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3-CONSECUTIVE C-COLORINGS OF GRAPHS*

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Abstract

A 3-consecutive C-coloring of a graph $G = (V, E)$ is a mapping $\varphi : V \to \mathbb{N}$ such that every path on three vertices has at most two colors. We prove general estimates on the maximum number $\bar{\chi}_{3CC}(G)$ of colors in a 3-consecutive C-coloring of G, and characterize the structure of connected graphs with $\bar{\chi}_{3CC}(G) \geq k$ for $k = 3$ and $k = 4$.

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1. Introduction

Motivated by various new developments in the theory of graph and hypergraph coloring, in this note we introduce the notion of 3-consecutive C-coloring of graphs. For a given graph $G = (V, E)$, a mapping

$$\varphi : V \to \mathbb{N}$$

is called a 3-*consecutive C-coloring* — abbreviated as 3CC-*coloring* for short — if there exists no 3-colored path on three vertices; that is, among every three consecutive vertices there exist two having the same color.

Obviously, the trivial coloring that assigns the same color to all vertices of G is 3-consecutive. Therefore, we are interested in the *maximum* number of colors that can occur in a 3CC-coloring of G. This number will be called the 3-*consecutive upper chromatic number* of G, denoted by $\bar{\chi}_{3CC}(G)$.

It is immediate by definition that, in any 3CC-coloring with more than one color, replacing two color classes with their union results in a 3CC-coloring and the number of colors decreases by precisely one. Consequently, for each integer k between 1 and $\bar{\chi}_{3CC}(G)$ there exists a 3CC-coloring of G with exactly k colors. Moreover, by the pigeon-hole principle, every coloring with only one or two colors is a 3CC-coloring; that is, $\bar{\chi}_{3CC}(G) \geq 2$ holds for each graph having at least two vertices. Note further that $\bar{\chi}_{3CC}(G) = |V(G)|$ if and only if each connected component of G is an isolated vertex or an isolated edge.

If H is a spanning subgraph of a graph G, then any three consecutive vertices of H are also consecutive in G. Hence any 3CC-coloring of G is also a 3CC-coloring of H, thus

$$\bar{\chi}_{3CC}(G) \leq \bar{\chi}_{3CC}(H).$$

But a maximum 3CC-coloring of G need not be a maximum 3CC-coloring of H, therefore strict inequality may hold.

Moreover, $\bar{\chi}_{3CC}$ is additive with respect to vertex-disjoint union; that is, the 3-consecutive upper chromatic number of a disconnected graph G is equal to the sum of $\bar{\chi}_{3CC}$ over the connected components of G.

Previous models. Our new coloring model is closely related to the following earlier ones, which also served as motivation for the present study.

- A 3-*consecutive coloring* of a graph $G = (V, E)$ is a coloring of vertices of G such that if uvw is a path on 3 vertices, then v receives the color of u or w. The 3-*consecutive coloring number* $\chi_{3c}(G)$ is the maximum number of colors which can be used in such a coloring. This invariant was introduced in [7] and studied in some detail in [8]. Clearly, any 3-consecutive coloring is a 3CC-coloring, and hence

$$\chi_{3c}(G) \le \bar{\chi}_{3CC}(G).$$

- A hypergraph is a pair $\mathcal{H} = (V, \mathcal{E})$, where $\mathcal{E}$ is a set system over V, and $\emptyset \notin \mathcal{E}$. The elements of V and $\mathcal{E}$ are called vertices and edges of $\mathcal{H}$, respectively. A C-*coloring* of $\mathcal{H}$ is a mapping $\varphi : V \to \mathbb{N}$ such that every edge $E \in \mathcal{E}$ contains at least two vertices with a common color; that is, $|\varphi(E)| < |E|$. The *upper chromatic number*, denoted by $\bar{\chi}(\mathcal{H})$, is the largest possible number of colors that can be used in a C-coloring of $\mathcal{H}$.

  The roots of this notion date back to the early 1970's in the works of Berge (unpublished, cf. [1, p. 151]) and Sterboul [11]; moreover, C-coloring is a particular case of Voloshin's *mixed hypergraph* model (introduced in [12]) where it exactly means coloring of C-hypergraphs.

