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ON EDGE DETOUR GRAPHS

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Abstract

For two vertices u and v in a graph $G = (V, E)$, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a u–v *detour*. A set $S \subseteq V$ is called an *edge detour set* if every edge in G lies on a detour joining a pair of vertices of S. The *edge detour number* $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is an *edge detour basis* of G. A connected graph G is called an *edge detour graph* if it has an edge detour set. It is proved that for any non-trivial tree T of order p and detour diameter D, $dn_1(T) \leq p - D + 1$ and $dn_1(T) = p - D + 1$ if and only if T is a caterpillar. We show that for each triple D, k, p of integers with $3 \leq k \leq p - D + 1$ and $D \geq 4$, there is an edge detour graph G of order p with detour diameter D and $dn_1(G) = k$. We also show that for any three positive integers R, D, k with $k \geq 3$ and $R < D \leq 2R$, there is an edge detour graph G with detour radius R, detour diameter D and $dn_1(G) = k$. Edge detour graphs G with detour diameter $D \leq 4$ are characterized when $dn_1(G) = p - 2$ or $dn_1(G) = p - 1$.

Keywords: detour, edge detour set, edge detour basis, edge detour number.

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1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies, we refer to [1, 5].

For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. It is known that the detour distance is a metric on the vertex set V. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D G$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D G$ of G is the maximum detour eccentricity among the vertices of G. These concepts were studied by Chartrand *et al.* [2].

A vertex x is said to *lie on* a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. These concepts were studied by Chartrand *et al.* [3, 4].

An edge e of G is said to *lie on* a u–v detour P if e is an edge of the detour P. In general, there are graphs G for which there exist edges which do not lie on a detour joining any pair of vertices of V. For the graph G given in Figure 1.1, the edge v_1v_2 does not lie on a detour joining any pair of vertices of V. This motivated us to introduce the concepts of *weak edge detour set* of a graph [6] and *edge detour graphs* [7].

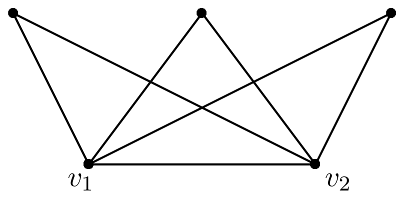

Figure 1.1. G

A set $S \subseteq V$ is called a *weak edge detour set* of G if every edge in G has both its ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is called a *weak edge detour basis* of G. Weak edge detour sets and weak edge detour number of a graph were introduced and studied by Santhakumaran and Athisayanathan in [6].

A set $S \subseteq V$ is called an *edge detour set* of G if every edge in G lies on a detour joining a pair of vertices of S. The *edge detour number* $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is an *edge detour basis* of G. A graph G is called an *edge detour graph* if it has an edge detour set. Edge detour graphs were introduced and studied by Santhakumaran and Athisayanathan in [7]. It is proved in [7] that every edge detour set of an edge detour graph contains its end-vertices and no edge detour basis contains its cut vertices.

For the graph G given in Figure 1.2(a), the sets $S_1 = \{u, x\}, S_2 = \{u, w, x\}$ and $S_3 = \{u, v, x, y\}$ are a detour basis, weak edge detour basis and edge detour basis of G respectively and hence $dn(G) = 2$, $dn_w(G) = 3$ and $dn_1(G) = 4$. For the graph G given in Figure 1.2(b), the set $S = \{u_1, u_2\}$ is a detour basis, weak edge detour basis and an edge detour basis so that $dn(G) = dn_w(G) = dn_1(G) = 2$. The graphs G given in Figure 1.2 are edge detour graphs. For the graph G given in Figure 1.1, the set $S = \{v_1, v_2\}$ is a detour basis and also a weak edge detour basis, but it does not contain an edge detour set and so G is not an edge detour graph.

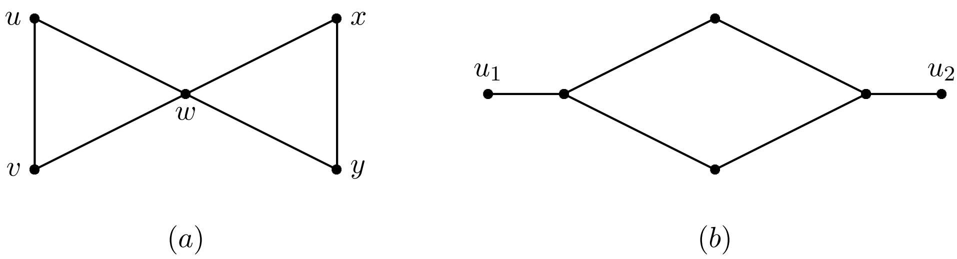

Figure 1.2. G

A *caterpillar* is a tree for which the removal of all end-vertices leaves a path. A *wounded spider* is the graph formed by subdividing at most $t-1$ of the edges of a *star* $K_{1,t}$ for $t \geq 0$. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v to $V(H)$ is called a *branch* of G at v. An *end-block* of G is a block containing exactly one cut-vertex of G. The following theorems are used in the sequel.

Theorem 1.1 ([7]). *For any edge detour graph G of order $p \geq 2$, $2 \leq dn_1(G) \leq p$.*

Theorem 1.2 ([7]). *Each end-vertex of an edge detour graph G belongs to every edge detour set of G. Also if the set S of all end-vertices of G is an edge detour set, then S is the unique edge detour basis for G.*

Theorem 1.3 ([7]). *If T is a tree with k end-vertices, then $dn_1(T) = k$.*

Theorem 1.4 ([7]). *Any cycle G is an edge detour graph and $dn_1(G) = 2$ if G is an even cycle, and $dn_1(G) = 3$ if G is an odd cycle.*

Theorem 1.5 ([7]). *Let $G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $p \geq 5$ such that $r \geq 2$, each $n_i \geq 2$ and $n_1+n_2+\cdots+n_r+k = p-1$. Then G is an edge detour graph and $dn_1(G) = 2r + k$.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. EDGE DETOUR NUMBER AND DETOUR DIAMETER OF AN EDGE DETOUR GRAPH

The following Theorem gives an upper bound for the detour number of a graph in terms of its order and detour diameter.

