g(C_4) = V$. If $g(v_1) = \delta$ then $g(v_3) = (1 - \delta)$

and if $g(v_2) = \Delta$ then $g(v_4) = (1 - \Delta)$. By equating the function values at each vertex, we get the following system of equations.

$$\lambda_1 + \lambda_4 = \delta,$$

$$\lambda_1 + \lambda_2 = \Delta,$$

$$\lambda_2 + \lambda_3 = (1 - \delta) \quad \text{and}$$

$$\lambda_3 + \lambda_4 = (1 - \Delta).$$

This system is consistent. To get the solution, assign an arbitrary value to one of the λ_is. By Theorem 1.5, g is a non-basic MTDF. Convex combinations of the BMTDFs taken two, three or four at a time have same pairs of boundary and positive sets except for the convex combination of the pairs of functions (f_1, f_2), (f_2, f_3), (f_3, f_4) and (f_4, f_1). So the set $\mathfrak{F}_T(C_4)$ is isomorphic to I^2.

When $n = 8$, we claim that P_8 has no MTDF f such that $B_f \neq V$. Otherwise, it must have a BMTDF g such that $B_g \neq V$. But by Lemma 2.2, it must be a 0-1 BMTDF. Without loss of generality let us assume that, $v_1 \notin B_g$. Then the function values at the vertices v_2, v_4, v_8 and v_6 are $1, 0, 1$ and 0 respectively. Consequently we get $g(N(v_5)) = 0$, which is a contradiction. Thus it is clear that $\mathfrak{F}_T(C_8)$ is a convex set. To know more about the structure of this set, we have to consider all 0-1 MTDFs of the graph. The 0-1 MTDFs are $f_1 = (1, 1, 0, 0, 1, 1, 0, 0)$, $f_2 = (0, 1, 1, 0, 0, 1, 1, 0)$, $f_3 = (0, 0, 1, 1, 0, 0, 1, 1)$ and $f_4 = (1, 0, 0, 1, 1, 0, 0, 1)$. Their convex combinations taken two, three or four at a time, have same pairs of boundary and positive sets except for the pairs of functions (f_1, f_2), (f_2, f_3), (f_3, f_4) and (f_4, f_1). So the set $\mathfrak{F}_T(C_8)$ is isomorphic to I^2. ∎

Theorem 2.4. *For a path P_n,*

1. *if $n = 2$ or 4, then $\mathfrak{F}_T(P_n)$ is a 0-simplex,*
2. *if $n = 3$ or 5, then $\mathfrak{F}_T(P_n)$ is a 1-simplex,*
3. *$\mathfrak{F}_T(P_n) \cong I^2$ if $n = 6$ or 8,*
4. *if $n = 7$, then $\mathfrak{F}_T(P_n)$ is a 2-simplex and*
5. *the set $\mathfrak{F}_T(P_n)$ is not a convex set if $n = 9$ or $n \geq 11$.*

Proof. Let the vertices of P_n be labeled as $v_1, v_2, \ldots, v_n$. One can easily verify that P_n, when $n = 2$ and 4, has unique MTDF. The case $n = 3$ follows from Theorem 1.15. When $n = 5$, let g be an arbitrary MTDF of P_5. Then $g(v_2) = g(v_4) = 1$. Let $g(v_1) = \Delta$. Then $v_2 \in B_g$ and $g(v_3) = (1-\Delta)$. Consequently, $g(v_5) > 0$ and hence $v_4 \in B_g$. So $g(v_5) = \Delta$. Clearly $g = \lambda f_1 + (1-\lambda) f_2$, where f_1 and f_2 are defined by $f_1(v_1) = f_1(v_5) = 1$, $f_1(v_3) = 0$, $f_2(v_1) = f_2(v_5) = 0$, $f_2(v_3) = 1$ and $f_i(v_2) = f_i(v_4) = 1$ for $i = 1$ and 2.