  The 3CC-colorings of a graph $G = (V, E)$ can be interpreted as C-colorings of the hypergraph $\mathcal{H} = (V, \mathcal{E})$ where $\mathcal{E}$ consists of all 3-element sets $\{u, v, w\} \subseteq V$ inducing a connected subgraph of G.

- A *color-bounded hypergraph* $\mathcal{H} = (V, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ is a hypergraph with $\mathcal{E} = \{E_1, \ldots, E_m\}$ where each edge E_i is associated with two integers s_i and t_i such that $1 \le s_i \le t_i \le |E_i|$. A vertex coloring $\varphi : X \to \mathbb{N}$ is considered to be proper if for each edge E_i there occur at least s_i and at most t_i different colors on it; that is, the inequalities $s_i \le |\varphi(E_i)| \le t_i$ are satisfied, for every $E_i \in \mathcal{E}$, [2, 3].

  If $G = (V, E)$ is a graph with $V = \{v_1, \ldots, v_n\}$, then its 3CC-colorings are precisely the colorings of the color-bounded hypergraph $\mathcal{H} = (V, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ with $\mathcal{E} = \{E_1, \ldots, E_n\}$, where each hyperedge E_i $(1 \le i \le n)$ is the closed neighborhood of vertex v_i in G, and the color bounds are $\boldsymbol{s}(E_i) = 1$ and $\boldsymbol{t}(E_i) = 2$.

Our results. In this paper we study 3CC-colorings of connected graphs. In Section 2 we prove upper bounds on $\bar{\chi}_{3CC}(G)$ in terms of several parameters of G and, particularly, we obtain tight bounds for trees and unicyclic graphs. In Section 3 we give characterizations for graphs admitting proper 3CC-colorings with exactly 3 and exactly 4 colors. These theorems also yield necessary conditions for graphs having 3-consecutive colorings with exactly 3 and 4 colors, respectively, because of the inequality $\chi_{3c}(G) \leq \bar{\chi}_{3CC}(G)$.

Standard notation. As usual, we write $N[v]$ for the *closed neighborhood* of vertex v, and $d(x, y)$ for the *distance* of vertices x and y. In the latter, we sometimes put subscript as $d_G(x, y)$, if the graph under consideration has to be emphasized.

2. Bounds

Theorem 1. *For any graph $G = (V, E)$ of order p and minimum degree δ, we have $\bar{\chi}_{3CC}(G) \leq \lfloor \frac{2p}{\delta+1} \rfloor$.*

Proof. Consider a 3CC-coloring of G with exactly k colors. Let us call a color class or its color "small" if it contains fewer than $\frac{\delta+1}{2}$ vertices, otherwise call it "big". If all colors are big, then we immediately obtain that the number of colors is at most $2p/(\delta + 1)$.

Hence, we can assume that there are $\ell \geq 1$ small color classes. Choose one vertex from each. In this way we have vertices $v_1, \ldots, v_\ell$ with small colors $c_1, \ldots, c_\ell$, respectively. The closed neighborhood $N[v_i]$ of each v_i contains at least $\delta + 1$ vertices, from exactly two colors, namely c_i and another one, say α_i. Since c_i is a small color, α_i is a big one. Moreover, the sets $N[v_1], \ldots, N[v_\ell]$ are mutually disjoint. Indeed, a common vertex with a small color would imply the identity $\alpha_i = c_j$ for a big and a small color, whilst a common vertex with a big color would yield a polychromatic P_3.

Now, the set $A = \bigcup_{1 \leq i \leq \ell} N[v_i]$ contains at least $\ell(\delta + 1)$ vertices and at most 2ℓ different colors. Thus, the average size of color classes intersecting A is at least $\frac{\delta+1}{2}$, and all the remaining classes are big, of size at least $\frac{\delta+1}{2}$ each. This implies $\bar{\chi}_{3CC}(G) \leq \lfloor \frac{2p}{\delta+1} \rfloor$. ■

In a graph $G = (V, E)$ a set $S \subseteq V$ is a *neighborhood set* if $\bigcup_{v \in S}(\langle N[v] \rangle) = G$, where $\langle N(v) \rangle$ is the subgraph induced by $N[v]$, the closed neighborhood of v. The *neighborhood number* of a graph G, denoted by $n_0(G)$, is the minimum cardinality of a neighborhood set in G (see [9]). For short, we shall write *N-set* for neighborhood set in general, and *N_0-set* for neighborhood set of minimum cardinality.

