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RADIO NUMBER FOR SOME THORN GRAPHS

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Abstract

For a graph G and any two vertices u and v in G, let $d(u, v)$ denote the distance between u and v and let $\mathrm{diam}(G)$ be the diameter of G. A multilevel distance labeling (or radio labeling) for G is a function f that assigns to each vertex of G a positive integer such that for any two distinct vertices u and v, $d(u, v)+ \mid f(u) - f(v) \mid\geq \mathrm{diam}(G) + 1$. The largest integer in the range of f is called the span of f and is denoted $span(f)$. The radio number of G, denoted $rn(G)$, is the minimum span of any radio labeling for G. A thorn graph is a graph obtained from a given graph by attaching new terminal vertices to the vertices of the initial graph. In this paper the radio numbers for two classes of thorn graphs are determined: the caterpillar obtained from the path P_n by attaching a new terminal vertex to each non-terminal vertex and the thorn star $S_{n,k}$ obtained from the star S_n by attaching k new terminal vertices to each terminal vertex of the star.

Keywords: multilevel distance labeling, radio number, caterpillar, diameter.

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1. Introduction

Radio labeling (or multilevel distance labeling) of graphs is motivated by restrictions in assigning channel frequencies for radio transmitters [6]. More precisely, for a set of given stations, it is required to assign to each station a channel such that interference is avoided and the span of assigned channels is minimized. Channels are positive integers, and the level of interference is related to the distances between stations. For small distances the interference is stronger, so the stations that are geographically close must be assigned channels with large frequency difference, while for stations that are further apart this difference can be small. This type of problem can be modeled by a graph, where vertices represent stations and every vertex has a positive number assigned, representing a channel. Every pair of close stations is connected by an edge.

Let G be a connected graph with vertex set $V(G)$ and diameter $\mathrm{diam}(G)$. For any two vertices u and v of G, $d(u,v)$ represents the distance between them. A vertex u for which there exists a vertex v such as $d(u,v) = \mathrm{diam}(G)$ is called a *peripheral vertex*. A *radio labeling* (or *multilevel distance labeling*) of G is a one-to-one mapping $f : V(G) \to \mathbb{Z}^+$ which assigns to each vertex a positive integer, satisfying the condition

$$d(u,v) + \mid f(u) - f(v) \mid \geq \mathrm{diam}(G) + 1$$

for every two distinct vertices u, v. This condition is referred to as *radio condition* (or *multilevel distance labeling condition*). The *span* of f, denoted by *span*(f), is the maximum integer in the range of f. The *radio number* of G, denoted $rn(G)$, is the smallest span in all radio labelings of G. Since the radio condition contains only the difference of the labels, a radio labeling realizing $rn(G)$ must have the minimum label equal to 1.

For many classes of graphs is not easy to determine their radio number. For radio numbers of paths and cycles in [2] and [3] only upper bounds were obtained. Later, in [8], Liu and Zhu determined the exact values of these radio numbers. In [9] Rahim and Tomescu considered radio labelings for helm graphs (a helm graph H_n is obtained from the wheel W_n by attaching a vertex of degree one to each of the n vertices of the cycle of the wheel).

Liu [7] determined a lower bound for the radio number of trees and characterized the trees achieving this bound. To be able to discuss these results, we introduce the following notions.

Let T be a tree rooted at a vertex w. For any two vertices u and v, if u is on the path connecting w and v, then u is an *ancestor* of v and v is a *descendent* of u. The *level function* on $V(T)$, for a fixed root w, is defined by

$$L_w(u) = d(w,u), \ \forall\, u \in V(T).$$

For any $u, v \in V(T)$, define

$$\Phi_w(u,v) = \max\{L_w(t) \mid t \text{ is a common ancestor of } u \text{ and } v\}.$$

Let w' be a neighbor of w. The subtree induced by w' together with all the descendents of w' is called a *branch*.

Remark 1.1 ([7])**.** Let T be a tree rooted at w. For any vertices u and v we have:

(1) $\Phi_w(u,v) = 0$ if and only if u and v belong to different branches, unless one of them is w;

(2) $d(u,v) = L_w(u) + L_w(v) - 2\Phi_w(u,v)$.

For any vertex w in T, the *status of w in T* is defined by

$$s_T(w) = \sum_{u \in V(T)} L_w(u) = \sum_{u \in V(T)} d(u,w).$$

The *status of T* is the minimum status among all vertices of T:

$$s(T) = \min\{s_T(w) \mid w \in V(T)\}.$$

A vertex w^* of T is called a *weight center* of T if $s_T(w^*) = s(T)$.

Remark 1.2. The set of all weight centers of a tree T is known as the median of T ([1]).

Because in [7] a radio labeling is also allowed to take value 0, the radio numbers and limits determined in [7] are one less than the radio numbers previously defined in this article. We will recall the results from [7], making the necessary adjustments by adding one to the bounds and radio numbers arising from these results.

Theorem 1.3 [7]**.** *Let T be a tree with n vertices and diameter d. Then*

$$rn(T) \geq (n-1)(d+1) + 2 - 2s(T).$$

Moreover, the equality holds if and only if for every weight center w^* there exists a radio labeling f with $f(u_1) = 1 < f(u_2) < \cdots < f(u_n)$ for which all the following properties hold, for every i with $1 \leq i \leq n-1$:

(1) *u_i and u_{i+1} belong to different branches, unless one of them is w^*;*

(2) *$\{u_1, u_n\} = \{w^*, v\}$, where v is some vertex with $L_{w^*}(v) = 1$;*

(3) *$f(u_{i+1}) = f(u_i) + d + 1 - L_{w^*}(u_i) - L_{w^*}(u_{i+1})$.*

Thorn graphs were introduced by Gutman in [5]. For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, a *thorn graph* of G with nonnegative parameters $p_1, p_2, \ldots, p_n$ is obtained by attaching p_i new vertices of degree one to the vertex v_i, for each $1 \leq i \leq n$. A thorn path is called *caterpillar*. In the following sections we will determine the radio number for two classes of thorn graphs: a particular class of caterpillars and the thorn star.

2. Radio Labeling and Radio Number for a Class of Caterpillars

For $n \geq 2$ we denote by CP_n the caterpillar obtained from the path with n vertices P_n by attaching a new terminal vertex to each non-terminal vertex of P_n. CP_n has $m = 2n - 2$ vertices and diameter $d = n - 1$.

In this section we will determine the radio number for this type of caterpillar, more precisely we will show that: $rn(CP_3) = 5$, $rn(CP_n) = 4k^2 - 6k + 4$ for $n = 2k$ and $rn(CP_n) = 4k^2 - 2k + 4$ for $n = 2k + 1$, $k \geq 2$.