When $n = 7$, let g be an arbitrary MTDF of the graph. Then $v_1, v_2, v_6, v_7 \in B_g$. The vertex $v_3 \notin B_g$. Otherwise, $g(v_4) > 0$ and there is no vertex in B_g to dominate v_4 and this contradicts the assumption that g is an MTDF. Similarly, $v_5 \in B_g$. The set $\bigcap_g B_g$ can dominate any vertex in P_7. Hence the set $\mathfrak{F}_T(P_7)$ is convex. To find all BMTDFs of P_7, we have to consider two cases.

First case: when $v_4 \in B_g$. We have two 0-1 MTDFs, having this property. They are $f_1 = (0,1,1,0,0,1,1)$ and $f_2 = (1,1,0,0,1,1,0)$. Let g be an arbitrary MTDF such that $v_4 \in B_g$ and $g(v_3) = \Delta$ and $g(v_5) = (1-\Delta)$. We get f_1 and f_2 when $\Delta = 1$ and $\Delta = 0$ respectively. If $0 < \Delta < 1$, then $g(v_1) = (1-\Delta)$ and $g(v_7) = \Delta$. Hence, $g = \Delta f_1 + (1-\Delta) f_2$. Second case: when $v_4 \notin B_g$. There exists only one 0 - 1 MTDF having this property. Let that function be $f_3 = (0,1,1,0,1,1,0)$. If g is not a 0-1 MTDF, we take $g(v_3) = \delta$ and $g(v_5) = \Delta$. Consequently, $g(v_1) = (1-\delta)$ and $g(v_7) = (1-\Delta)$. Next, assume that $g = \sum_i \lambda_i f_i$. Then by equating the function values at different vertices, we get the system of equations.

$$\begin{aligned}
&\lambda_1 + \lambda_3 = \delta, \\
&\lambda_2 + \lambda_3 = \Delta, \\
&\lambda_2 = (1-\delta) \quad \text{and} \\
&\lambda_1 = (1-\Delta).
\end{aligned}$$

Solving them, we get $\delta = \Delta$ and subsequently the values of λ_is. So $\mathfrak{F}_T(P_7)$ is isomorphic to a two simplex.

When $n = 6$, take an arbitrary MTDF, say g. If $g(v_1) = \Delta$, then $g(v_3) = (1-\Delta)$. Similarly, if $g(v_4) = \delta$, then $g(v_6) = (1-\delta)$. The function g is a convex combination of the BMTDFs $f_1 = (1,1,0,0,1,1)$, $f_2 = (1,1,0,1,1,0)$, $f_3 = (0,1,1,0,1,1)$ and $f_4 = (0,1,1,1,1,0)$. The boundary and positive sets of all possible convex combinations of these functions are same, except for the pairs of functions (f_1, f_2), (f_2, f_4), (f_4, f_3) and (f_3, f_1). So $\mathfrak{F}_T(P_6)$ is isomorphic to I^2.

When $n = 8$, the set of vertices $\{v_1, v_2, v_7, v_8\} \subseteq B_f$ for all MTDF f of P_8. Next let g be an arbitrary MTDF of P_8 and $g(v_4) = \Delta$. If $\Delta = 0$

then $g(v_6) = 1$ and $g(v_8) = 0$. If $\Delta = 1$ then $g(v_6) = 0$ and $g(v_8) = 1$. If $0 < \Delta < 1$ then $v_5 \in B_g$. So $g(v_6) = (1 - \Delta)$ and $g(v_8) = \Delta$. Let as define f_1 and f_2 such that, $f_1(v_4) = 1$, $f_1(v_6) = 0$, $f_1(v_8) = 1$, $f_2(v_4) = 0$, $f_2(v_6) = 1$, $f_2(v_8) = 0$ and $f_i(v) = g(v)$ for all other vertices. The functions f_1 and f_2 are MTDFs. Also $g = \Delta f_1 + (1 - \Delta)f_2$. Next by considering the function f_1 at the place of g and starting with the vertex v_5 and applying the same procedure, we can show that $f_1 = \delta f_{11} + (1-\delta)f_{12}$. The functions f_{11} and f_{12} are defined as, $f_{11}(v_5) = 1$, $f_{11}(v_3) = 0$, $f_{11}(v_1) = 1$, $f_{12}(v_5) = 0$, $f_{12}(v_3) = 1$ and $f_{12}(v_1) = 0$. Also when $i = 1$ or 2, $f_{1i}(v) = f_1(v)$ for all remaining vertices.