Theorem 2. *Let G be a connected graph. Then, $\bar{\chi}_{3CC}(G) \leq n_0(G) + 1$. Further, for a tree T, $\bar{\chi}_{3CC}(T) = n_0(T) + 1$.*

Proof. Let $k = n_0(G)$ and $S = \{v_1, \ldots, v_k\}$ be an N_0-set. Suppose that the vertices of S are labeled in such a way that $(N[v_i] \cap \bigcup_{1\leq j<i} N[v_j]) \neq \emptyset$ for all $2 \leq i \leq k$. Such an order on S exists because G is connected.

Since each $N[v_i]$ can have at most two colors, and at least one of them occurs in $\bigcup_{1\leq j<i} N[v_j]$ if $i \neq 1$, G cannot be colored with more than $|S|+1 = n_0(G)+1$ colors. This completes the proof of the first statement.

To prove the second part, we first fix a root in the tree T and choose a smallest N-set S^* with $|S^*| = n_0(T)$. If S^* contains some vertex v all of whose children also belong to S^* (or, in particular, if v is a leaf) then v can be replaced by its parent in the N-set. Repeatedly applying this replacement, an N-set S is obtained, in which every vertex has at least one child not contained in S.

Next, we show a procedure which yields a proper 3CC-coloring of T with exactly $n_0(T)+1$ colors. First, assign color 1 to the root, and then in every step choose a vertex v which has already got a color but its children have not yet. To color its children, we apply the following rules:

(i) If $v \notin S$ then every child of v receives the color of v.

(ii) If $v \in S$ then we choose precisely one child u not contained in S. In the coloring, u will receive its dedicated color, whilst all the remaining children will get the color of v.

The number of used colors remains the same when we color the children of a vertex $v \notin S$, whilst it increases by precisely one when $v \in S$. Moreover, there is no leaf belonging to S. Taking also into account the color of the root, this means exactly $n_0(T)+1$ colors.

The obtained vertex coloring of T is a proper 3CC-coloring. Indeed, for every vertex v, in the neighborhood $N[v]$ there occur at most two different colors because if u gets its dedicated color then $u \notin S$ is assumed, hence its parent $v \in S$ has monochromatic $N[v] \setminus \{u\}$.

This coloring algorithm proves that for a tree T the inequality $\bar{\chi}_{3CC}(T) \geq n_0(T)+1$ holds. We have already proved that also $\bar{\chi}_{3CC}(T) \leq n_0(T)+1$ is valid, hence the second statement follows. ■

We remark that the algorithm described in the proof actually yields a 3-consecutive coloring. Moreover, it is known that $n_0(G)$ does not exceed the vertex covering number $\alpha_0(G)$, moreover $n_0(G) = \alpha_0(G)$ for every triangle-free graph (see [9]; characterizations for other graph classes and complexity results on $n_0(G)$ can be found in [6] and [4]). As a consequence, we have

Corollary 1.

(i) *For a connected graph* G, $\bar{\chi}_{3CC}(G) \leq \alpha_0(G) + 1$.

(ii) *For a tree* T, $\chi_{3c}(T) = \bar{\chi}_{3CC}(T) = \alpha_0(T) + 1 = \beta_1(T) + 1$ *where* $\beta_1(T)$ *is the edge independence number.*

(iii) *For trees, both* χ_{3c} *and* $\bar{\chi}_{3CC}$ *can be determined and an optimal coloring can be found in linear time.*

A set $S \subset V$ of vertices in a connected graph $G = (V, E)$ is called a *connected dominating set* if (i) every vertex $v \in V \setminus S$ is adjacent to at least one vertex in S, and (ii) the subgraph $G[S]$ induced by S is connected (see [10]). The minimum cardinality of a connected dominating set S is called the *connected domination number*, and is denoted by $\gamma_c(G)$; such a set S is called a γ_c-*set*. Condition (i) alone defines the notion of *dominating set*, the minimum cardinality of which is called *domination number* and is denoted by $\gamma(G)$.