We will consider two cases, in accordance with the parity of n.

Case 1. n is even.
Let $n = 2k$, $k \geq 1$. In this case we denote by $v_1, \ldots, v_{2k}$ the vertices of P_n from which the caterpillar CP_n is obtained, by v'_{i-1} the terminal vertex attached to v_i, for $2 \leq i \leq k$, and by v'_{i+1} the terminal vertex attached to v_i, for $k + 1 \leq i \leq 2k - 1$ (Figure 1).

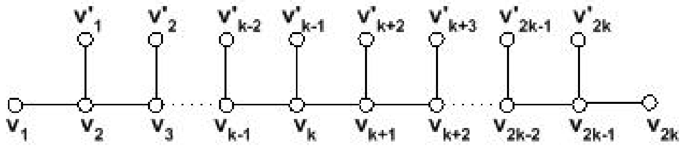

Figure 1. CP_{2k}

We have $m = 4k - 2$ and $d = 2k - 1$.

Theorem 2.1. *For $n = 2k$, $k \geq 1$, the radio number for CP_n is $rn(CP_n) = 4k^2 - 6k + 4$.*

Proof. We use Theorem 1.3.
Let $n = 2k$, $k \geq 1$. In this case CP_n has two weight centers, v_k and v_{k+1}. We have

$$\begin{aligned} s(CP_n) &= s_{CP_n}(v_k) = \sum_{u \in V(CP_n)} d(u, v_k) \\ &= 3 \cdot 1 + 4(2 + \cdots + k - 1) + 2 \cdot k \\ &= 2k^2 - 1. \end{aligned}$$

By Theorem 1.3,

$$\begin{aligned} rn(CP_n) &\geq (m-1)(d+1) + 2 - 2s(T) = (4k-3)(2k) + 2 - 2(2k^2 - 1) \\ &= 4k^2 - 6k + 4. \end{aligned}$$

Moreover, in order to prove equality, it suffices to find a radio labeling f for CP_n with $span(f) = 4k^2 - 6k + 4$ (or, equivalently, a radio labeling that satisfies the properties (1)–(3) in Theorem 1.3 for every weight center of CP_n; furthermore, since CP_n is symmetrical, it suffices to give a radio labeling for CP_n with these properties only for weight center v_k).

For that, we order the vertices of CP_n as follows: alternate v_j and v'_{k+j} for $j = k, k-1, \ldots, 2$, then v_1, v_{2k}, then alternate v'_j and v_{k+j} for $j = k-1, k-2, \ldots, 1$. We rename the vertices of CP_n in the above ordering by u_1, u_2, …, u_m.

We define a labeling f for CP_n using the rules given by (2) and (3) from Theorem 1.3 as follows: $f(u_1) = 1$, $f(u_{i+1}) = f(u_i) + d + 1 - d(u_{i+1}, u_i)$ for $1 \leq i \leq m - 1$.

For example, if $k = 4$, the order in which the vertices are labeled and their labels are shown in Figure 2.

Since we have the following distances: $d(v_j, v'_{k+j}) = k$, for $2 \leq j \leq k$; $d(v'_{k+j}, v_{j-1}) = k+1$, for $2 \leq j \leq k$; $d(v_1, v_{2k}) = 2k-1$, $d(v_{2k}, v'_{k-1}) = k+1$; $d(v'_j, v_{k+j}) = k$, for $1 \leq j \leq k-1$ and $d(v_{k+j}, v'_{j-1}) = k+1$, for $2 \leq j \leq k-1$, we obtain:

$$\begin{aligned} span(f) &= f(u_m) = f(v_{k+1}) = f(u_1) + (m-1)(d+1) - \sum_{i=1}^{m-1} d(u_{i+1}, u_i) \\ &= 4k^2 - 6k + 4. \end{aligned}$$

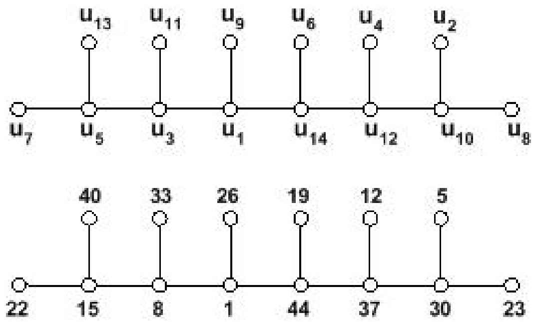

Figure 2. CP_{2k}

The following relations also hold:
$f(v_j) = f(v_{j+1}) + 2k - 1$ for $1 \leq j \leq k-1$;
$f(v_j) = f(v_{j+1}) + 2k - 1$ for $k+1 \leq j \leq 2k-1$;
similary,
$f(v'_j) = f(v'_{j+1}) + 2k - 1$ for $k+2 \leq j \leq 2k-1$;
$f(v'_j) = f(v'_{j+1}) + 2k - 1$ for $1 \leq j \leq k-2$;
$f(v'_{k-1}) = f(v'_{k+2}) + 2k - 1$;
$f(v_{2k}, v'_{k+2}) = k$;
$|f(v'_i) - f(v_j)| \geq 2k-1$ if v'_i and v_j are not consecutive in the order previously established.

Consecutive vertices in the ordering verify the radio constraint by construction. Then it is easy to check that for every two distinct vertices u and v the radio condition is verified, considering each particular case of pairs of vertices (both vertices are from P_n, both are terminal or they are of different type), so f is a radio labeling for CP_n. Moreover, from the way f was defined, it satisfies the properties (1)–(3) in Theorem 1.3 for the weight center v_k, since the vertices u_i and u_{i+1} belong to different branches for $2 \leq i \leq m-1$, $u_1 = v_k$ and $u_m = v_{k+1}$, with $L_{v_k}(v_{k+1}) = d(v_k, v_{k+1}) = 1$.