Similarly we can express the function f_2 as a convex combination of two MTDFs f_{21} and f_{22} having functions values, $f_{21} = (1, 1, 0, 0, 1, 1, 1, 0)$ and $f_{22} = (0, 1, 1, 1, 0, 0, 1, 1)$. Now, the MTDF $g = \Delta(\delta f_{11} + (1 - \delta)f_{12}) + (1 - \Delta)(\delta f_{21} + (1 - \delta)f_{22})$ and hence g is the convex combination of the MTDFs f_{ij} where $i, j = 1, 2$. Exactly as in the case of P_6, we can verify that the boundary and positive sets of all possible convex combinations of these functions are same, except for the pairs of functions (f_{11}, f_{12}), (f_{11}, f_{21}), (f_{21}, f_{22}) and (f_{12}, f_{22}). Hence the result.

When $n = 10$, the vertices v_1, v_2, v_9 and v_{10} are in B_f for every MTDF f of P_{10}. We shall show that, the vertices v_5 and v_6 are also in B_f. Suppose that $v_5 \notin B_f$. Then $f(v_4) > 0$ and $f(v_6) > 0$ and the vertex v_3 must be in the boundary of f. Otherwise B_f cannot dominate P_f. But this is impossible as $f(v_2) = 0$ for all MTDF f of P_{10}. Similarly we can show that $v_6 \in B_f$ for all MTDFs f of P_{10}. Consequently, the set $\{v_1, v_2, v_5, v_6, v_9, v_{10}\} \subseteq \bigcap_f B_f$, where the intersection is taken over all MTDFs of P_{10}. Since the set $\{v_1, v_2, v_5, v_6, v_9, v_{10}\}$ dominates $V(P_{10})$, the convex combination of any two MTDFs is an MTDF and hence $\mathfrak{F}_T(P_{10})$ is convex.

Next we proceed to prove that P_9 and P_{12} have two MTDFs, whose convex combinations are not MTDFs. For any MTDF of a path, the function values of odd labeled vertices are independent of the function values of even labeled vertices. In other words, if f and g are any two MTDFs of P_n, the new function h — defined by $h(x) = f(x)$ if x is an odd vertex and $h(x) = g(x)$ if x is an even vertex — is an MTDF. So, if necessary we can concentrate on either odd vertices or even vertices, without mentioning the other set of vertices. In P_9, let f and g be any two MTDFs such that $f(v_1) = 0$, $f(v_3) = 1$, $f(v_5) = 1$, $f(v_7) = 0$, $f(v_9) = 1$, $g(v_1) = 1$, $g(v_3) = 0$, $g(v_5) = 1$, $g(v_7) = 1$ and $g(v_9) = 0$. Convex combinations of f and g are not MTDFs. So $\mathfrak{F}_T(P_9)$ is not a convex set. Similarly in P_{12}, consider any two MTDFs f and g such that, $f(v_1) = 1$, $f(v_3) = 0$, $f(v_5) = 1$, $f(v_7) = 1$, $f(v_9) = 0$, $f(v_{11}) = 1$, $g(v_1) = 0$, $g(v_3) = 1$, $g(v_5) = 1$, $g(v_7) = 0$, $g(v_9) = 1$ and $g(v_{11}) = 1$. Again the convex combination of f and g is not an MTDF, implying that $\mathfrak{F}_T(P_{12})$ is not a convex set.

