A NOTE ON CYCLIC CHROMATIC NUMBER

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Abstract

A cyclic colouring of a graph $G$ embedded in a surface is a vertex colouring of $G$ in which any two distinct vertices sharing a face receive distinct colours. The cyclic chromatic number $\chi_c(G)$ of $G$ is the smallest number of colours in a cyclic colouring of $G$. Plummer and Toft in 1987 conjectured that $\chi_c(G) \leq \Delta^* + 2$ for any 3-connected plane graph $G$ with maximum face degree $\Delta^*$. It is known that the conjecture holds true for $\Delta^* \leq 4$ and $\Delta^* \geq 18$. The validity of the conjecture is proved in the paper for some special classes of planar graphs.

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1. Introduction

Graphs, which we are dealing with, are plane, 3-connected and simple. Consider such a graph $G = (V, E, F)$ and let us present notations used in this article. The degree $\deg(x)$ of $x \in V \cup F$ is the number of edges incident to $x$. A vertex of degree $k$ is a $k$-vertex, a face of degree $k$ is a $k$-face. By $V(x)$ we denote the set of all vertices incident to $x \in E \cup F$; similarly, $F(y)$ is the set of all faces incident to $y \in V \cup E$. If $e \in E$, $F(e) = \{f_1, f_2\}$ and $\deg(f_1) \leq \deg(f_2)$, then the pair $(\deg(f_1), \deg(f_2))$ is called the type of $e$. A cycle in $G$ is facial if its vertex set is equal to $V(f)$ for some $f \in F$.

A vertex $x_1$ is cyclically adjacent to a vertex $x_2 \neq x_1$ if there is a face $f$ with $x_1, x_2 \in V(f)$. The cyclic neighbourhood $N_c(x)$ of a vertex $x$ is the set of all vertices that are cyclically adjacent to $x$ and the closed cyclic...
The neighbourhood of \( x \) is \( N_c(x) = N_c(x) \cup \{ x \} \). (The usual neighbourhood of \( x \) is denoted by \( N(x) \).) The cyclic degree of \( x \) is \( cd(x) = |N_c(x)| \). A cyclic colouring of \( G \) is a mapping \( \varphi : V \rightarrow C \) in which \( \varphi(x_1) \neq \varphi(x_2) \) whenever \( x_1 \) is cyclically adjacent to \( x_2 \) (elements of \( C \) are colours of \( \varphi \)). The cyclic chromatic number \( \chi_c(G) \) of the graph \( G \) is the minimum number of colours in a cyclic colouring of \( G \).

For \( p, q \in \mathbb{Z} \) let \( [p, q] = \{ z \in \mathbb{Z} : p \leq z \leq q \} \) and \( [p, \infty) = \{ z \in \mathbb{Z} : p \leq z \} \).

Let \( G \) be an embedding of a 2-connected graph and let \( v \) be its vertex of degree \( n \). Consider a sequence \( (f_1, \ldots, f_n) \) of faces incident to \( v \) in a cyclic order around \( v \) (there are altogether \( 2n \) such sequences) and the sequence \( D = (d_1, \ldots, d_n) \) in which \( d_i = \deg(f_i) \) for \( i \in [1, n] \). The sequence \( D \) is called the type of the vertex \( v \) provided it is the lexicographical minimum of the set of all such sequences corresponding to \( v \).

It is easy to see that \( cd(v) = \sum_{i=1}^{n} (d_i - 2) \). A contraction of an edge \( xy \in E(G) \) consists in a continuous identification of the vertices \( x \) and \( y \) forming a new vertex \( x \leftrightarrow y \) and the removal of the created loop together with all possibly created multiedges; let \( G/xy \) be the result of such a contraction. An edge \( xy \) of a 3-connected plane graph \( G \) is contractible if \( G/xy \) is again 3-connected.