Case 2. n is odd.
Let $n = 2k+1$. For $k = 1$, it is easy to see that $rn(CP_3) = 5$, CP_3 being the star S_3. If $k \geq 2$, in order to label the vertices of CP_n, we denote by $v_1, \ldots, v_{2k+1}$ the vertices of P_n from which the caterpillar is obtained and by v'_i the terminal vertex attached to v_i, for $2 \leq i \leq 2k$ (Figure 3). We have $m = 4k$ and $d = 2k$. ■

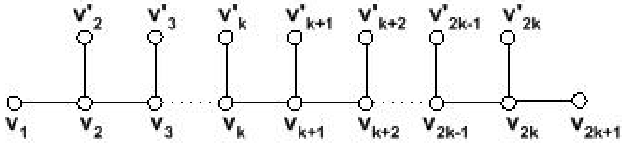

Figure 3. CP_{2k+1}

Theorem 2.2. *For $n = 2k+1$, $k \geq 2$ the radio number for CP_n is $rn(CP_n) = 4k^2 - 2k + 4$.*

Proof. We shall see that it is not sufficient to use only Theorem 1.3 to prove the equality. Let $n = 2k+1$, $k \geq 2$. In this case CP_n has a single weight center, v_{k+1}. We have:

$$\begin{aligned} s(CP_n) &= s_{CP_n}(v_{k+1}) = \sum_{u \in V(CP_n)} d(u, v_{k+1}) \\ &= 3 \cdot 1 + 4(2 + \cdots + k) = 2k^2 + 2k - 1. \end{aligned}$$

By Theorem 1.3,

$$\begin{aligned} rn(CP_n) &\geq (m-1)(d+1) + 2 - 2s(T) \\ &= (4k-1)(2k+1) + 2 - 2(2k^2 + 2k - 1) \\ &= 4k^2 - 2k + 3. \end{aligned}$$

We will prove that there is no radio labeling for CP_n that satisfies the properties (1)–(3) for the weight center v_{k+1} of CP_n, so the inequality is strict.

Suppose that there exists a radio labeling f for CP_n with these properties. We order the vertices of CP_n by their labels and rename the vertices in this ordering by $u_1, u_2, \ldots, u_m$: $1 = f(u_1) < f(u_2) < \cdots < f(u_m)$. Let u_i, u_{i+1}, u_{i+2} be three consecutive vertices in this ordering, $1 \leq i \leq m-2$. We can assume, without loss of generality, that $d(u_i, u_{i+1}) \geq d(u_{i+1}, u_{i+2})$. We shall prove the following claims:

(a) If one of the vertices u_i, u_{i+1}, u_{i+2} belongs to the path that connects the other two, then $\min\{d(u_i, u_{i+1}), d(u_{i+1}, u_{i+2})\} \leq k$;

(b) $\min\{d(u_i, u_{i+1}), d(u_{i+1}, u_{i+2})\} \leq k+1$;

(c) If v is a peripheral vertex in CP_{2k+1} and $v = u_i$, then i is different from 1 and m. Moreover, if its neighboring vertices u_{i-1} and u_{i+1} are different from v_{k+1}, then one of the vertices u_{i-1} or u_{i+1} is v'_{k+1}.

Claim (a). We assume first that one of the vertices u_i, u_{i+1}, u_{i+2} belongs to the path that connects the other two.

Suppose that $\min\{d(u_i, u_{i+1}), d(u_{i+1}, u_{i+2})\} > k$. Then $d(u_i, u_{i+1}) > \frac{n}{2}$ and $d(u_{i+1}, u_{i+2}) > \frac{n}{2}$. Because we assumed that $d(u_i, u_{i+1}) \geq d(u_{i+1}, u_{i+2})$, u_{i+2} must lie on the path connecting u_i and u_{i+1}, hence $d(u_i, u_{i+2}) = d(u_i, u_{i+1}) - d(u_{i+1}, u_{i+2})$. By property (3) from Theorem 1.3, $f(u_{i+1}) - f(u_i) = d + 1 - d(u_i, u_{i+1}) = n - d(u_i, u_{i+1})$, hence we have:

$$\begin{aligned}
f(u_{i+2}) - f(u_i) &= f(u_{i+2}) - f(u_{i+1}) + f(u_{i+1}) - f(u_i) \\
&= n - d(u_i, u_{i+1}) + n - d(u_{i+1}, u_{i+2}) \\
&= 2n - (d(u_i, u_{i+1}) + d(u_{i+1}, u_{i+2})) \\
&= 2n - (d(u_i, u_{i+1}) - d(u_{i+1}, u_{i+2}) + 2d(u_{i+1}, u_{i+2})) \\
&= 2n - d(u_i, u_{i+2}) - 2d(u_{i+1}, u_{i+2}) \\
&< 2n - d(u_i, u_{i+2}) - 2\tfrac{n}{2} = n - d(u_i, u_{i+2}).
\end{aligned}$$

This contradicts that f is a radio labeling. It follows that Claim (a) is true.

Claim (b). In order to prove Claim (b) it suffices to consider the case when no vertex belongs to the path connecting the other two.

Suppose that $\min\{d(u_i, u_{i+1}), d(u_{i+1}, u_{i+2})\} > k + 1$. It results that $d(u_i, u_{i+1}) \geq k+2$ and $d(u_{i+1}, u_{i+2}) \geq k+2$. Since $d(u_i, u_{i+1}) \geq d(u_{i+1}, u_{i+2})$, by Theorem 1.3 (1), we can only have the following situation: u_{i+2} does not belong to the path connecting u_i and u_{i+1}, but there exists a vertex u'_{i+2} adjacent to u_{i+2} that belongs to this path. Then

$$\begin{aligned}
d(u_i, u_{i+2}) &= d(u_i, u_{i+1}) - d(u_{i+1}, u'_{i+2}) + 1 \\
&= d(u_i, u_{i+1}) - (d(u_{i+1}, u_{i+2}) - 1) + 1 \\
&= d(u_i, u_{i+1}) - d(u_{i+1}, u_{i+2}) + 2.
\end{aligned}$$

Hence

$$\begin{aligned}
f(u_{i+2}) - f(u_i) &= 2n - (d(u_i, u_{i+1}) + d(u_{i+1}, u_{i+2})) \\
&= 2n - (d(u_i, u_{i+2}) + 2d(u_{i+1}, u_{i+2}) - 2) \\
&= 2n - d(u_i, u_{i+2}) - 2d(u_{i+1}, u_{i+2}) + 2 \\
&\leq 2n - d(u_i, u_{i+2}) - 2(k + 2) + 2
\end{aligned}$$

$$= 2n - d(u_i, u_{i+2}) - (n+3) + 2$$
$$= n - 1 - d(u_i, u_{i+2})$$
$$< n - d(u_i, u_{i+2}),$$

which is a contradiction since

$$f(u_{i+2}) - f(u_i) \geq d + 1 - d(u_i, u_{i+2}) = n - d(u_i, u_{i+2}),$$

so Claim (b) follows.

Claim (c). Let v be a peripheral vertex in CP_{2k+1} (v is one of vertices v_1, v_{2k+1}, v'_2, v'_{2k}). For any vertex u not belonging to the same branch as v we have $d(v,u) \geq k+1$. Also, $d(v,u) = k+1$ holds only for those two vertices u which are also adjacent to the center v_{k+1}. Let i be an index between 1 and m such that $v = u_i$. By property (1) from Theorem 1.3, $\{u_1, u_m\} = \{v_{k+1}, v^*\}$, with $d(v_{k+1}, v^*) = 1$, hence i is different from 1 and m and we have $f(u_{i-1}) < f(v) < f(u_{i+1})$. Moreover, if u_{i-1} and u_{i+1} are different from center v_{k+1}, since $\min\{d(u_{i-1}, v), d(v, u_{i+1})\} \geq k+1$, the assumptions from Claim (a) are not verified, so none of the vertices u_{i-1}, v, u_{i+1} belongs to the path connecting the other two, and $\min\{d(u_{i-1}, v), d(v, u_{i+1})\} = k+1$. It follows that one of the vertices u_{i-1} or u_{i+1} is v'_{k+1}.