Bounds on $\bar{\chi}_{3CC}(G)$ involving γ_c and γ are as follows.

Theorem 3. *For any connected graph* G, $\bar{\chi}_{3CC}(G) \leq \gamma_c(G) + 1$ *holds, moreover* $\bar{\chi}_{3CC}(G) \leq 2\gamma(G)$.

Proof. If S is a connected dominating set of G, its vertices have an ordering $v_1, \ldots, v_{|S|}$ such that $(N[v_i] \cap \bigcup_{1 \leq j < i} N[v_j]) \neq \emptyset$ for all $2 \leq i \leq |S|$. Similarly to the proof of the first part of Theorem 2, it can be proved that $\bar{\chi}_{3CC}(G) \leq |S| + 1$. Hence, choosing S to be a γ_c-set, the first inequality follows.

Since the closed neighborhood of each vertex contains at most two colors in any 3CC-coloring, and the closed neighborhoods of the vertices in a dominating set cover the entire vertex set, the second upper bound also holds. ■

A relation between $\bar{\chi}_{3CC}(G)$ and the chromatic number $\chi(G)$ is as follows: It is known that for a connected graph G of order p, $\gamma_c(G) \leq p - \Delta(G)$, where $\Delta(G)$ is the maximum degree of a vertex in G (cf. [5]) and $\chi(G) \leq \Delta(G) + 1$. Therefore we have the following:

Corollary 2. *For any connected graph* G *of order* $p \geq 3$, $\bar{\chi}_{3CC}(G) \leq p - \chi(G) + 2$.

Theorem 4. *For a unicyclic graph of order* $p \geq 3$,

$$\alpha_0(G) - 1 \leq \bar{\chi}_{3CC}(G) \leq \alpha_0(G) + 1.$$

Proof. In view of Corollary 1 we need to establish only the lower bound. Let C be the cycle in G and e be an edge of C. Then $G-e$ is a tree and, again by Corollary 1, $\bar{\chi}_{3CC}(G-e) = \alpha_0(G-e)+1$. Since G is unicyclic, either $\alpha_0(G-e) = \alpha_0(G)-1$, or $\alpha_0(G-e) = \alpha_0(G)$. Therefore, $\alpha_0(G-e) \geq \alpha_0(G)-1$. Also, $\bar{\chi}_{3CC}(G) = \bar{\chi}_{3CC}(G-e)$ or $\bar{\chi}_{3CC}(G) = \bar{\chi}_{3CC}(G-e)-1$ and hence $\bar{\chi}_{3CC}(G) \geq \bar{\chi}_{3CC}(G-e)-1$. Thus, $\bar{\chi}_{3CC}(G) \geq \alpha_0(G)-1$. ■

The upper bound is attained e.g. if G is a cycle with exactly one pendant edge at each of its vertices. The lower bound is attained e.g. if G is an odd cycle of length at least 5.

3. Characterizations

3.1. Three-colorability

Theorem 5. *A connected graph $G = (V, E)$ has a 3-consecutive C-coloring with exactly three colors — that is, $\bar{\chi}_{3CC}(G) \geq 3$ — if and only if its diameter is at least 3.*

Proof. To prove necessity by a contradiction, assume a connected graph G with diameter at most 2 and its proper 3CC-coloring φ using exactly 3 colors.

Since G is connected, there exist two adjacent vertices x and y having different colors, say $\varphi(x) = 1$ and $\varphi(y) = 2$. Moreover, consider a vertex z colored differently from each of them: $\varphi(z) = 3$. There cannot occur multicolored P_3 and hence, vertex z is adjacent neither with x nor with y. Since the diameter equals 2, there exist common neighbors x' and y' for the vertex pairs x, z and y, z, respectively. Now, consider the P_3 subgraphs $x'xy$ and $xx'z$. The former one forces that x' does not have color 3, whilst the latter forbids color 2. Therefore, the color of x' should be 1, and by a similar argument we obtain that y' has color 2. (This implies that x' and y' are different vertices.) But in this case the forbidden multicolored P_3 $x'zy'$ would appear. Consequently, if $\bar{\chi}_{3CC}(G) \geq 3$, then the diameter is at least 3.