Finally, we shall show that if f is a 0-1 MTDF of P_n, then it can be extend to an MTDF of P_{n+2}. The values of $f(v_{(n+1)})$ and $f(v_{(n+2)})$ are decided depending upon $f(v_{(n-1)})$ and $f(v_n)$. As an example, let $f(v_{(n-1)}) = 0$ and $f(v_{(n)}) = 1$. Then $f(v_{(n+1)}) = 1$ and $f(v_{(n+2)}) = 0$. Similarly we can always find two values for the vertices v_{n+1} and v_{n+2}. So the set of all MTDFs of the paths $P_{(9+i)}$ and $P_{(12+i)}$ for $i = 2, 4, 6, \ldots$ are not convex. ∎

3. GRAPHS HAVING $\mathfrak{F}$ ISOMORPHIC TO A PRODUCT OF SIMPLICIAL COMPLEXES

Let $A \subseteq V(G)$. The subgraph of G induced by A is denoted by $\langle A \rangle$. Let G be a graph and $V_1, V_2 \subset V(G)$ such that $V_1 \cup V_2 = V(G)$. Note that, the possibility of $V_1 \cap V_2 \neq \emptyset$ is not eliminated. Let $W \subset V$ and f is an MTDF of G. We denote the *restriction* of f to W by f/W. A graph G is a *function reducible graph* with respect to a partition V_1 and V_2, if $\mathfrak{F}(\langle V_i \rangle) = \{f/V_i : f$ is an MTDF of $G\}$ for $i = 1$ and 2 and if for any $f_1 \in \mathfrak{F}(V_1)$ and $f_2 \in \mathfrak{F}(V_2)$ the new function defined by

$$f(v) = \begin{cases} f_1(v), & \text{if } v \in V_1, \\ f_2(v), & \text{if } v \in V_2 \end{cases}$$

is an MTDF of the whole graph.

Clearly for all $v \in (V_1 \cap V_2)$ and any MTDF f of G, $f(v)$ must be a constant. Disconnected graphs are examples. But the following example shows that some connected graphs also possess this property.

Lemma 3.1. *If $f(v)$ is a constant for all MTDF f of a graph G, then $f(v) = 0$ or 1.*

Proof. Suppose that, $0 < f(v) = \Delta < 1$. We know that every graph has at least one MTDS. If we take the characteristic function of an MTDS, then the function value at v is either 0 or 1. This is a contradiction. ∎

Lemma 3.2. *If the graph G is function reducible with respect to the pair V_1, V_2 and $V_1 \cap V_2 \neq \emptyset$, then $V_1 \cap V_2 \subseteq C_0 \cup C_1$.*

Proof. By the definition of function reducible graphs, if f is an arbitrary MTDF of G then $f(v) = 0$ or 1 for all $v \in V_1 \cap V_2$. ■

Example 3.3. *Take two star graphs $K_{(1,n)}$ and $K_{(1,m)}$ where $n, m \geq 2$. The graph G is made by joining one pendant vertex from each star, by an edge.*

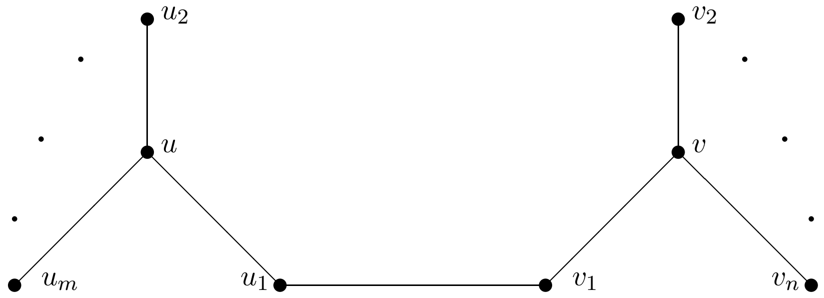

Figure 3.1

Consider the partition $V_1 = \{u, u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v, v_1, v_2, \ldots, v_n\}$ of the graph. The vertices $u, v \in C_1$. We can change the function values at the vertices u_i's without affecting the function values at v_i's, such that $u \in B_g$ for any MTDF g of the graph.