If the graph \( G \) is 2-connected, any face \( f \) of \( G \) is incident to \( \deg(f) \) vertices. In such a case \( \chi_c(G) \) is naturally lower bounded by \( \Delta^*(G) \), the maximum face degree of \( G \).

By a classical result of Whitney [9] all plane embeddings of a 3-connected planar graph are essentially the same. This means that \( \chi_c(G_1) = \chi_c(G_2) \) if \( G_1, G_2 \) are plane embeddings of a fixed 3-connected planar graph \( G \); thus, we can speak simply about the cyclic chromatic number of \( G \). Plummer and Toft in [8] conjectured that if \( G \) is a 3-connected plane graph, then \( \chi_c(G) \leq \Delta^*(G) + 2 \). They showed a weaker inequality \( \chi_c(G) \leq \Delta^*(G) + 9 \). Let \( PTC(d) \) denote the conjecture by Plummer and Toft restricted to graphs with \( \Delta^*(G) = d \). By the Four Colour Theorem, for a triangulation \( G \) we have \( \chi_c(G) \leq 4 = \Delta^*(G) + 1 \). PTC(4) is known to be true by the work of Borodin [2]. Horňák and Jendroľ [5] proved PTC(4) for any \( d \geq 24 \). The bound was improved to 22 by Morita [7], but to the best of our knowledge, the proof was never published. Horňák and Zlámalová [6] proved PTC(4) for any \( d \geq 18 \). Enomoto et al. [4] obtained for \( \Delta^*(G) \geq 60 \) even a stronger result, namely that \( \chi_c(G) \leq \Delta^*(G) + 1 \). The example of the (graph of) \( d \)-sided prism with maximum face degree \( d \) and cyclic chromatic number \( d + 1 \)
A Note on Cyclic Chromatic Number 117
shows that the bound is best possible. The best known general result (with
no restriction on $\Delta^*(G)$) is the inequality $\chi_c(G) \leq \Delta^*(G) + 5$ of Enomoto
and Hornák [3].

Conjecture by Plummer and Toft is still open. This means that we do
not know any $G$ with $\chi_c(G) - \Delta^*(G) \geq 3$. On the other hand, all $G$'s with
$\chi_c(G) - \Delta^*(G) = 2$ we are aware of satisfy $\Delta^*(G) = 4$. Therefore, the
conjecture could even be strengthened: If $G$ is a 3-connected plane graph $G$
with $\Delta^*(G) \neq 4$, then $\chi_c(G) \leq \Delta^*(G) + 1$.

In this paper we show that PTC($d$) is true for 3-connected plane graphs
of minimum degree 5 or of minimum degree 4 and maximum face degree at
least 6.

2. Auxiliary Results

In the proof of the result of this paper we shall need a special information
on the structure of 3-connected plane graphs contained in Lemma 1 that
follows by results of Ando et al. [1].

Lemma 1. If a vertex of degree at least four of a 3-connected plane graph
$G$ with $|V(G)| \geq 5$ is not incident to a contractible edge, then it is adjacent
to three 3-vertices.

Let $d \in [5, \infty)$. A 3-connected plane graph $G$ is said to be $d$-minimal if
$\Delta^*(G) \leq d$ and $\chi_c(G) > d + 2$, but $\Delta^*(H) \leq d$ implies $\chi_c(H) \leq d + 2$
for any 3-connected plane graph $H$ such that the pair $(|V(H)|, |E(H)|)$ is
lexicographically smaller than the pair $(|V(G)|, |E(G)|)$.

The next lemma shows that a $d$-minimal graph cannot contain some
configurations.

Lemma 2. Let $d \in [5, \infty)$ and let $G$ be a $d$-minimal graph. Then $G$ does
not contain any of the following configurations:

1. a vertex $x$ with $\deg(x) \geq 4$ and $\cd(x) \leq d + 1$ that is incident to a
contractible edge;
2. an edge of type $(3, d_2)$ with $d_2 \in [3, 4]$;
3. the configuration $C_i$ of Figure $i$, $i \in [1, 2]$, where $d = 6$ and the configura-
tion $C_3$ of Figure 3, where $d = 7$ and where encircled numbers represent
degrees of corresponding vertices and vertices without degree specification
are of an arbitrary degree.
proof. 1. The statement has already been proved in [5] (Lemma 3.1(e)).