Proving the opposite direction, we consider a graph G containing two vertices x and y at distance 3 apart. Let the vertex coloring φ assign color 1 to x, color 3 to y, and all the remaining vertices receive color 2. Since x and y cannot belong to a common P_3, every three consecutive vertices can get at most two different colors. Therefore, φ is a proper 3CC-coloring and $\bar{\chi}_{3CC}(G) \geq 3$ holds. ■

3.2. Four-colorability

Here we characterize the graphs admitting a 3-consecutive C-coloring with at least four colors. In the proof, the following notion will be used.

Definition 1. Let φ be a vertex coloring of a connected graph G. The *color-graph of* G with respect to coloring φ, denoted by $C_\varphi(G)$, has the colors occurring in φ as its vertices — called *color-vertices* — and two distinct color-vertices are adjacent in $C_\varphi(G)$ if and only if there exist two adjacent vertices in G having the corresponding two colors.

Theorem 6. *A connected graph $G = (V, E)$ admits a 3-consecutive C-coloring with exactly four colors — that is, $\bar{\chi}_{3CC}(G) \geq 4$ — if and only if it satisfies at least one of the following properties:*

(i) *There exist three vertices $x, y, z \in V$ such that any two of them are at distance at least 3 apart.*

(ii) *G has diameter at least 5.*

(iii) *There exists a cycle C of length eight in G such that, for each vertex $v \in V$, there exists a vertex u in C for which $d_G(u, v) = 4$ holds.*

Proof. ($\Rightarrow$) To prove necessity, we assume a 3CC-coloring φ of G with exactly four colors. In this case, $C_\varphi(G)$ has vertex set $\{1, 2, 3, 4\}$. Since G is connected and all the four colors are used in φ, the graph $C_\varphi(G)$ is connected, too. We distinguish three cases on the basis of vertex degrees occurring in $C_\varphi(G)$.

1. First, we assume that there exists a color-vertex in $C_\varphi(G)$ whose degree is 3. We suppose without loss of generality that this color-vertex is 1.

That is, there exist three vertices x, y and z in G with colors 2, 3 and 4, respectively, such that each of them is adjacent to a vertex from color class 1. We will prove that any two of the vertices x, y, z have distance at least 3. In a 3CC-coloring the closed neighborhood of any vertex can contain at most two different colors. This implies that x and y cannot have a common neighbor with color 1 or 4. On the other hand, if their common neighbor had color 2 (or 3) then in $N[y]$ (or in $N[x]$) there would occur three colors 1, 2, and 3. Therefore, the distance between x and y cannot be smaller than 3. The analogous statement is true for the pairs (x, z) and (y, z) as well. Hence, condition (i) is fulfilled. In the sequel, we refer to three vertices with this property as *three distant vertices*.

2. Second, we suppose that the color-graph $C_\varphi(G)$ is a path, where we assume the order $1, 2, 3, 4$.

Choose a vertex $x \in V$ with color 1 and a vertex $y \in V$ with color 4. Due to the assumed structure of $C_\varphi(G)$, every path connecting x and y contains vertices with colors 2 and 3 as well. Moreover, there occur at least two vertices with color 2 in it. Indeed, assuming only one vertex from color-class 2 in this path, this would have neighbors colored with 1 and also with 3, yielding a forbidden multicolored P_3. Hence, every x–y path has at least two internal vertices from color class 2 and, similarly, there exist at least two internal vertices with color 3. Consequently, $d_G(x, y) \geq 5$, complying with condition (ii).

3. In the remaining cases, every vertex of $C_\varphi(G)$ has degree two; that is, $C_\varphi(G)$ is a cycle. We assume the cyclic order 1-2-3-4-1 of colors, and all additions concerning them will be taken modulo 4. Also in this case, (i) and/or (ii) may be satisfied. But we assume throughout that none of the first two properties is valid for G, and then prove that (iii) necessarily holds under this assumption.