Hence we proved Claim (c).

Since f satisfies theorem 1.3 (2), for at least three peripheral vertices their neighboring vertices u_{i-1} and u_{i+1} are different from the center v_{k+1}. It follows that at least three peripheral vertices have the property that one of the vertices u_{i-1} or u_{i+1} is v'_{k+1}, which is impossible.

Therefore there is no radio labeling f for CP_n that verifies the properties (1)–(3) in Theorem 1.3 for the weight center v_{k+1} of CP_n. Hence $rn(CP_n) > 4k^2 - 2k + 3$.

To prove that $rn(CP_n) = 4k^2 - 2k + 4$ it suffices to find a radio labeling f for CP_n with $span(f) = 4k^2 - 2k + 4$. For that, we order the vertices of CP_n as follows: v_{k+1}, v_1, v'_{k+1}, then alternate v'_{2k-j} and v'_{k-j} for $j = 0, 1, \ldots, k-2$, then v_{2k+1}, then alternate v_{k-j} and v_{2k-j} for $j = 0, 1, \ldots, k-2$. We rename the vertices of CP_n in the above ordering by $u_1, u_2, \ldots, u_m$.

We define a labeling f for CP_n using the rules given by (3) from Theorem 1.3 as follows: $f(u_1) = 1$, $f(u_{i+1}) = f(u_i) + d + 1 - d(u_{i+1}, u_i)$ for $1 \leq i \leq m-1$, with one exception: $f(v_{2k+1}) = f(v'_2) + 2$.

For example, if $k = 4$, the order in which the vertices are labeled and their labels are shown in Figure 4.

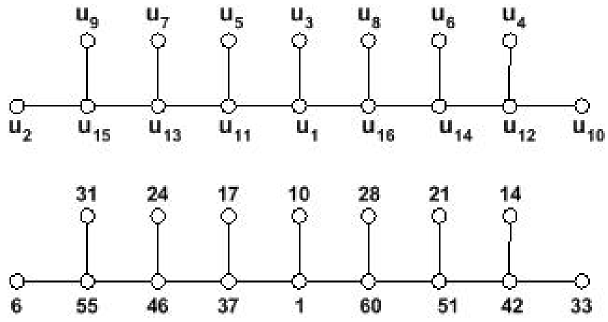

Figure 4. CP_{2k+1}

As in Case 1, from the definition of f and the distances between consecutive vertices in the above ordering, we obtain:

$$span(f) = 4k + 1 + (2k - 1)(k - 2) + 2 + (2k + 1)(k - 1) = 4k^2 - 2k + 4.$$

Also, it is easy to verify that for every two distinct vertices u and v the radio condition is verified, considering each particular pair of vertices as in Case 1 and taking in consideration the following facts: every consecutive vertex in the ordering considered above verify the radio constraint by construction, $|f(v_i) - f(v_j)| \geq 2k + 1$ if v_i and v_j are not consecutive; similary $|f(v'_i) - f(v'_j)| \geq 2k - 1$ if v'_i and v'_j are not consecutive; $f(v_k) - f(v'_2) = k + 2 \geq 2k + 1 - d(v_k, v'_2)$, $f(v'_i) - f(v_1) \geq f(v'_{2k}) - f(v_1) = 2k$ for $i \neq k + 1$, $f(v_{2k+1}) - f(v'_{k+2}) = k + 1 \geq 2k + 1 - d(v_{2k+1}, v'_{k+2})$ and the remaining differences for non-consecutive vertices v'_i and v_j are $|f(v'_i) - f(v_j)| \geq 2k + 1$.

So f is a radio labeling for CP_n, hence $rn(CP_{2k+1}) = 4k^2 - 2k + 4$. ■

3. Radio Labeling and Radio Number for a Thorn Star

The thorn star $S_{n,k}$ is the graph obtained from the star graph S_n by attaching k new terminal vertices to each terminal vertex of the star. We denote by z the center of the star, with $v_1, v_2, \ldots, v_n$ the terminal vertices from the initial star S_n and with $u_{i1}, u_{i2}, \ldots, u_{ik}$, $1 \leq i \leq n$ the new terminal vertices attached to the vertex v_i, for $1 \leq i \leq n$.

We have $|V(S_{n,k})| = 1 + n + nk = (k+1)n + 1$ and $\text{diam}(S_{n,k}) = 4$.
We will show that $rn(S_{n,k}) = (k+3)n + 2$ for $n \geq 3$ and $rn(S_{2,k}) = 3k + 8$.

Theorem 3.1. *For $n \geq 3$ and $k \geq 1$, $rn(S_{n,k}) = (k+3)n + 2$.*

Proof. We will first show that $rn(S_{n,k}) \geq (k+3)n + 2$. For that we use Theorem 1.3. The weight center of $S_{n,k}$ is z, hence we have

$$s(S_{n,k}) = s_{S_{n,k}}(z) = \sum_{i=1..n} d(z, v_i) + \sum_{\substack{i\,=\,1..n \\ j\,=\,1..k}} d(z, u_{ij}) = n + 2nk.$$

It follows that $rn(S_{n,k}) \geq 5(k+1)n + 2 - 2(n + 2nk) = (k+3)n + 2$.

To prove equality, it suffices to find a radio labeling f for $S_{n,k}$ with $span(f) = (k+3)n + 2$.

We define a label f for $S_{n,k}$ as follows:
$f(z) = 1$, $f(v_n) = 5$, $f(v_j) = (k+3)n + 2 - 3(n - j - 1)$ for $1 \leq j \leq n-1$ (vertices v_j have as labels numbers starting with $kn+8$, the maximum label of these vertices being $(k+3)n+2$), and terminal vertices are labeled with values from 7 to $kn + 6$ as follows: $f(u_{jt}) = 7 + (j-1) + (t-1)n$, for $1 \leq j \leq n$, $1 \leq t \leq k$. For $n = 4$ and $k = 3$ the labeling is shown in Figure 5.

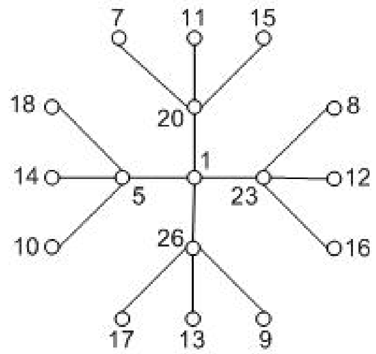

Figure 5. A radio labeling for $S_{4,3}$.