Theorem A [3]. *If G is a non-trivial connected graph of order $p \geq 3$ and detour diameter D, then $dn(G) \leq p - D + 1$.*

Remark 2.1. Theorem A is not true for the edge detour number $dn_1(G)$ of an edge detour graph G. There are edge detour graphs G for which $dn_1(G) = p - D + 1$, $dn_1(G) < p - D + 1$ and $dn_1(G) > p - D + 1$. For an even cycle C of order $p \geq 4$, $D = p - 1$ and by Theorem 1.4, $dn_1(C) = 2$ so that $dn_1(C) = p-D+1$. For the graph G in Figure 1.2(b), $p = 6$, $D = 4$ and $dn_1(G) = 2$ so that $dn_1(G) < p - D + 1$. For an odd cycle C of order $p \geq 3$, $D = p - 1$ and by Theorem 1.4, $dn_1(C) = 3$ so that $dn_1(C) > p - D + 1$.

Theorem 2.2. *If G is an edge detour graph of order $p \geq 2$ with $D = p-1$, then $dn_1(G) \geq p - D + 1$.*

Proof. For any edge detour graph G, $dn_1(G) \geq 2$. Since $D = p - 1$, we have $p - D + 1 = 2$ and so $dn_1(G) \geq p - D + 1$. ■

Remark 2.3. The converse of Theorem 2.2 is not true. For the edge detour graph G given in Figure 2.1, $p = 6$ and $D = 4$ so that $p - D + 1 = 3$ and $dn_1(G) = 4$. Thus $dn_1(G) > p - D + 1$ and $D \neq p - 1$.

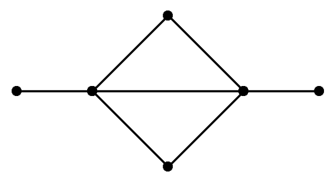

Figure 2.1. G

Theorem 2.4. *If G is a non-trivial tree of order p, then $dn_1(G) \leq p - D + 1$.*

Proof. Let u and v be the vertices of G for which $D(u, v) = D$ and let $P : u = v_0, v_1, \ldots, v_{D-1}, v_D = v$ be u–v detour of length D. Let $S = V(G) - \{v_1, v_2, \ldots, v_{D-1}\}$. It is clear that S is an edge detour set of G and so $dn_1(G) \leq |S| = p - D + 1$. ■

We give below a characterization theorem for trees.

Theorem 2.5. *For every non-trivial tree T of order p, $dn_1(T) = p - D + 1$ if only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $D = D(u, v)$ and $P : u = v_0, v_1, \ldots, v_{D-1}, v_D = v$ be a detour diameteral path. Let k be the number of end-vertices of T and l be the number of internal vertices of T other than $v_1, v_2, \ldots, v_{D-1}$. Then $D - 1 + l + k = p$. By Theorem 1.3, $dn_1(T) = k = p - D - l + 1$. Hence $dn_1(T) = p - D + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the detour diameteral path P, if and only if T is a caterpillar. ■

Corollary 2.6. *For a wounded spider T of order p, $dn_1(T) = p - D + 1$ if and only if T is obtained from $K_{1,t}$ $(t \geq 1)$ by subdividing at most two of its edges.*

Proof. It is clear that a wounded spider T is a caterpillar if and only if T is obtained from $K_{1,t}$ ($t \geq 1$) by subdividing at most two of its edges. Then the result follows from Theorem 2.5. ■

The following two theorems give realization results under certain conditions.

Theorem 2.7. *For each triple D, k, p of integers with $3 \leq k \leq p-D+1$ and $D \geq 4$, there exists an edge detour graph G of order p with detour diameter D and $dn_1(G) = k$.*

Proof. *Case* 1. When D is even, let G be the graph obtained from the cycle $C_D : u_1, u_2, \ldots, u_D, u_1$ of order D by adding $k-1$ new vertices v_1, v_2, $\ldots$, v_{k-1} and joining each vertex v_i ($1 \leq i \leq k-1$) to u_1 and adding $p-D-k+1$ new vertices $w_1, w_2, \ldots, w_{p-D-k+1}$ and joining each vertex w_i ($1 \leq i \leq p-D-k+1$) to both u_1 and u_3. The graph G is connected of order p and detour diameter D and is shown in Figure 2.2(a).

Now, we show that $dn_1(G) = k$. Let $S = \{v_1, v_2, \ldots, v_{k-1}\}$ be the set of all end-vertices of G. No edge of G other than the edges u_1v_i ($1 \leq i \leq k-1$) lies on a detour joining a pair of vertices of S and so S is not an edge detour set of G. Let $T = S \cup \{v\}$, where v is the antipodal vertex of u_1 in C_D. Then every edge of G lies on a detour joining a vertex v_i ($1 \leq i \leq k-1$) and v so that T is an edge detour set of G. Now, it follows from Theorem 1.2 that T is an edge detour basis of G and so $dn_1(G) = k$.

Case 2. When D is odd, let G be the graph obtained from the cycle C_D : $u_1, u_2, \ldots, u_D$, u_1 of order D by adding $k-2$ new vertices v_1, $v_2, \ldots, v_{k-2}$ and joining each vertex v_i ($1 \leq i \leq k-2$) to u_1 and adding $p-D-k+2$ new vertices w_1, $w_2, \ldots, w_{p-D-k+2}$ and joining each vertex w_i ($1 \leq i \leq p-D-k+2$) to both u_1 and u_3. The graph G is connected of order p and detour diameter D and is shown in Figure 2.2(b).