Next we consider the graphs having at least two MTDFs. We call the graph G a *function separable graph*, if $V(G)$ has at least one partition, say $\{V_1, V_2\}$ such that for any two MTDFs f and g of G, the functions defined by

$$(f, g)(v) = \begin{cases} f(v), & \text{if } v \in V_1, \\ g(v), & \text{if } v \in V_2 \end{cases}$$

and

$$(g, f)(v) = \begin{cases} g(v), & \text{if } v \in V_1, \\ f(v), & \text{if } v \in V_2 \end{cases}$$

are MTDFs of G. Let the graph G be function separable with respect to a partition $\{V_1, V_2\}$, we shall call the function $f_{V_1} : V_1 \to [0, 1]$, a *basic function* of V_1 if and only if there exists a BMTDF f of G such that, $f/V_1 = f_{V_1}$.

We avoid the words "minimal dominating" because, the basic functions may not always be a minimal dominating function. Next we proceed to prove some properties of basic functions. We define a convex combination of the functions f/V_1 and g/V_1 if and only if the convex combination of the MTDFs f and g of G is minimal.

Theorem 3.4. *The function f_{V_1} is a basic function if and only if there does not exist functions $f^i_{V_1}$ where $i = 1, 2, \ldots, r$ such that, $f_{V_1} = \sum_i \lambda_i f^i_{V_1}$, $\sum_i \lambda_i = 1$ and $0 < \lambda_i < 1$.*

Proof. Let the function f_{V_1} be a basic function on the set V_1. Suppose that, there exist functions $f^i_{V_1}$ where $i = 1, 2, \ldots, r$ such that, $f_{V_1} = \sum_i \lambda_i f^i_{V_1}$, where $\sum_i \lambda_i = 1$ and $0 < \lambda_i < 1$. By the definition of basic function, there exists a BMTDF f of G such that $f/V_1 = f_{V_1}$. The functions $f_i : V \to [0, 1]$ are defined such that

$$f_i(v) = \begin{cases} f^i_{V_1}(v), & \text{if } v \in V_1, \\ f(v), & \text{if } v \in V_2. \end{cases}$$

It is an easy exercise to show that $f = \sum_i (\lambda_i f_i)$, where $\sum_i \lambda_i = 1$ and $0 < \lambda_i < 1$. This contradicts the fact that f is a BMTDF.

To prove the converse, we have to show that if f_{V_1} cannot be expressed as a convex combination of a set of functions over V_1, then f is a BMTDF. Suppose on the contrary that f is not basic. Then there exist MTDFS $f_1, f_2, \ldots, f_r$ such that $f = \sum_i \lambda_i f_i$, where $\sum_i \lambda_i = 1$ and $0 < \lambda_i < 1$. This implies that f_{V_1} is a convex combination of f_i/V_1, a contradiction. ■

Theorem 3.5. *Let G be a function separable graph with respect to a partition $\{V_1, V_2\}$. If the sets of all basic functions over V_1 and V_2 are $A = \{f_1, f_2, \ldots, f_r\}$ and $B = \{g_1, g_2, \ldots, g_s\}$ respectively, then the set of all BMTDFs of G is $A \times B = \{(f_i, g_j) : i = 1, 2, \ldots r$ and $j = 1, 2, \ldots s\}$.*

Proof. If $f \in \mathfrak{F}_{BT}(G)$, then f/V_1 and f/V_2 are basic functions and $f = (f/V_1, f/V_2)$. So $f \in \mathfrak{F}_{BT}(G) \subseteq A \times B$. To prove the converse, let $(f, g) \in A \times B$. We have to show that (f, g) is a BMTDF of G. Suppose not. Since (f, g) is an MTDF, there exists MTDFs $f_1, f_2, \ldots, f_r$ such that f is a convex combination of these functions. Then f/V_1 is a convex combination of the restrictions, $f_1/V_1, f_2/V_1, \ldots, f_r/V_1$ and by Theorem 3.4 we get a contradiction. ■