Hence $span(f) = f(v_{n-1}) = (k+3)n + 2$. It remains to verify the radio condition for each pair of vertices. We have the following cases:

- $d(u_{jt}, v_j) = 1$, $1 \leq j \leq n$, $1 \leq t \leq k$. It suffices to show that in this case we have $|f(v_j) - f(u_{jt})| \geq 4$.

For $1 \le j \le n-1$

$$\begin{aligned}|f(v_j) - f(u_{jt})| &= f(v_j) - f(u_{jt}) \\ &= (k+3)n + 2 - 3(n-j-1) - 7 - (j-1) - (t-1)n \\ &= (k+1-t)n + 2j - 1 \ge n + 2 - 1 \ge 4.\end{aligned}$$

For $j = n$

$$\begin{aligned}|f(v_n) - f(u_{nt})| &= f(u_{nt}) - f(v_n) \\ &= 7 + (n-1) + (t-1)n - 5 = tn + 1 \ge n + 1 \ge 4.\end{aligned}$$

- $d(u_{jt}, u_{js}) = 2$, $1 \le j \le n$, $1 \le t \ne s \le k$. In this case we have

$$|f(u_{jt}) - f(u_{js})| = |(t-1)n - (s-1)n| = |(t-s)n| \ge n \ge 3.$$

- $d(u_{jt}, u_{ls}) = 4$, $1 \le j \ne l \le n$, $1 \le t, s \le k$. From the way f was defined we have $f(u_{jt}) \ne f(u_{ls})$.
- $d(z, u_{jt}) = 2$, $1 \le j \le n$, $1 \le t \le k$. We have

$$f(u_{jt}) - f(z) \ge 7 - 1 = 6.$$

- $d(z, v_j) = 1$, $1 \le j \le n$. We then deduce

$$f(v_j) - f(z) \ge f(v_n) - f(z) = 5 - 1 = 4.$$

- $d(v_k, v_j) = 2$, $1 \le j \ne k \le n$. In this case the following relations hold:

$$|f(v_k) - f(v_j)| \ge 3 \text{ for } k, j < n,$$

$$f(v_j) - f(v_n) \ge f(v_1) - f(v_n) = (k+3)n + 2 - 3(n-2) - 5 = kn + 3 \ge 3.$$

- $d(v_j, u_{it}) = 3$, for $1 \le j \ne i \le n$, $1 \le t \le k$.
  For $j = n$ we obtain

$$|f(u_{it}) - f(v_n)| \ge 7 - 5 = 2.$$

For $j \neq n$

$$|f(u_{it}) - f(v_j)| \geq (kn+8) - (kn+6) = 2.$$

In all cases the radio condition is satisfied. ■

As in case of caterpillars CP_n with n odd, in order to prove that $rn(S_{2,k}) = 3k+8$ it is not sufficient to find a suitable labeling of the vertices of $S_{2,k}$ and then apply Theorem 1.3 for the reverse inequality; we also need some additional results.

Forwards we say that the vertex v_1 and the terminal vertices attached to it are vertices of type 1, vertex v_2 and the terminal vertices attached to it are vertices of type 2, and the center z is of type 3.

For a radio labeling f of $S_{2,k}$, we order the vertices ascending by their labels and rename the terminal vertices u_{jt} in this order by $y_1, y_2, \ldots, y_{2k}$; we have

$$f(y_1) < f(y_2) < \cdots < f(y_{2k}).$$

We denote by Y the sequence $y_1, y_2, \ldots, y_{2k}$, by f_Y the sequence of the labels attached to vertices of Y: $f(y_1), f(y_2), \ldots, f(y_{2k})$, and with d_{f_Y} the sequence of differences between consecutive labels from f_Y, where the i-th element of the sequence is denoted by $d^i_{f_Y} = f(y_{i+1}) - f(y_i)$, for $1 \leq i \leq 2k-1$.

On the class of radio labeling of $S_{2,k}$ we define the function Δ_Y as follows:

$$\Delta_Y(f) = \sum_{i=1}^{2k-1} d^i_{f_Y} = f(y_{2k}) - f(y_1).$$

Remark 3.2.

1. In the sequence d_{f_Y} it is not possible to have two consecutive elements with value 1.
2. Δ_Y attains a minimum only for radio labelings f^* of $S_{2,k}$ with the sequence of differences

$$d_{f^*_Y} = \{1, 2, 1, 2, \ldots, 1, 2, 1\}.$$

   For those labelings $\Delta_Y(f^*) = 3k - 2$.
3. $f(y_{2k}) = f(y_1) + \Delta_Y(f)$.

Proof. 1. Suppose that there exists an index i such that $d^i_{f_Y} = 1$ and $d^{i+1}_{f_Y} = 1$. It follows that

$$f(y_{i+1}) - f(y_i) = f(y_{i+2}) - f(y_{i+1}) = 1.$$

Since the pairs of vertices y_i and y_{i+1}, respectively y_{i+1} and y_{i+2} must satisfy the radio condition, it follows that $d(y_i, y_{i+1}) = d(y_{i+1}, y_{i+2}) = 4$, hence y_i and y_{i+2} are of the same type. We obtain $d(y_i, y_{i+2}) = 2$. Since the radio condition must be satisfied for the vertices y_i and y_{i+2}, it follows that $f(y_{i+2}) - f(y_i) \geq 5 - d(y_i, y_{i+2}) = 3$. But

$$f(y_{i+2}) - f(y_i) = f(y_{i+2}) - f(y_{i+1}) + f(y_{i+1}) - f(y_i) = 1 + 1 = 2,$$

a contradiction.

2. Using the first remark, it is obvious that the minimum can be obtained only in the conditions stated in this remark. In this conditions we have

$$\Delta_Y(f^*) = 1 \cdot k + 2 \cdot (k-1) = 3k - 2.$$

■

We denote $\Delta^*_Y = \Delta_Y(f^*) = 3k - 2$.

Lemma 3.3. *Let f be a radio labeling for $S_{2,k}$. If for a type t, with $t \in \{1, 2\}$ there exists an index i between 1 and $2k$ such that $f(y_i) < f(v_t) < f(y_{i+1})$, then the following properties hold:*

1. $d^i_{f_Y} \geq 4$;
2. *If* $d^i_{f_Y} \leq 5$, *then* y_i *and* y_{i+1} *are of type* $3 - t$;
3. *If* $i + 2 \leq 2k$, *then* $d^i_{f_Y} + d^{i+1}_{f_Y} \geq 6$;
4. *If* $i - 1 \geq 1$, *then* $d^{i-1}_{f_Y} + d^i_{f_Y} \geq 6$.