Now, we show that $dn_1(G) = k$. Let $S = \{v_1, v_2, \ldots, v_{k-2}\}$ be the set of all end-vertices of G. As in Case 1, S is not an edge detour set of G. Let $S_1 = S \cup \{v\}$, where v is any vertex of G such that $v \neq v_i$ ($1 \leq i \leq k-2$). It is easy to see that S_1 is not an edge detour set of G. Now, the set $T = S \cup \{u_2, u_D\}$ is clearly an edge detour set of G. Hence it follows from Theorem 1.2 that T is an edge detour basis of G and so $dn_1(G) = k$. ■

Chartrand *et al.* [2] proved that the detour radius and detour diameter of a connected graph G satisfy $rad_D G \leq diam_D G \leq 2rad_D G$. They also proved that every pair a, b of positive integers can be realized as the detour radius and detour diameter respectively of some connected graph provided $a \le b \le 2a$. We extend this theorem so that the edge detour number can be prescribed as well when $a < b \le 2a$.

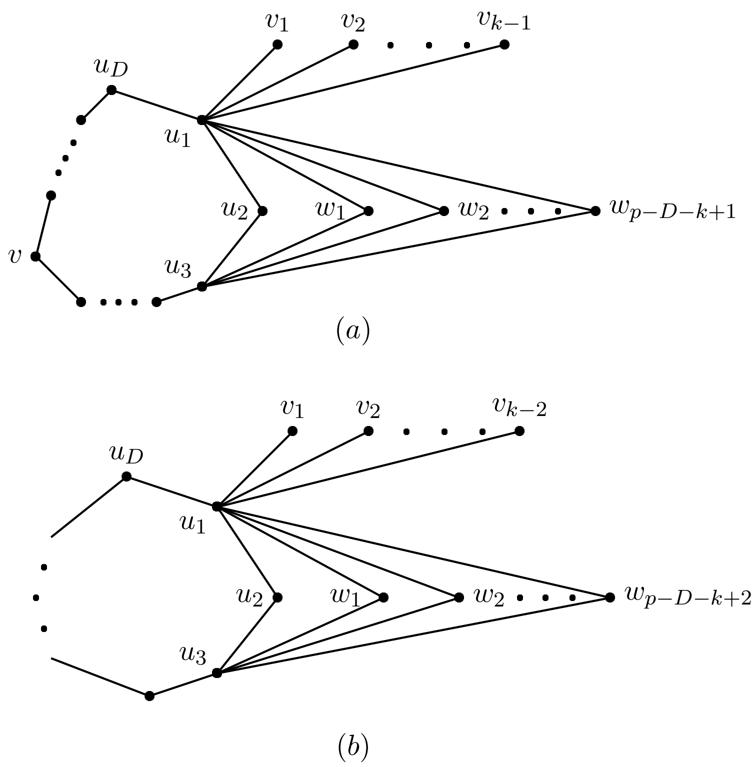

Figure 2.2. G

Theorem 2.8. *Let R, D, k be three positive integers such that $k \ge 3$ and $R < D \le 2R$. Then there exists an edge detour graph G such that $rad_D G = R$, $diam_D G = D$ and $dn_1(G) = k$.*

Proof. *Case* 1. Let R be an odd integer. When $R = 1$, let $G = K_{1,k}$. Clearly, $rad_D G = 1$, $diam_D G = 2$ and by Theorem 1.3, $dn_1(G) = k$. When $R \ge 3$ and $R < D \le 2R$, we construct a graph G with the desired properties as follows: Let $C_{R+1} : v_0, v_1, \ldots, v_R, v_0$ be a cycle of order $R+1$ and let $P_{D-R+1} : u_0, u_1, \ldots, u_{D-R}$ be a path of order $D - R + 1$. Let H be the graph obtained from C_{R+1} and P_{D-R+1} by identifying v_0 of C_{R+1} with u_0 of P_{D-R+1}. The required graph G is obtained from H by adding $k-2$ new vertices w_1, $w_2, \ldots, w_{k-2}$ to H and joining each w_i $(1 \le i \le k-2)$ to the vertex u_{D-R-1} and is shown in Figure 2.3(a). It is clear that G is connected with $rad_D G = R$ and $diam_D G = D$.

Now, we show that $dn_1(G) = k$. Let $S = \{w_1, w_2, \ldots, w_{k-2}, u_{D-R}\}$ be the set of all end-vertices of G. No edge of G other than the edges $w_i u_{D-R-1}$ $(1 \le i \le k-2)$ and the edge $u_{D-R} u_{D-R-1}$ lies on a detour joining a pair of vertices of S and so S is not an edge detour set of G. Let $T = S \cup \{v\}$, where v is the antipodal vertex of v_0 in C_{R+1}. Then T is an edge detour set of G and hence it follows from Theorem 1.2 that T is an edge detour basis of G so that $dn_1(G) = k$.

Case 2. Let R be an even integer. Construct the graph H as in Case 1. Then G is obtained from H by adding $k-3$ new vertices w_1, $w_2, \ldots, w_{k-3}$ to H and joining each w_i $(1 \le i \le k-3)$ to the vertex u_{D-R-1} and is shown in Figure 2.3(b). It is clear that G is connected with $rad_D G = R$ and $diam_D G = D$.