2. The statement has already been proved in [6] (Lemma 3.6).

3. For the rest of the proof suppose that $G$ contains a configuration $C_i$, $i \in [1, 3]$, described in Lemma 2.3. Then 4-vertex $x_0$ of the configuration $C_i$, $i \in [1, 3]$, is incident to a contractible edge (because of Lemma 1). The graph $G'$ obtained by contracting of this edge is a 3-connected plane graph satisfying $\Delta^*(G') \leq \Delta^*(G) \leq d$ and $|V(G')| = |V(G)| - 1$, hence there is a cyclic colouring $\varphi : V(G') \to C$. This colouring will be used to find a cyclic colouring $\psi : V(G) \to C$ in order to obtain a contradiction with $\chi_c(G) > d + 2$. If not stated explicitly otherwise, we put $\psi(u) = \varphi(u)$ for any $u \in V(G) - \{x_0\}$.

$i \in \{1, 3\}$: First note that $cd(x_0) = d + 2$. If there is a colour $c \in C - \varphi(N(x_0))$, then we put $\psi(x_0) = c$, else, by assumptions, there is a colour $c^* \not\in \varphi(N(x_1) \cup N(x_2) - N(x_0))$. Therefore we can put $\psi(x_1) = c^*$ ($\psi(x_2) = c^*$) and $\psi(x_0) = \varphi(x_1)$ ($\psi(x_0) = \varphi(x_2)$).

$i = 2$: If there is a colour $c \in C - \varphi(N(x_0))$, then we put $\psi(x_0) = c$, else there is exactly one $j \in C$ such that $|\{\varphi(u) = j : u \in N(x_0)\}| = 2$. Without loss of generality we can suppose that $j \neq \varphi(x_2)$.

If $\varphi(x_1) \neq j$, then $C - \varphi(N(x_1)) \neq \emptyset$, so we can put $\psi(x_0) = \varphi(x_1)$ and colour properly $x_1$.

Now let us suppose that $\varphi(x_1) = j$. If $\varphi(x_3) \neq j$, then $C - \varphi(N(x_3)) \neq \emptyset$ and we can recolour $x_3$ and put $\psi(x_0) = \varphi(x_3)$.

If $\varphi(x_3) = j$, then we put $\psi(x_2) = \psi(x_4) = j$, $\psi(x_0) = \varphi(x_2)$, $\psi(x_3) = \varphi(x_4)$ and $\psi(x_1) = c$, where $c \in C - \varphi(N(x_1))$.

The result of this paper will be proved by contradiction, using the Discharging Method. For any vertex $v$ of 3-connected graph $G = (V, E, F)$ let
A Note on Cyclic Chromatic Number

\[
c_0(v) = 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{\deg(f)}
\]

be the initial charge of vertex \(v\). Then, using Euler’s formula and the handshaking lemma, is easy to see that \(\sum_{v \in V} c_0(v) = 2\).

In this section we shall establish (Lemma 2) that the structure of a \(d\)-minimal graph \(G = (V, E, F)\) is restricted. In the next section we use the Discharging Method to distribute the initial charges of vertices of \(G\) such that every vertex \(v \in V(G)\) will have a nonpositive new charge \(c_1(v)\), but the sum of all charges will be the same. Then we will show that the restriction of structure of \(G\) is so strong that the existence of \(G\) is incompatible with \(\sum_{v \in V} c_1(v) = 2\).