Proof. From the radio condition we have:

$$f(v_t) - f(y_i) \geq 5 - d(v_t, y_i),$$

$$f(y_{i+1}) - f(v_t) \geq 5 - d(v_t, y_{i+1}).$$

It follows that

$$f(y_{i+1}) - f(y_i) \geq 10 - [d(v_t, y_i) + d(v_t, y_{i+1})].$$

But $d(v_t, y_i)$ has value 1 if y_i is of type t, and 3 otherwise. We then obtain $d^i_{f_Y} = f(y_{i+1}) - f(y_i) \geq 10 - (3 + 3) = 4$. Moreover, if $d^i_{f_Y} \leq 5$, then $d(v_t, y_i) + d(v_t, y_{i+1}) \geq 5$, from which it follows that $d(v_t, y_i) = d(v_t, y_{i+1}) = 3$, hence y_i and y_{i+1} are of type $3 - t$.

In order to prove properties 3 and 4 of the lemma it suffices to consider the case $d^i_{f_Y} = 4$, since for greater values of $d^i_{f_Y}$ the inequalities are obvious. In this case from the property 2 of the lemma it follows that y_i and y_{i+1} are of type $3 - t$ and $d(v_t, y_i) = d(v_t, y_{i+1}) = 3$.

If y_{i+2} has the same type as y_i and y_{i+1}, then $d(y_{i+1}, y_{i+2}) = 2$ and from the radio condition we have

$$f(y_{i+2}) \geq f(y_{i+1}) + 5 - d(y_{i+1}, y_{i+2}) \geq f(y_{i+1}) + 3.$$

Then

$$d^i_{f_Y} + d^{i+1}_{f_Y} = 4 + f(y_{i+2}) - f(y_{i+1}) \geq 4 + 3 = 7.$$

Otherwise, if y_{i+2} has type t, $d(v_t, y_{i+2}) = 1$ and from the radio condition we obtain

$$f(y_{i+2}) \geq f(v_t) + 5 - d(v_t, y_{i+2}) = f(v_t) + 4 = f(y_{i+1}) + 2,$$

hence

$$d^i_{f_Y} + d^{i+1}_{f_Y} \geq 4 + 2 = 6.$$

Property 4 can be proved analogously. ■

Remark 3.4. Let f be a radio labeling for $S_{2,k}$. If there exists an index i between 1 and $2k - 1$ such that $f(y_i) < f(z) < f(y_{i+1})$, then $d^i_{f_Y} \geq 6$.

Proof. From the radio condition we have:

$$f(y_{i+1}) - f(y_i) \geq 10 - [d(z, y_i) + d(z, y_{i+1})] = 10 - (2 + 2) = 6.$$ ■

Using these results we can determine a lower bound for $rn(S_{2,k})$.

Theorem 3.5. *For $k \geq 1$, $rn(S_{2,k}) \geq 3k + 8$.*

Proof. Let f be a radio labeling for $S_{2,k}$. We prove that $span(f) \geq 3k + 8$. We consider the following cases, by comparing the labels $f(z)$, $f(v_1)$, $f(v_2)$ with the labels from f_Y.

Case 1. None of the labels $f(z)$, $f(v_1)$, $f(v_2)$ are between $f(y_1)$ and $f(y_{2k})$.

In this case the sequence of all vertices ordered by their labels is obtained starting from the sequence Y by adding, in turn, each of the vertices z, v_1, v_2 at the beginning or at the end of the current sequence. We denote by z', v_1', respectively v_2' the vertex near which z, v_1, respectively v_2 are added in the sequence. Then, using the radio condition, we obtain:

$$\begin{aligned} span(f) &\geq 1 + \Delta_Y(f) + |f(z) - f(z')| + |f(v_1) - f(v_1')| + |f(v_2) - f(v_2')| \\ &\geq 1 + \Delta_Y(f) + 5 - d(z, z') + 5 - d(v_1, v_1') + 5 - d(v_2, v_2') \\ &= \Delta_Y(f) + 16 - [d(z, z') + d(v_1, v_1') + d(v_2, v_2')]. \end{aligned}$$

Let $S = d(z, z') + d(v_1, v_1') + d(v_2, v_2')$. For $t \in \{1, 2\}$ and $1 \leq i \leq 2k$ we have: $d(z, y_i) = 2$, $d(v_t, y_i) = 1$ if y_i is of type t, $d(v_t, y_i) = 3$ if y_i is of type t, $d(v_1, v_2) = 2$ and $d(v_t, z) = 1$. Moreover, at most two of the vertices z', v_1', v_2' are in Y. It follows that $S \leq 7$.

If $S \leq 6$, then

$$span(f) \geq \Delta_Y(f) + 16 - S \geq \Delta_Y^* + 16 - S \geq 3k - 2 + 16 - 6 = 3k + 8.$$

If $S = 7$, then at least one of the vertices v_t' with $t \in \{1, 2\}$ is y_1 or y_{2k} and $d(v_t, v_t') = 3$. We can assume, without loss of generality, that $v_1' = y_1$. We have $f(v_1) < f(y_1) < f(y_2)$. We will prove that $f(y_{2k}) \geq f(v_1) + 3k + 1$.

From the radio condition for v_1 and y_1 we obtain

$$f(y_1) \geq f(v_1) + 5 - d(v_1, y_1) = f(v_1) + 2$$

and then $f(y_2) \geq f(v_1) + 4$.

If $f(y_1) \geq f(v_1) + 3$, then

$$f(y_{2k}) = f(y_1) + \Delta_Y(f) \geq f(y_1) + \Delta_Y^* \geq f(v_1) + 3 + 3k - 2 = f(v_1) + 3k + 1.$$

Otherwise we have $f(y_1) = f(v_1) + 2$ and it follows that $d(v_1, y_1) = 3$ and y_1 is of type 2.

Moreover, if y_2 is of type 1, from the radio condition we have

$$f(y_2) \geq f(v_1) + 5 - d(v_1, y_2) = f(v_1) + 4 = f(y_1) + 2.$$

Otherwise

$$f(y_2) \geq f(y_1) + 5 - d(y_1, y_2) = f(y_1) + 3(\geq f(v_1) + 4).$$

In both situations we obtain $d^1_{f_Y} = f(y_2) - f(y_1) \geq 2$, hence $\Delta_Y(f) > \Delta^*_Y$, and the following relation holds:

$$\begin{aligned} f(y_{2k}) &= f(y_1) + \Delta_Y(f) \geq f(y_1) + \Delta^*_Y + 1 \\ &\geq f(v_1) + 2 + 3k - 2 + 1 = f(v_1) + 3k + 1. \end{aligned}$$

Then

$$\begin{aligned} span(f) &\geq f(y_{2k}) + |f(z) - f(z')| + |f(v_2) - f(v'_2)| \\ &\geq f(v_1) + 3k + 1 + 10 - [d(z, z') + d(v_2, v'_2)] \\ &\geq 1 + 3k + 1 + 10 - [S - d(v_1, v'_1)] \\ &\geq 3k + 12 - (7 - 1) = 3k + 8. \end{aligned}$$

Case 2. Only one of the values $f(v_1)$ and $f(v_2)$ is between $f(y_1)$ and $f(y_{2k})$ ($f(z)$ is not between $f(y_1)$ and $f(y_{2k})$).