Now, we show that $dn_1(G) = k$. Let $S = \{w_1, w_2, \ldots, w_{k-3}, u_{D-R}\}$ be the set of all end-vertices of G. As in Case 1, S is not an edge detour set of G. Let $S_1 = S \cup \{v\}$, where v is any vertex of G such that $v \notin S$. It is easy to see that S_1 is not an edge detour set of G. Now the set $T = S \cup \{v_1, v_R\}$ is clearly an edge detour set of G. Hence it follows from Theorem 1.2 that T is an edge detour basis of G and so $dn_1(G) = k$. ■

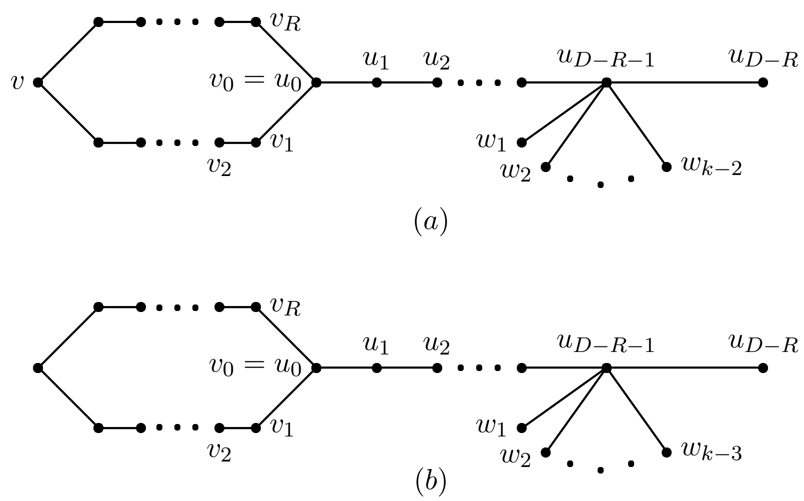

Figure 2.3. G

3. Edge Detour Graphs with Detour Diameter $D \leq 4$

It is proved in [7] (see Theorem 1.1) that for any edge detour graph G of order $p \geq 2$, $2 \leq dn_1(G) \leq p$. The bounds in this inequality are sharp. For the complete graph K_p ($p = 2$ or 3), $dn_1(K_p) = p$. The set of two end-vertices of a path P_n ($n \geq 2$) is its unique edge detour set so that $dn_1(P_n) = 2$. Thus the complete graph K_p ($p = 2$ or 3) has the largest possible edge detour number p and the non-trivial paths have the smallest edge detour number 2.

The following problem seems to be a difficult one and we leave it open.

Problem 3.1. Does there exist a graph G of order $p \geq 4$ for which $dn_1(G) = p$?

In this section we characterize edge detour graphs G with detour diameter $D \leq 4$ for which $dn_1(G) = p - 2$ or $dn_1(G) = p - 1$. First, we characterize graphs G with detour diameter $D \leq 4$ for which $dn_1(G) = p - 2$. For this purpose we introduce the collection $\mathscr{K}$ of graphs given in Figure 3.1.

Theorem 3.2. *Let G be an edge detour graph of order $p \geq 5$ with detour diameter $D \leq 4$. Then $dn_1(G) = p - 2$ if and only if G is a double star or $G \in \mathscr{K}$.*

Proof. It is straightforward to verify that if G is a double star or $G \in \mathscr{K}$, then $dn_1(G) = p - 2$. For the converse, let G be an edge detour graph of order $p \geq 5$, $D \leq 4$ and $dn_1(G) = p - 2$.

If $D \leq 2$, then it is clear that there are no graphs G for which $dn_1(G) = p - 2$.

Suppose $D = 3$. If G is a tree, then G is a double star and the result follows from Theorem 1.3. Assume that G is not a tree. Let $c(G)$ denote the length of a longest cycle in G. Since $D = 3$, it follows that $c(G) \leq 4$. We consider two cases.

Case 1. Let $c(G) = 4$. Let C: v_1, v_2, v_3, v_4, v_1 be a 4-cycle in G. Since $p \geq 5$ and G is connected there exists a vertex x not on C such that it is adjacent to some vertex, say v_1 of C. Then x, v_1, v_2, v_3, v_4 is a path of length 4 in G so that $D \geq 4$, which is a contradiction.

Case 2. Let $c(G) = 3$. If G contains two or more triangles, then $c(G) = 4$ or $D \geq 4$, which is a contradiction. Hence G contains a unique triangle