Theorem 3.6. *Let f and g be two MTDFs of a function separable graph G. These functions are basic if and only if both MTDFs (f, g) and (g, f) are BMTDFs.*

Proof. Let the function f and g be BMTDFs of G and let G be function separable with respect to the partition $\{V_1, V_2\}$. Then f/V_1 is a basic function over V_1. Suppose that (f, g) is not basic. Then there exists MTDFs $f_1, f_2, \ldots f_r$ of G such that $(f, g) = \sum_i \lambda_i f_i$. Using the Theorem 3.4, we get that f/V_1 is not a basic function. This is a contradiction. Proof of the function (g, f) and that of the converse are similar. ∎

Theorem 3.7. *All bipartite graphs are function separable.*

Proof. Let V_1 and V_2 be the partition of $V(G)$. Consider any two MTDFs say, f and g of G and the new function (f, g). We claim that this function is an MTDF of G. First note that, $(f, g)(N(v)) = g(N(v))$ if $v \in V_1$ and $(f, g)(N(v)) = f(N(v))$ if $v \in V_2$. Also $V_2 \cap B_f \to V_1 \cap P_f$ and $V_1 \cap B_g \to V_2 \cap P_g$. Thus $B_{(f,g)} \to P_{(f,g)}$ and hence the result. ∎

It is interesting to note that if G is a function reducible graph, then it is a function separable graph. The previous theorem provides an example, which shows the converse is not always true.

Theorem 3.8. *Function reducible graphs are function separable graphs.*

Proof. Let G be a function reducible graph with respect to the vertex subsets V_1 and V_2. Then consider the new vertex subsets $V_1' = V_1 - (V_1 \cap V_2)$ and $V_2' = V_2$. With respect to the partition $\{V_1', V_2'\}$, the graph is function separable. ∎

Theorem 3.9. *Cycles are not function reducible.*

Proof. Suppose that the cycle G is function reducible with respect to the vertex subsets V_1 and V_2. Since we consider only connected induced subgraphs, $\langle V_1 \rangle$ and $\langle V_2 \rangle$ must be paths. Then $V(\langle V_i \rangle) \cap C_1(\langle V_i \rangle) \neq \emptyset$. But $C_1(G) = \emptyset$. These are not possible simultaneously. ∎

This result shows that, bipartite graphs are not function reducible graphs in general, because even cycles are bipartite graphs. Next we discuss some necessary conditions for a graph to be function reducible. In the result we use the following sets. $C_1' = \{x \in C_1 : x \text{ is adjacent to at least one vertex in } C_1\}$, $L' = \{x \in L : x \in N(y) \text{ and } y \in C_1'\}$ and $C = C_1' \cup L'$.

Theorem 3.10. *Let G be a connected graph containing at least three vertices. It has two disjoint vertex subsets V_1 and V_2, where $V_1 \cup V_2 \cup C = V$ such that, for any path between the vertices $u \in V_1$ and $v \in V_2$, there exist at least two adjacent vertices which are common to the path and the set C_1'. Then the graph G is function reducible.*

Proof. Let G be a connected graph containing at least three vertices. Also let V_1 and V_2 be two vertex subsets, satisfying the condition given above. There exists $C' \subseteq C$ such that C' is a cut set of G. We claim that, $V_1' = V_1 \cup C$ and $V_2' = V_2 \cup C$ are two partitions of G, with respect to which the graph is function reducible.

Claim 1. If f is an MTDF of G, then $f_{V_1'}$ is an MTDF of $\langle V_1' \rangle$.
Take a vertex $x \in V_1'$.