We can assume, without loss of generality, that $f(v_1) \in \{f(y_1), \ldots, f(y_{2k})\}$. Then there exists an index p between 1 and $2k - 1$ such that $f(y_p) < f(v_1) < f(y_{p+1})$. From lemma 3.3 we have $d^p_{f_Y} \geq 4$.

If $d^p_{f_Y} \geq 6$, using remark 3.2 we obtain:

$$\Delta_Y(f) \geq 6 + 1 \cdot k + 2 \cdot (k - 2) = 3k + 2.$$

Otherwise we have $4 \leq d^p_{f_Y} \leq 5$, and, from Lemma 3.3, it follows that y_p and y_{p+1} are of type 2 and $k \geq 2$. Then $p - 1 \geq 1$ or $p + 2 \geq 2k$. We assume $p + 2 \geq 2k$, since the case $p - 1 \geq 1$ can be treated analogously. Using lemma 3.3 it follows that $d^p_{f_Y} + d^{p+1}_{f_Y} \geq 6$. Moreover, since y_p and y_{p+1} are of type 2, there exists an index q between 1 and $2k - 1$ such that y_q and y_{q+1} are of type 1, and then

$$d^q_{f_Y} = f(y_{q+1}) - f(y_q) \geq 5 - d(y_{q+1}, y_q) = 5 - 2 = 3.$$

It follows that $\Delta_Y(f) \geq 6 + 3 + 1 \cdot (k - 1) + 2 \cdot (k - 3) = 3k + 2$.

In all cases we obtain $\Delta_Y(f) \geq 3k + 2$, and it follows that

$$\begin{aligned} span(f) &\geq f(y_{2k}) + |f(z) - f(z')| + |f(v_2) - f(v_2')| \\ &\geq 1 + \Delta_Y(f) + |f(z) - f(z')| + |f(v_2) - f(v_2')| \\ &\geq 1 + 3k + 2 + 10 - [d(z, z') + d(v_2, v_2')] \\ &\geq 3k + 3 + 10 - (2 + 3) = 3k + 8. \end{aligned}$$

Case 3. $f(v_1)$ and $f(v_2)$ are between $f(y_1)$ and $f(y_{2k})$, but $f(z)$ is not. Then there exist two indices p and q between 1 and $2k-1$ such that

$$f(y_p) < f(v_1) < f(y_{p+1}) \text{ and } f(y_q) < f(v_2) < f(y_{q+1}).$$

By Lemma 3.3 we have $d^p_{f_Y} \geq 4$ and $d^q_{f_Y} \geq 4$. We prove that $\Delta_Y(f) \geq 3k+4$.

If $d^p_{f_Y} \geq 5$ and $d^q_{f_Y} \geq 5$, then, from Remark 3.2, it follows that

$$\Delta_Y(f) \geq 5 + 5 + 1 \cdot k + 2 \cdot (k - 3) = 3k + 4.$$

If $d^p_{f_Y} = 4$ and $d^q_{f_Y} \geq 5$, then, using the same lemma, for $p+1 \leq 2k$ we have $d^p_{f_Y} + d^{p+1}_{f_Y} \geq 6$ and for $p-1 \geq 1$ we have $d^{p-1}_{f_Y} + d^p_{f_Y} \geq 6$. Hence, if there exists, $d^{p+1}_{f_Y} \geq 2$ and $d^{p-1}_{f_Y} \geq 2$ we obtain

$$\Delta_Y(f) \geq 4 + 5 + 1 \cdot (k - 1) + 2 \cdot (k - 2) = 3k + 4$$

since in the sequence d_{f_Y} it is not possible to have two consecutive elements with value 1. Analogously we can prove that, if $d^p_{f_Y} \geq 5$ and $d^q_{f_Y} = 4$, then $\Delta_Y(f) \geq 3k + 4$.

It remains to consider the situation when $d^p_{f_Y} = d^q_{f_Y} = 4$. Using an argument similar to the previous one, it can be proved that in the sequence d_{f_Y} the value 1 cannot be on one of the positions $p-1$, $p+1$, $q-1$, $q+1$, if such a position exist. Then $\Delta_Y(f) \geq 4+4+1 \cdot (k-2)+2 \cdot (k-1) = 3k+4$.

In all situations we have $\Delta_Y(f) \geq 3k + 4$, hence

$$\begin{aligned} span(f) &\geq 1 + \Delta_Y(f) + |f(z) - f(z')| \\ &\geq 1 + 3k + 4 + 5 - d(z, z') \geq 3k + 10 - 2 = 3k + 8. \end{aligned}$$

Case 4. $f(z)$ is between $f(y_1)$ and $f(y_{2k})$, but $f(v_1)$ and $f(v_2)$ are not. Then there exists an index p between 1 and $2k-1$ such that $f(y_p) < f(z) < f(y_{p+1})$. By remark 3.4 we have $d^p_{f_Y} \geq 6$. We assume, without loss of generality, that $f(v_1) < f(y_1)$ and $f(v_2)$ satisfies one of the relations: $f(v_2) < f(v_1)$ or $f(v_2) > f(y_{2k})$.

If $p = 1$, then the smallest labels are $f(v_1) < f(y_1) < f(z) < f(y_2)$ and we obtain $\Delta_Y(f) \geq 6 + 1 \cdot (k-1) + 2 \cdot (k-1) = 3k+3$, hence it follows

$$\begin{aligned} span(f) &\geq 1 + \Delta_Y(f) + |f(v_1) - f(y_1)| + |f(v_2) - f(v_2')| \\ &\geq 1 + 3k + 3 + 5 - d(v_1, y_1) + 5 - d(v_2, v_2') \\ &\geq 1 + 3k + 3 + 5 - 3 + 5 - 3 = 3k + 8. \end{aligned}$$

If $p > 1$, then, using same type of arguments as in case 1, we will prove that $f(y_2) \geq f(v_1) + 4$. Since in d_{f_Y} is not possible to have two consecutive elements with value 1, it will follow that

$$\begin{aligned} f(y_{2k}) &\geq f(y_2) + 6 + 1 \cdot (k-1) + 2 \cdot (k-2) \\ &\geq f(v_1) + 4 + 3k + 1 \geq 3k + 6 \end{aligned}$$

and so

$$span(f) \geq f(y_{2k}) + |f(v_2) - f(v_2')| \geq 3k + 6 + 5 - d(v_2, v_2') \geq 3k + 8.$$

From the radio condition, $f(y_1) \geq f(v_1) + 2$. If the inequality is strict, then it is obvious that $f(y_2) \geq f(y_1) + 1 \geq f(v_1) + 4$. Otherwise we have $f(y_1) = f(v_1) + 2$ and, using the radio condition, we obtain $d(v_1, y_1) = 3$, which implies that y_1 is of type 2. As in case 1, it follows that $f(y_2) \geq f(v_1) + 4$.