If a vertex \(v\) is of type \((d_1, \ldots, d_n)\), then

\[
c_0(v) = \gamma(d_1, \ldots, d_n) = 1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{d_i}.
\]

Clearly, if \(\pi\) is a permutation of the set \([1, n]\), then \(\gamma(d_{\pi(1)}, \ldots, d_{\pi(n)}) = \gamma(d_1, \ldots, d_n)\). Let the weight of a sequence \(D = (d_1, \ldots, d_n) \in \mathbb{Z}^n\) be defined by \(\text{wt}(D) = \sum_{i=1}^{n} d_i\). For \(n \in [2, \infty)\), \(q \in [0, n - 2]\), \((d_1, \ldots, d_{n-1}) \in [1, \infty)^{n-1}\) and \(w \in [\sum_{i=1}^{n-1} d_i + 1, \infty)\) let \(S_q(d_1, \ldots, d_{n-1}; w)\) be the set of all sequences \(D = (d_1, \ldots, d_q, d_{q+1}', \ldots, d_n') \in \mathbb{Z}^n\) satisfying \(d_i \geq d_i'\) for any \(i \in [q + 1, n - 1]\) and \(\text{wt}(D) \geq w\). The following lemma has been proved as Lemma 4 in [6].

**Lemma 3.** The maximum of \(\gamma(d_1, \ldots, d_q, d_{q+1}', \ldots, d_n')\) over all sequences \((d_1, \ldots, d_q, d_{q+1}', \ldots, d_n') \in S_q(d_1, \ldots, d_{n-1}; w)\) is equal to \(\gamma(d_1, \ldots, d_{n-1}, w - \sum_{i=1}^{n-1} d_i)\). \(\blacksquare\)

**Claim 1.** 1. If \(c_0(v) > 0\) for a vertex \(v\) of a 3-connected graph \(G = (V, E, F)\) with \(\Delta^*(G) \geq 5\), then \(\deg(v) \leq 4\).

2. If \(c_0(v) > 0\) for a 4-vertex \(v\) of a 3-connected graph \(G = (V, E, F)\), then the type of \(v\) is from the set \\{(3, 5, 3, 5), (3, 5, 3, 6), (3, 5, 3, 7)\}.

**Proof.** 1. Clearly, for vertices of degree at least 6 it holds

\[
c_0(v) = 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{\deg(f)} \leq 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{3}
\]
\[
= 1 - \frac{\deg(v)}{2} + \frac{\deg(v)}{3} = 1 - \frac{\deg(v)}{6} \leq 0.
\]

By Lemmas 2.2 and 3, for vertices of degree 5 it holds \( c_0(v) \leq \gamma(3, 5, 3, 5, 5) \leq 0 \).

2. The statement can be derived from Lemmas 2.2 and 3 and the following facts:

If a 4-vertex \( v \) is not adjacent to a 3-face, then \( c_0(v) \leq \gamma(4, 4, 4, 4) \leq 0 \).

If a 4-vertex \( v \) is adjacent to exactly one 3-face, then \( c_0(v) \leq \gamma(3, 5, 4, 5) \leq 0 \).

If a 4-vertex \( v \) is adjacent to exactly two 3-faces, but no 5-face, then \( c_0(v) \leq \gamma(3, 6, 3, 6) \leq 0 \).

If a 4-vertex \( v \) is adjacent to exactly two 3-faces, 5-face and face of degree at least 8, then \( c_0(v) \leq \gamma(3, 5, 3, 8) \leq 0 \).

A vertex \( v \in V \) is positive if \( c_0(v) > 0 \), otherwise it is nonpositive. For a vertex \( v \in V \) let \( n(v) \) denote the number of all neighbours of \( v \) of positive initial charge.

3. Discharging

**Theorem 4.** For every 3-connected plane graph \( G \) with \( \delta(G) = 4 \) and \( \Delta^*(G) \geq 6 \) or with \( \delta(G) \geq 5 \) it holds \( \chi_e(G) \leq \Delta^*(G) + 2 \).