Case 1. $x \in V_1$. Then $N(x) \subset V_1'$. So $f_{V_1'}(N(x)) = f(N(x)) \geq 1$. Suppose on the contrary that $y \in N(x) \cap V_2$. We get a contradiction because there exists a path connecting x and y which does not contain any elements of C_1'. Also for any $y \in N(x)$, $N(y) \subset V_1'$. Suppose not. Let $z \in N(y) \cap V_2$. Then the path xyz can contain at most one vertex from C_1'. We get contradiction again. Thus for all $x \in V_1$, $f_{V_1'}(N(x)) \geq 1$ and if $f_{V_1'}(x) \geq 0$ then $B_{f_{V_1'}} \to_t \{x\}$. Hence $f_{V_1'}$ is an MTDF of $\langle V_1' \rangle$. Similarly we can prove that $f_{V_2'}$ is an MTDF of $\langle V_2' \rangle$.

Case 2. $x \in C_1'$. Since $N(x) \cap C_1' \neq \emptyset$, $f_{V_1'}(N(x)) \geq 1$ and $N(x) \cap L' \neq \emptyset$. As in case one we get $B_{f_{V_1'}} \to_t \{x\}$. Hence $f_{V_1'}$ is an MTDF of $\langle V_1' \rangle$. Similarly we can prove that $f_{V_2'}$ is an MTDF of $\langle V_2' \rangle$.

Next let f and g be any MTDFs of $\langle V_1' \rangle$ and $\langle V_2' \rangle$ respectively. First we shall prove that $f(x) = g(x)$ for all $x \in C$. Take an arbitrary MTDF f of $\langle V_1' \rangle$. If $x \in C_1'$, there exists $y \in C_1'$ such that x and y are adjacent. Since $N(x) \cap L' \neq \emptyset$ and $N(y) \cap L' \neq \emptyset$, $x \in C_1(\langle V_1 \rangle)$. If $x \in L'$, then $f(x) = 0$ because, the only vertex adjacent to x, in $\langle V_1' \rangle$ is not an element of B_f. So $f(x) = 1$ for all $x \in C_1'$ and $f(x) = 0$ for all $x \in L'$. The same is true for any MTDF g of $\langle V_2' \rangle$. So $f(x) = g(x)$ for all $x \in C$. Define a function $h : V \to [0, 1]$ as follows.

$$h(v) = \begin{cases} f(v), & \text{if } v \in V_1', \\ g(v), & \text{if } v \in V_2. \end{cases}$$

Claim 2. h is an MTDF of G. Since f and g are MTDFs of the induced subgraphs, h is an MDF of G. By suitably reproducing the steps in the above paragraph, we can show that $B_h \rightarrow_t P_h$. ■

Theorem 3.11. *Let G be a function reducible graph with respect to the vertex subsets V_1 and V_2. Then the set $\mathfrak{F}(G)$ is the product of the sets $\mathfrak{F}(\langle V_1 \rangle)$ and $\mathfrak{F}(\langle V_2 \rangle)$.*

Proof. Obvious. ■

4. Problems for Further Research

Structure of the set of all minimal total dominating functions of many families of graphs are still unknown. A characterization is known for only those graphs G, for which $\mathfrak{F}_T(G)$ is isomorphic to one simplex. Characterization of graphs, such that $\mathfrak{F}(G)$ is isomorphic to other higher dimensional simplexes is quite open. The first step to study the structure of $\mathfrak{F}(G)$ is to find all BMTDFs of a graph. Only a little research is done in this area.

Acknowledgement

A part of this research work is completed when the first author visited the Department of Mathematics and Statistics, University of Victoria, BC, Canada with the financial support of the Department of Science and Technology, India in BOYSCAST fellowship scheme. He expresses his gratitude for all facilities provided and support given by the Department of Mathematics and Statistics, University of Victoria, to complete this research.

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Received 19 February 2009
Revised 1 September 2009
Accepted 1 September 2009