We can partition each color class α $(1 \leq \alpha \leq 4)$ into two parts:

- $V_{\alpha,\alpha+1}$ contains the vertices colored with α and having a neighbor of color $\alpha + 1$.
- $V_{\alpha,\alpha-1}$ contains the vertices colored with α and having a neighbor of color $\alpha - 1$.

Both $V_{\alpha,\alpha-1}$ and $V_{\alpha,\alpha+1}$ are nonempty, but $V_{\alpha,\alpha-1} \cap V_{\alpha,\alpha+1} = \emptyset$. Now, suppose for a contradiction that there exists a vertex v of color α which is not contained in $V_{\alpha,\alpha-1} \cup V_{\alpha,\alpha+1}$. Choose a vertex $u \in V_{\alpha+1,\alpha+2}$. All the neighbors of v have color α, therefore any v–u path contains an internal vertex with color α, while u has no neighbor of color α, forcing one more internal vertex. Consequently, $d_G(v, u) \geq 3$. Similarly, any vertex $w \in V_{\alpha-1,\alpha-2}$ has distance at least 3 from both vertices v and u. This would mean three distant vertices complying with property (i), but this contradicts our present assumption.

Hence, $\bigcup_{\alpha=1}^{4}(V_{\alpha,\alpha-1} \cup V_{\alpha,\alpha+1}) = V$ holds. In other words, the vertex set of G is partitioned into eight nonempty disjoint sets admitting a cyclic order.

For the sake of simpler discussion, let us introduce the notation $Q_1 = V_{1,2}$, $Q_2 = V_{2,1}$, $Q_3 = V_{2,3}$, $Q_4 = V_{3,2}$, $Q_5 = V_{3,4}$, $Q_6 = V_{4,3}$, $Q_7 = V_{4,1}$, $Q_8 = V_{1,4}$. Subscripts of the sets Q_i will be considered modulo 8.

Every edge of G must have its endpoints either in the same class Q_i or in two cyclically consecutive classes. Thus, any two vertices $x \in Q_i$ and $y \in Q_j$ $(i < j)$ are at distance at least $\min\{j - i, 8 - (j - i)\}$ apart. On the other hand, we claim that any two vertices from the same or from two consecutive classes have distance at most 2. Indeed, if $d_G(x, y) \geq 3$ for some $x \in Q_i$ and $y \in Q_j$, where $j = i$ or $j = i + 1$, then x and y together with any vertex from Q_{j+3} would be three distant vertices, what does not meet the present requirements.

Now, choose one vertex q_i from each class Q_i. Due to the previous observation, any two vertices q_i and q_{i+1} are adjacent or have a common neighbor which received the color of q_i or q_{i+1}. Hence, joining every two consecutive vertices by a shortest path, we obtain a cycle (or a closed walk), where all the four colors occur and, for each $1 \leq \alpha \leq 4$, the vertices having color α form a connected arc (subpath). Hence, we can consider a shortest cycle with this property. This cycle C contains some vertex from each class Q_i; thus, its length is at least 8. By minimality and the structure of $C_\varphi(G)$, for any two $x, y \in C$, the equality $d_C(x, y) = d_G(x, y)$ holds. Consequently, if cycle C had nine or more vertices, we could choose three distant vertices, and they would have distances at least 3 not only in C but also in G. Hence, under the assumed conditions, $C \cong C_8$ and it involves exactly one vertex r_i from each class Q_i.

Any vertex $v \in V$ is contained in a uniquely determined class Q_j. Since property (ii) is not valid for G, we have $d_G(v, r_{j+4}) \leq 4$. On the other hand, every path from Q_j to Q_{j+4} has to involve vertices either from Q_{j+1}, Q_{j+2} and Q_{j+3}, or from Q_{j-1}, Q_{j-2} and Q_{j-3}, hence we obtain $d_G(v, r_{j+4}) = 4$, what completes the proof of necessity.

($\Leftarrow$) To prove sufficiency, we will construct appropriate 3CC-colorings for graphs G having property (i) or (ii), and also for graphs G satisfying only (iii) from the three constraints.