Case 5. Only $f(z)$ and one of the labels $f(v_1)$ or $f(v_2)$ are between $f(y_1)$ and $f(y_{2k})$; assume $f(v_2)$ is between $f(y_1)$ and $f(y_{2k})$.

Then there exist two indices p and q between 1 and $2k-1$ such that $f(y_p) < f(z) < f(y_{p+1})$ and $f(y_q) < f(v_2) < f(y_{q+1})$. By Lemma 3.3 and Remark 3.4 we have $d_{f_Y}^p \geq 6$ and $d_{f_Y}^q \geq 4$. We will prove that $\Delta_Y(f) \geq 3k+5$. It will follow that

$$span(f) \geq 1 + \Delta_Y(f) + |f(v_2) - f(v_2')| \geq 1 + 3k + 5 + 5 - 3 = 3k + 8.$$

Thus, if $d_{f_Y}^q \geq 5$, then

$$\Delta_Y(f) \geq 6 + 5 + 1 \cdot k + 2 \cdot (k-3) = 3k + 5.$$

Otherwise, if $d_{f_Y}^q = 4$, using arguments similar to the previous cases, it follows that in the sequence d_{f_Y} value 1 cannot be on positions $q-1$, $q+1$, if these positions exist. Then $\Delta_Y(f) \geq 4 + 6 + 1 \cdot (k-1) + 2 \cdot (k-2) = 3k + 5$.

Case 6. All of the labels $f(z)$, $f(v_1)$, $f(v_2)$ are between $f(y_1)$ and $f(y_{2k})$. Then there exist three indices p, q and r between 1 and $2k-1$ such that $f(y_p) < f(z) < f(y_{p+1})$, $f(y_q) < f(v_1) < f(y_{q+1})$ and $f(y_r) < f(v_2) < f(y_{r+1})$ and we have $d^p_{f_Y} \geq 6$, $d^q_{f_Y} \geq 4$ and $d^r_{f_Y} \geq 4$.

If one of the values $d^q_{f_Y}$ or $d^r_{f_Y}$ is strictly greater than 4, then $\Delta_Y(f) \geq 6+5+4+1\cdot k+2\cdot(k-4) = 3k+7$. It follows that in the sequence d_{f_Y} value 1 cannot be on positions $q-1$, $q+1$, $r-1$, $r+1$, if these positions exist, hence $\Delta_Y(f) \geq 4+4+6+1\cdot(k-2)+2\cdot(k-2) = 3k+8$.

In both situations we have $span(f) \geq 1+\Delta_Y(f) \geq 3k+8$. ■

Theorem 3.6. *For $k \geq 1$, $rn(S_{2,k}) = 3k+8$.*

Proof. By Theorem 3.5, it suffices to build a radio labeling f for $S_{2,k}$ with $span(f) = 3k+8$. Let f be a labeling defined as follows:
$f(z) = 1$,
$f(u_{11}) = 4$, $f(u_{1j}) = 4+3(j-1)$, for $2 \leq j \leq k$,
$f(u_{21}) = 5$, $f(u_{2j}) = 5+3(j-1)$, for $2 \leq j \leq k$,
$f(v_1) = 1+f(u_{2k})+5-d(v_1,u_{2k}) = f(u_{2k})+3$,
$f(v_2) = f(v_1)+5-d(v_1,v_2) = f(u_{2k})+3+3 = f(u_{2k})+6$.
Then $f(u_{2k}) = 5+3(k-1) = 3k+2$ and $span(f) = f(v_2) = 3k+8$.

For $n = 4$ and $k = 3$ the labeling is shown in Figure 6.

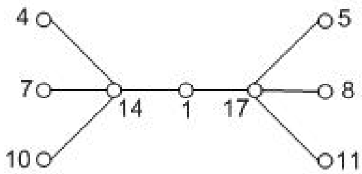

Figure 6. A radio labeling for $S_{2,3}$.

We prove that f is a radio labeling for $S_{2,k}$, by considering each possible type of pairs of vertices and verifying the radio condition for it.

- For any $1 \leq t \leq 2$, $1 \leq j \leq k$ we have

$$|f(u_{tj}) - f(z)| \geq f(u_{11}) - f(z) = 3 = 5 - d(z, u_{tj}).$$

- For any $1 \leq t \leq 2$, $1 \leq i < j \leq k$ we have

$$|f(u_{tj}) - f(u_{ti})| = 3(j - i) \geq 3 = 5 - d(u_{tj}, u_{ti}).$$

Moreover, $d(u_{tj}, u_{t'i}) = 4$ for $t' = 3 - t$ and $f(u_{tj}) \neq f(u_{t'i})$ from the way f was defined.

- For any $1 \leq i < j \leq k$ the following relations hold

$$f(v_1) - f(u_{1j}) \geq f(v_1) - f(u_{1k}) = 3k + 5 - (3k + 1) = 4 = 5 - d(v_1, u_{1j}).$$

$$f(v_1) - f(u_{2j}) \geq f(v_1) - f(u_{2k}) = 3k + 5 - (3k + 2) = 3 \geq 5 - d(v_1, u_{2j}).$$

$$f(v_2) - f(u_{1j}) \geq f(v_2) - f(u_{1k}) = 3k + 8 - (3k + 1) = 7 > 5 - d(v_2, u_{1j}).$$

$$f(v_2) - f(u_{2j}) \geq f(v_2) - f(u_{2k}) = 3k + 8 - (3k + 2) = 6 \geq 5 - d(v_2, u_{2j}).$$

- We have the relations:

$$\begin{aligned} f(v_2) - f(z) &\geq f(v_1) - f(z) = f(u_{2k}) + 3 - 1 \\ &= 3k + 4 \geq 5 - d(z, v_1) = 5 - d(z, v_2), \\ f(v_2) - f(v_1) &= 3. \end{aligned}$$

■

For related problems see the survey paper [4].

References

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