**Proof.** Let \( G \) be a \( \Delta^* \)-minimal graph.

Case A. If \( \delta(G) \geq 5 \), then by the definition of the initial charge and Claim 1.1 we have \( c_0(v) \leq 0 \) for any \( v \in V(G) \), contradicting Euler’s formula. If \( \delta(G) = 4 \) and \( \Delta^*(G) \geq 9 \), then, by Lemmas 1 and 2.1, \( G \) does not contain positive 4-vertices. Thus, by the definition of initial charge and Claim 1.1, we have \( c_0(v) \leq 0 \) for every \( v \in V(G) \), contradicting Euler’s formula.

Case B. Let \( \delta(G) = 4 \) and \( \Delta^*(G) \in [6, 8] \). Let us state the only redistribution rule \( R \): A vertex \( v \) with \( c_0(v) < 0 \) sends to its neighbour \( w \) with \( c_0(w) > 0 \) the amount \( \frac{c_0(v)}{n(v)} \).

Now our aim is to show that \( c_1(v) \leq 0 \) for any \( v \in V(G) \) (where \( c_1(v) \) is the charge of \( v \) after using \( R \)).

- **(1)** If \( c_0(v) \leq 0 \), then obviously \( c_0(v) \leq c_1(v) \leq 0 \).
- **(2)** If \( c_0(v) > 0 \), then \( v \) is either of type \( (3, 5, 3, 6) \) with \( c_0(v) = \frac{1}{30} \) or of type \( (3, 5, 3, 7) \) with \( c_0(v) = \frac{1}{105} \) for the case \( \Delta^*(G) \in \{7, 8\} \) (because of
Lemmas 1 and 2. $G$ does not contain vertices of type $(3,5,3,5)$ and $v$ is either of type $(3,5,3,5)$ with $c_0(v) = \frac{1}{15}$ or of type $(3,5,3,6)$ with $c_0(v) = \frac{1}{30}$ for the case $\Delta^*(G) = 6$.

(21) If $v$ is of type $(3,5,3,5)$, then:

(211) If there exist two distinct neighbours $t_1, t_2$ of vertex $v$ such that $\deg(t_1), \deg(t_2) \geq 5$, then $c_1(v) \leq \frac{1}{15} + 2 \cdot \frac{1}{5} \cdot \gamma(3,5,3,5) \leq 0$.

(212) If at most one neighbour of vertex $v$ is of degree at least 5, then, by absence of $C_4$ in $G$, $c_1(v) \leq \frac{1}{15} + 4 \cdot \gamma(3,5,4,5) = 0$.

(22) If $v$ is either of type $(3,5,3,6)$ or of type $(3,5,3,7)$, then let $t_2, t_3$ be the neighbours of $v$ incident with 5-face, let $t_1, t_4$ be the other two neighbours of $v$, where $t_1$ is a common neighbour of vertices $v$ and $t_2$ and $t_4$ is a common neighbour of vertices $v$ and $t_3$.

(221) If there exists $i \in \{1, 4\}$ such that $\deg(t_i) \geq 5$, then $c_0(t_i) + \frac{1}{30} n(t_i) \leq 1 - \frac{2}{30} \delta(t_i) + \frac{1}{30} n(t_i) \leq 1 - \frac{7}{30} \delta(t_i) + \frac{1}{30} \delta(t_i) = 1 - \frac{1}{5} \delta(t_i) \leq 0$, and so $\frac{c_0(t_i)}{\delta(t_i)} \leq -\frac{1}{30}$. Therefore $c_1(v) \leq \frac{1}{30} - \frac{1}{30} = 0$.

(222) If $\deg(t_i) = 4$ for any $i \in \{1, 4\}$, then let $g_1$ be another face incident with the edge $t_1 t_2$ (and not incident with vertex $v$); similarly let $g_2$ be another face incident with the edge $t_3 t_4$ (and not incident with vertex $v$). By Lemma 2.2 we have $\deg(g_i) \geq 5$, $i \in \{1, 2\}$. Finally, let $f_i$ be the fourth face incident with the vertex $t_i$ (thus $f_i$ is not incident with $v$ and $f_i \notin \{g_1, g_2\}$).