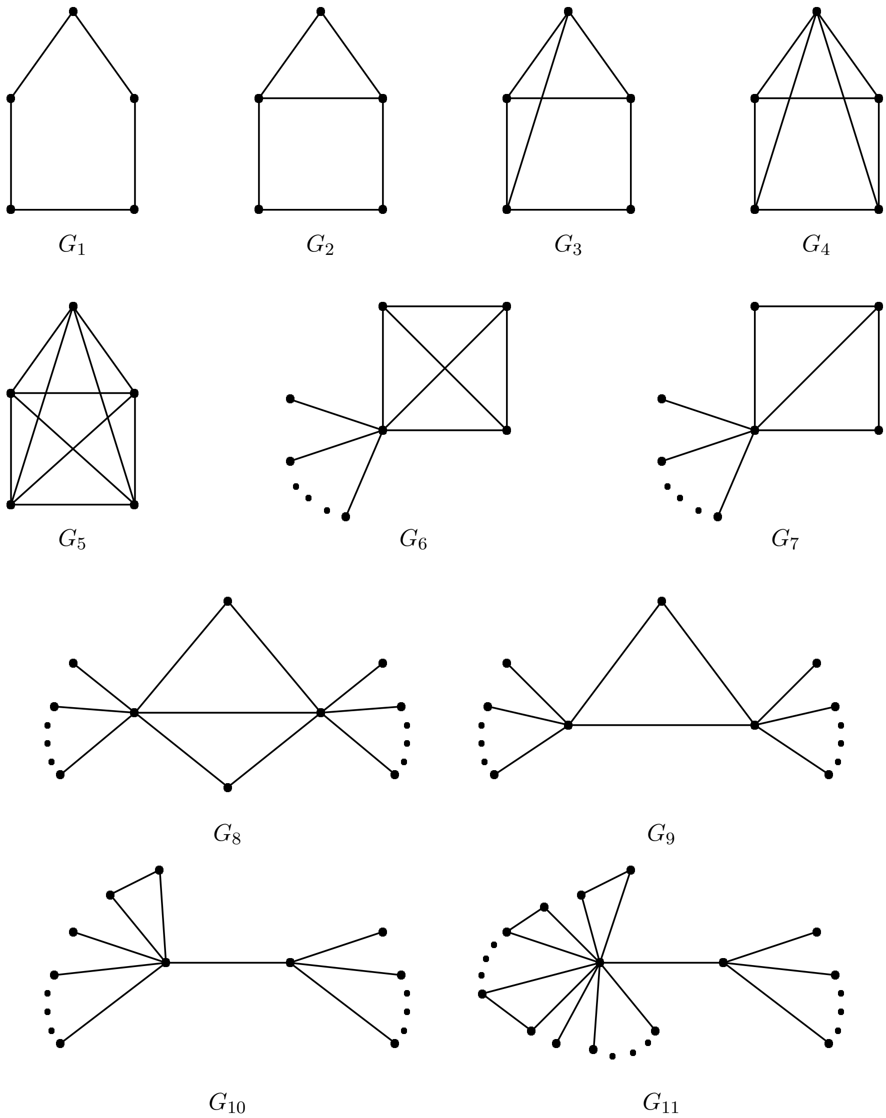

Figure 3.1. Graphs in family $\mathscr{K}$.

C_3: v_1, v_2, v_3, v_1. Now, if there are two or more vertices of C_3 having degree 3 or more, then $D \geq 4$, which is a contradiction. Thus exactly one vertex in C_3 has degree 3 or more. Since $D = 3$, it follows that $G = K_{1,p-1} + e$ and so $dn_1(K_{1,p-1}+e) = p-1$, which is a contradiction. Thus it follows that G is a double star.

Suppose $D = 4$. If G is a tree, then there exists a path of length 4 so that there are at least 3 internal vertices of G. Hence there are at most $(p-3)$ end-vertices of G, so that by Theorem 1.3, $dn_1(G) \le p-3$, which is a contradiction. So, assume that G is not a tree. Let $c(G)$ denote the length of a longest cycle in G. Since $D = 4$, it follows that $c(G) \le 5$. We consider three cases.

Case 1. Let $c(G) = 5$. Then, since $D = 4$, it is clear that G has exactly five vertices. Now, it is easily verified that the graphs G_1, G_2, G_3, G_4 and $G_5 \in \mathscr{K}$ given in Figure 3.1 are the only graphs with $dn_1(G_i) = p-2$ $(1 \le i \le 5)$ among all graphs on five vertices having a largest cycle of length 5.

Case 2. Let $c(G) = 4$. Suppose that G contains K_4 as an induced subgraph. Since $p \ge 5$, $D = 4$ and $c(G) = 4$, every vertex not on K_4 is pendant and adjacent to exactly one vertex of K_4. Thus the graph reduces to the graph $G_6 \in \mathscr{K}$ given in Figure 3.1. Also since $dn_1(G_6) = p-2$, G_6 is the only graph in this case satisfying the requirements of the theorem.

Now, suppose that G does not contain K_4 as an induced subgraph. We claim that G contains exactly one 4-cycle C_4. Suppose that G contains two or more 4-cycles. If two 4-cycles in G have no edges in common, then it is clear that $D \ge 5$, which is a contradiction. If two 4-cycles in G have exactly one edge in common, then G must contain the graphs given in Figure 3.2 as subgraphs or induced subgraphs. In any case $D \ge 5$ or $c(G) \ge 5$, which is a contradiction.

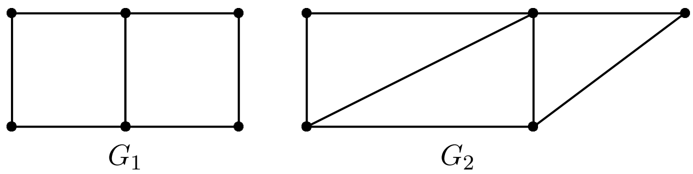

Figure 3.2. G

If two 4-cycles in G have exactly two edges in common, then G must contain the graphs given in Figure 3.3 as subgraphs. It is easily verified that all other subgraphs having two edges in common will have cycles of length ≥ 5, which is a contradiction.

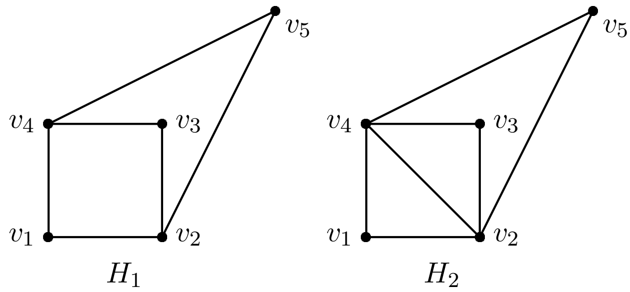

Figure 3.3. G

Now, if $G = H_1$, then $dn_1(G) = p - 3$, which is a contradiction. Assume first that G contains H_1 as a proper subgraph. Then there is a vertex x such that $x \notin V(H_1)$ and x is adjacent to at least one vertex of H_1. If x is adjacent to v_1, we get a path x, v_1, v_2, v_3, v_4, v_5 of length 5 so that $D \geq 5$, which is a contradiction. Hence x cannot be adjacent to v_1. Similarly x cannot be adjacent to v_3 and v_5. Thus x is adjacent to v_2 or v_4 or both. If x is adjacent only to v_2, then x must be a pendant vertex of G, for otherwise, we get a path of length 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph G reduces to the one given in Figure 3.4.