(I) Assume that there exist three vertices x, y, z having mutual distances at least 3. By this property, there is no P_3 involving at least two of them. Therefore, we can assign colors $1, 2$ and 3 to the vertices x, y and z, respectively. If all the remaining vertices receive color 4, the assignment obtained is a proper 3CC-coloring with four colors, hence $\bar{\chi}_{3CC}(G) \geq 4$.

(*II*) Assuming property (ii), there exist two vertices x and y at distance 5 apart. Consider the following coloring φ:

- $\varphi(x) = 1; \quad \varphi(y) = 4;$
- $\varphi(v) = 2$ if $1 \le d_G(x, v) \le 2;$
- $\varphi(v) = 3$ if $d_G(x, v) > 2$ and $v \ne y$.

This yields a proper 3CC-coloring, since color 1 can occur together only with color 2 in a P_3, and similarly, color 4 appears only with color 3. Consequently, no multicolored P_3 can arise.

(*III*) Assume that the graph G satisfies (iii) but none of the conditions (i) and (ii). Let the cycle corresponding to (iii) be $C_8 = r_1r_2r_3r_4r_5r_6r_7r_8$.

Consider a vertex $v \in V$. By condition (iii), there exists a vertex r_i in the cycle whose distance from v equals 4. Since $4 = d_G(v, r_i) \le d_G(v, r_{i+1}) + 1$, we obtain $d_G(v, r_{i+1}) \ge 3$ and, similarly, $d_G(v, r_{i-1}) \ge 3$ must hold for the other neighbor of r_i, too. Let us assume that there exists a further vertex r_j in the cycle whose distance from v is at least 3. In this case r_{i-1}, r_j, v or r_{i+1}, r_j, v would be three distant vertices. Since (i) is supposed to be not valid, this is a contradiction. Therefore, for all vertices r_k distinct from r_{i-1}, r_i and r_{i+1}, the inequality $d_G(v, r_k) \le 2$ holds. Taking into account that the relations $d_G(v, r_i) \ge 4$ and $d_G(v, r_{i\pm2}) \le 2$ imply $d_G(v, r_{i\pm1}) = 3$, the above argument also yields that for every v there exists precisely one $r_i \in C_8$ at distance 4.

This uniqueness makes it possible to define the partition of the vertex set into eight disjoint classes (subscript addition taken modulo 8):

$$v \in Q_j \quad \Longleftrightarrow \quad d_G(v, r_{j+4}) = 4, \qquad \text{for all } 1 \le j \le 8.$$

Since $r_j \in Q_j$ holds for every j, none of the partition classes is empty. Summarizing the previous observations:

If $v \in Q_j$,

- $d_G(v, r_{j+4}) = 4;$
- $d_G(v, r_{j+3}) = d_G(v, r_{j+5}) = 3;$
- $d_G(v, r_k) \le 2$ otherwise.

Next, we prove that any two adjacent vertices x and y belong either to the same Q_i or to two consecutive partition classes.

Assume $x \in Q_i$, $y \in Q_j$, and $d_G(x, y) = 1$. By the properties of the distance function we obtain

$$d_G(x, y) + d_G(y, r_{i+4}) \geq d_G(x, r_{i+4}) = 4,$$
$$d_G(y, r_{i+4}) \geq 3.$$

The inequalities can be fulfilled only if $i + 4$ equals either $j + 4$ or $j + 3$ or $j + 5$. These correspond to the cases where $i = j$, $i = j - 1$ or $i = j + 1$; that is, when x and y belong either to the same or two consecutive partition classes.

Now, we can define an appropriate 3CC-coloring φ with four colors:

- $\varphi(v) = k$ if $v \in Q_{2k-1} \cup Q_{2k}$, for all $1 \leq k \leq 4$.

As we have shown, there occur edges only between consecutive partition classes and inside one class, hence no multicolored P_3 can arise. This proves the assertion for the last case. ■

Remark 1. As it can be read out from the proof, in Theorem 6 the prescribed property (iii) can be replaced by other statements without changing validity:

- C_8 can be assumed to be an induced subgraph of G.
- We can prescribe that for every vertex $v \in V$ there exists precisely one vertex $u \in C_8$ at distance 4.

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