(2221) If there exists $i \in \{1, 4\}$ such that $\deg(f_i) \geq 5$, then $c_0(t_i) \leq \gamma(3,5,3,5) = -\frac{1}{15}$ and $n(t_i) \leq 2$. Therefore $c_1(v) \leq c_0(v) + 2 \cdot (-\frac{1}{15}) \leq 0$.

(2222) If there exists $i \in \{1, 4\}$ such that $\deg(f_i) = 4$, then let $j \in \{1, 2\}$ be such that face $g_j$ is neighbour of face $f_i$.

(22221) If $\deg(f_j) \geq 6$, then $c_0(t_i) \leq \gamma(3,6,4,6) = -\frac{1}{12}$ and $n(t_i) \leq 2$. Thus $c_1(v) \leq c_0(v) + 2 \cdot (-\frac{1}{12}) \leq 0$.

(22222) If $\deg(f_j) = 5$, then $c_0(t_i) \leq \gamma(3,5,4,6) = -\frac{1}{30}$. Simultaneously $n(t_i) = 1$, else either $G$ contains a vertex of type $(3,5,3,5)$ (for $\Delta^*(G) \in \{7, 8\}$) or $G$ contains a configuration $C_4$ (if $\Delta^*(G) = 6$). Then $c_1(v) \leq c_0(v) - \frac{1}{30} \leq 0$.

(223) Let now $\deg(f_1) = \deg(f_4) = 3$.

(2231) If $\Delta^*(G) \in \{7, 8\}$, then:

(22311) If there exists $i \in \{2, 3\}$ such that $\deg(f_i) = 3$, then, by $C_3$, $v$ is of type $(3,5,3,7)$ and $g_j$ adjacent to $f_i$, $j \in \{1, 2\}$, is of degree at least 6, because $G$ does not contain a vertex of type $(3,5,3,5)$. Then a vertex $t_k$, $k \in \{1, 4\}$, which is a common neighbour of vertices $v$ and $t_i$, has the initial charge $c_0(t_k) \leq \gamma(3,6,3,7) = -\frac{1}{42}$. Due to $R$, the vertex $t_k$ sends at
most $-\frac{1}{168}$ to the vertex $v$. If $\deg(f_{5-i}) = 3$, then also the vertex $t_{5-k}$ sends at most $-\frac{1}{168}$ to the vertex $v$, else $t_{5-i}$ sends at most $\frac{1}{2} \cdot (-\frac{1}{60})$ to $v$. Thus $c_1(v) \leq \max\{c_0(v) - 2 \cdot \frac{1}{168}, c_0(v) - \frac{1}{168} - \frac{1}{168}\} = c_0(v) - \frac{1}{64} \leq 0$.

(22312) Let now $\deg(f_2) = \deg(f_3) = 4$. Then $c_0(t_2), c_0(t_3) \leq \gamma(3, 5, 4, 5) = -\frac{1}{60}$. Now if $v$ is of type $(3, 5, 3, 7)$, then $c_1(v) \leq c_0(v) - 2 \cdot \frac{1}{2} \cdot \frac{1}{60} \leq 0$, else, by $C_3$, $d = 7$, $\deg(g_1), \deg(g_2) \geq 6$ and so $c_1(v) \leq c_0(v) + 2 \gamma(3, 5, 4, 6) \leq 0$.

(2232) If $\Delta^*(G) = 6$, then due to absence of configuration $C_2$ in $G$, there exists $i \in \{2, 3\}$ such that vertex $t_i$ is of type $(3, 5, 4, 6)$. Therefore $n(t_i) = 1$ and $c_1(v) \leq c_0(v) - \frac{1}{50} \leq 0$.

 References


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