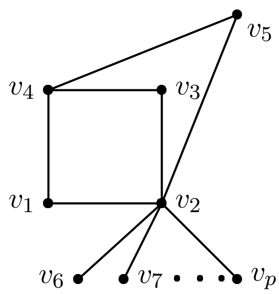

Figure 3.4. G

However, for this graph G, it follows from Theorem 1.2 that the set $\{v_4, v_6, v_7, \ldots, v_p\}$ is an edge detour basis so that $dn_1(G) = p - 4$, which is a contradiction. So, in this case there are no graphs satisfying the requirements of the theorem. If x is adjacent only to v_4, then we get a graph G isomorphic to the one given in Figure 3.4 and hence in this case also there are no graphs satisfying the requirements of the theorem. If x is adjacent to both v_2 and v_4, then the graph reduces to the one given in Figure 3.5.

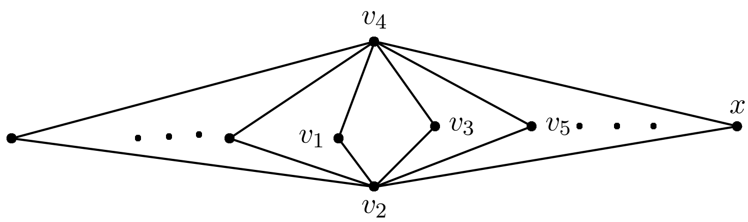

Figure 3.5. G

However, for this graph, $\{x, v_3\}$ is an edge detour basis so that $dn_1(G) = 2$ and hence $dn_1(G) \leq p - 4$, which is a contradiction. Thus a vertex not in H_1 cannot be adjacent to both v_2 and v_4.

Next, if a vertex x not on H_1 is adjacent only to v_2 and a vertex y not on H_1 is adjacent only to v_4, then x and y must be pendant vertices of G, for otherwise, we get either a path or a cycle of length ≥ 5 so that $D \geq 5$, which is a contradiction. Thus in this case, the graph reduces to the one given in Figure 3.6.

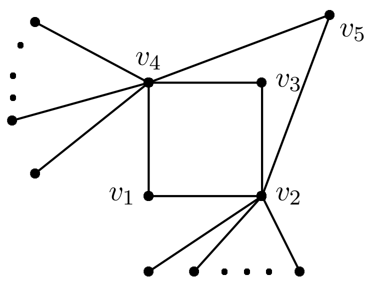

Figure 3.6. G

For this graph G, it follows from Theorem 1.2 that the set of all end-vertices is an edge detour basis so that $dn_1(G) = p - 5$. So, in this case also there are no graphs satisfying the requirements of the theorem. Thus we conclude that in this case there are no graphs G with H_1 as proper subgraph.

Next, if $G = H_2$, then the edge v_2v_4 does not lie on any detour joining a pair of vertices of G so that G is not an edge detour graph. If G contains H_2 as a proper subgraph, then as in the case of H_1, it is easily seen that the graph reduces to any one of the graphs given in Figure 3.7.

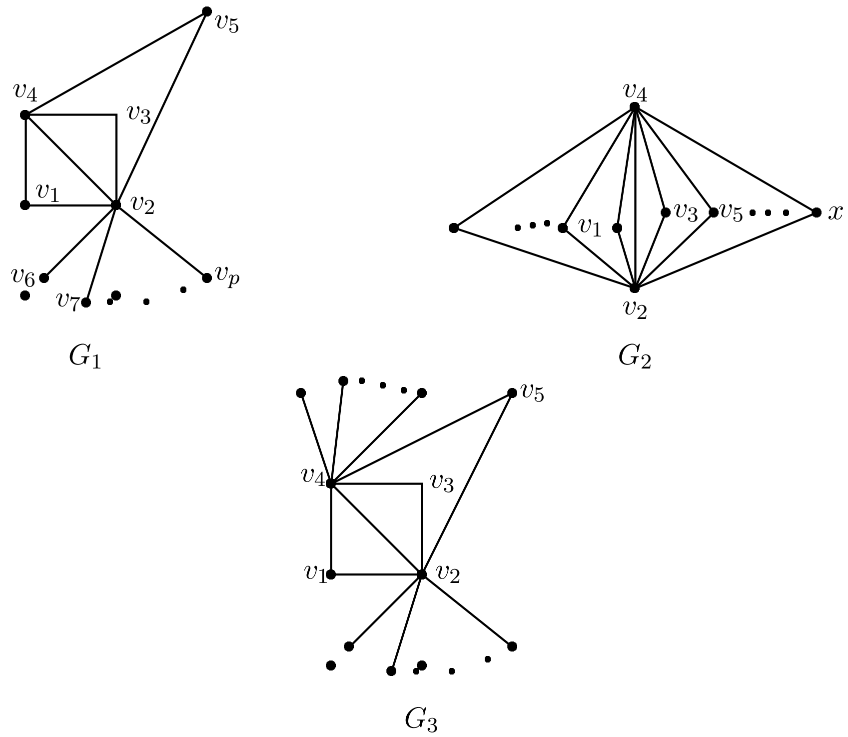

Figure 3.7. G

Since the edge v_2v_4 of G_i $(1 \leq i \leq 3)$ in Figure 3.7 does not lie on a detour joining any pair of vertices of G_i, these graphs are not edge detour graphs. Thus in this case there are no edge detour graphs G with H_2 as proper subgraph satisfying the requirements of the theorem. Thus we conclude that, if G does not contain K_4 as an induced subgraph, then G has a unique 4-cycle. Now we consider two subcases.

Subcase 1. The unique cycle C_4: v_1, v_2, v_3, v_4, v_1 contains exactly one chord v_2v_4. Since $p \geq 5$, $D = 4$ and G is connected, any vertex x not on C_4 is pendant and is adjacent to at least one vertex of C_4. The vertex x cannot be adjacent to both v_1 and v_3, for in this case we get $c(G) = 5$, which is a contradiction. Suppose that x is adjacent to v_1 or v_3, say v_1. Also if y is a vertex such that $y \neq x$, v_1, v_2, v_3, v_4, then y cannot be adjacent to v_2 or v_3 or v_4, for in each case $D \geq 5$, which is a contradiction. Hence y is a pendant vertex and cannot be adjacent to x or v_2 or v_3 or v_4 so that in this case the graph G reduces to the one given in Figure 3.8.

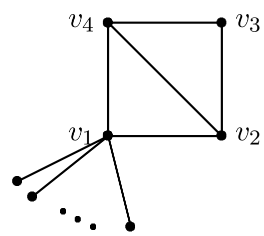

Figure 3.8. G

It follows from Theorem 1.2 that the set of all end vertices together with the vertex v_3 forms an edge detour basis for this graph G so that $dn_1(G) = p-3$. Similarly, if x is adjacent to v_3, we get a contradiction.

Now, if x is adjacent to both v_2 and v_4, we get the graph H given in Figure 3.9 as a subgraph which is isomorphic to the graph H_2 given in Figure 3.3. Then as in the first part of case 2, we see that there are no graphs which satisfy the requirements of the theorem.

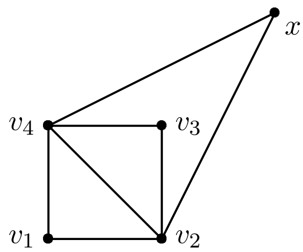

Figure 3.9. H

Thus x is adjacent to exactly one of v_2 or v_4, say v_2. Also if y is a vertex such that $y \neq x$, v_1, v_2, v_3, v_4, then y cannot be adjacent to x or v_1 or v_3, for in each case $D \geq 5$, which is a contradiction. If y is adjacent to v_2 and v_4, then we get the graph H given in Figure 3.10 as a subgraph. Then exactly as in the first part of case 2 it can be seen that there are no graphs satisfying the requirements of the theorem.

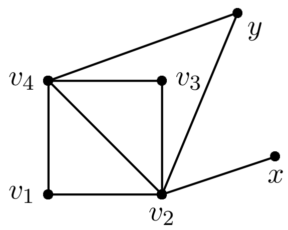

Figure 3.10. H

Thus y must be adjacent to v_2 or v_4 only. Hence we conclude that in either case the graph G must reduce to the graph G_7 or $G_8 \in \mathscr{K}$ as given in Figure 3.1. Similarly, if x is adjacent to v_4, then the graph G reduces to the graph G_7 or $G_8 \in \mathscr{K}$ as given in Figure 3.1. It is clear that $dn_1(G) = p - 2$ for these two classes of graphs G. Thus these two classes of graphs satisfy the requirements of the theorem. It is to be noted that the graph G_7 is nothing but $K_{1,p-1} + e + f$ where e and f are adjacent edges.

Subcase 2. The unique cycle C_4: v_1, v_2, v_3, v_4, v_1 has no chord. In this case we claim that G contains no triangle. Suppose that G contains a triangle C_3. If C_3 has no vertex in common with C_4 or exactly one vertex in common with C_4, we get a path of length at least 5 so that $D \geq 5$. If C_3 has exactly two vertices in common with C_4, we get a cycle of length 5. Thus, in all cases, we have a contradiction and hence it follows that G contains a unique chordless cycle C_4 with no triangles. Since $p \geq 5$, $D = 4$, $c(G) = 4$ and G is connected, any vertex x not on C_4 is pendant and is adjacent to exactly one vertex of C_4, say v_1. Also if y is a vertex such that $y \neq x$, v_1, v_2, v_3, v_4, then y cannot be adjacent to v_2 or v_4, for in this case $D \geq 5$, which is a contradiction. Thus y must be adjacent to v_3 only. Hence we conclude that in either case G must reduce to the graphs H_1 or H_2 as given in Figure 3.11.

For these graphs H_1 and H_2 in Figure 3.11, it follows from Theorem 1.2 that $dn_1(H_1) = p - 3$ and $dn_1(H_2) = p - 4$. Hence there are no graphs satisfying the requirements of the theorem. Thus when $D = 4$ and $c(G) = 4$, the graphs satisfying the requirements of the theorem are G_6, G_7, and $G_8 \in \mathscr{K}$ as in Figure 3.1.

Case 3. Let $c(G) = 3$.

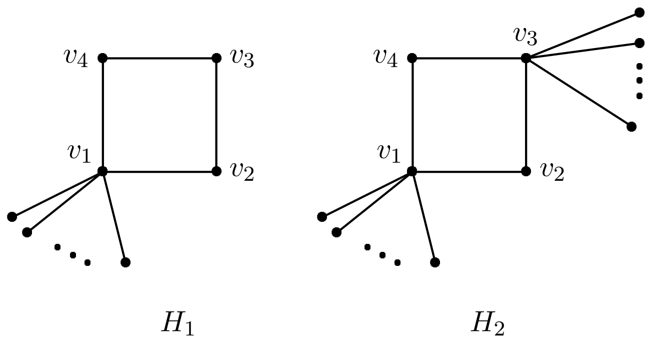

Figure 3.11. G

Case 3a. G contains exactly one triangle C_3: v_1, v_2, v_3, v_1. Since $p \geq 5$, there are vertices not on C_3. If all the vertices of C_3 have degree three or more, then since $D = 4$, the graph G must reduce to the one given in Figure 3.12.

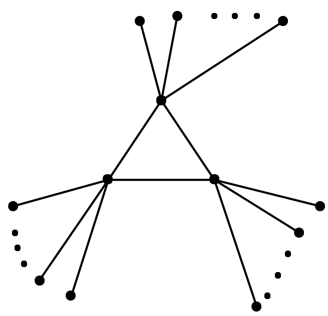

Figure 3.12. G

But, in this case $dn_1(G) = p-3$, which is a contradiction. Hence we conclude that at most two vertices of C_3 have degree ≥ 3.

Subcase 1. Exactly two vertices of C_3 have degree 3 or more. Let $deg_G v_3 = 2$. Now, since $p \geq 5$, $D = 4$, $c(G) = 3$ and G is connected, we see that the graph reduces to the graph $G_9 \in \mathscr{K}$ as given in Figure 3.1, for which $dn_1(G) = p - 2$. Thus in this case the graph $G_9 \in \mathscr{K}$ satisfies the requirements of the theorem.

Subcase 2. Exactly one vertex v_1 of C_3 has degree 3 or more. Since G is connected, $p \geq 5$, $D = 4$ and $c(G) = 3$, the graph reduces to the one given in Figure 3.13.

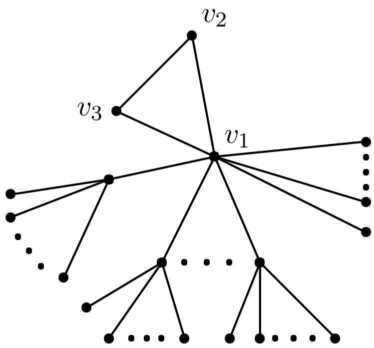

Figure 3.13. G

Now, we claim that exactly one neighbor of v_1 other than v_2 and v_3 has degree ≥ 2. If the claim is not true, then more than one neighbor of v_1 other than v_2 and v_3 has degree ≥ 2 and so the set of all end-vertices together with v_2 and v_3 forms an edge detour set of G. Hence $dn_1(G) \leq p - 3$, which is a contradiction. Thus in this case the graph reduces to the graph $G_{10} \in \mathscr{K}$ as in Figure 3.1, which satisfies the requirements of the theorem.

Case 3b. G contains more than one triangle. Since $D = 4$ and $c(G) = 3$, it is clear that all the triangles must have a vertex v in common. Now, if two triangles have two vertices in common then it is clear that $c(G) \geq 4$. Hence all triangles must have exactly one vertex in common. Since $p \geq 5$, $D = 4$, $c(G) = 3$ and G is connected, all the vertices of all the triangles are of degree 2 except v. Thus the graph reduces to the graphs given in Figure 3.14.

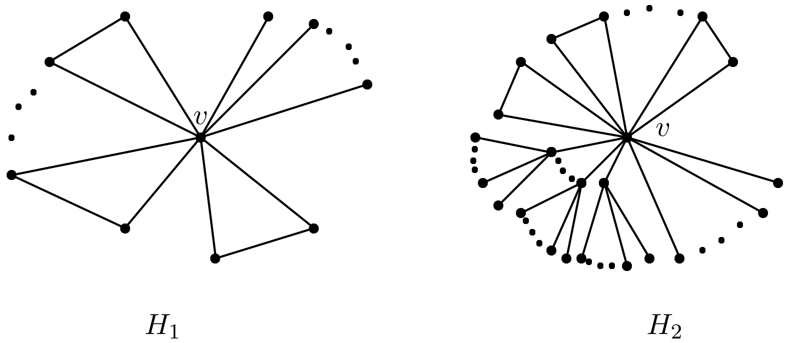

Figure 3.14. G

If $G = H_1$, then by Theorem 1.5, $dn_1(G) = p - 1$, which is a contradiction. If $G = H_2$, then we claim that exactly one neighbor of v not on the triangles has degree ≥ 2. If the claim is not true, then more than one neighbor of v not on the triangles has degree ≥ 2 and so the set of all end-vertices together with all the vertices of all triangles except v forms an edge detour set of G. Hence $dn_1(G) \leq p - 3$, which is a contradiction. Thus in this case the graph reduces to the graph $G_{11} \in \mathscr{K}$ as in Figure 3.1, which satisfies the requirements of the theorem. This completes the proof of the theorem. ■

Remark 3.3. For $p = 4$, the graphs are $G = P_4$, C_4 and $K_4 - e$ and $dn_1(G) = p-2$. For $p = 2$ and 3, there are no graphs G for which $dn_1(G) = p - 2$.

In view of Theorem 3.2 we leave the following problem as an open question.

Problem 3.4. Characterize edge detour graphs G with detour diameter $D \geq 5$ for which $dn_1(G) = p - 2$.

The following theorem characterizes trees T for which $dn_1(T) = p - 2$.

Theorem 3.5. *For any tree T of order $p \geq 5$, $dn_1(T) = p - 2$ if and only if T is a double star.*

Proof. If T is a double star, then by Theorem 1.3, $dn_1(T) = p - 2$. Conversely, assume that $dn_1(T) = p - 2$. If $D \leq 2$, then it is proved in Theorem 3.2 that there are no graphs G for which $dn_1(G) = p-2$. If $D = 3$, then it is proved in Theorem 3.2 that T is a double star. If $D \geq 4$, then there exist at least three internal vertices of T so that there are at most $p - 3$ end-vertices of T so that by Theorem 1.3, $dn_1(T) \leq p - 3$, which is a contradiction. This completes the proof. ■

The next theorem characterizes graphs G with detour diameter $D \leq 4$ for which $dn_1(G) = p - 1$. The proof is similar to that of Theorem 3.2 and hence we omit it.

Theorem 3.6. *Let G be an edge detour graph of order $p \geq 3$ with detour diameter $D \leq 4$. Then $dn_1(G) = p - 1$ if and only if G is K_4 or $K_{1,p-1}$ or $K_{1,p-1} + e_1 + e_2 + \cdots + e_t$ $(t \geq 1)$, where the edges e_i $(1 \leq i \leq t)$ are mutually nonadjacent.*

In view of Theorem 3.6 we leave the following problem as an open question.

Problem 3.7. Characterize edge detour graphs G with detour diameter $D \geq 5$ for which $dn_1(G) = p - 1$.

The following theorem characterizes trees T for which $dn_1(T) = p - 1$.

Theorem 3.8. *For any tree T of order $p \geq 3$, $dn_1(T) = p - 1$ if and only if T is the star $K_{1,p-1}$.*

Proof. The proof is similar to that of Theorem 3.5 and follows from Theorems 1.3 and 3.6. ■

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