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A NOTE ON CYCLIC CHROMATIC NUMBER

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Abstract

A cyclic colouring of a graph G embedded in a surface is a vertex colouring of G in which any two distinct vertices sharing a face receive distinct colours. The cyclic chromatic number $\chi_c(G)$ of G is the smallest number of colours in a cyclic colouring of G. Plummer and Toft in 1987 conjectured that $\chi_c(G) \leq \Delta^* + 2$ for any 3-connected plane graph G with maximum face degree Δ^*. It is known that the conjecture holds true for $\Delta^* \leq 4$ and $\Delta^* \geq 18$. The validity of the conjecture is proved in the paper for some special classes of planar graphs.

Keywords: plane graph, cyclic colouring, cyclic chromatic number.

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1. Introduction

Graphs, which we are dealing with, are plane, 3-connected and simple. Consider such a graph $G = (V, E, F)$ and let us present notations used in this article. The *degree* $\deg(x)$ of $x \in V \cup F$ is the number of edges incident to x. A vertex of degree k is a *k-vertex*, a face of degree k is a *k-face*. By $V(x)$ we denote the set of all vertices incident to $x \in E \cup F$; similarly, $F(y)$ is the set of all faces incident to $y \in V \cup E$. If $e \in E$, $F(e) = \{f_1, f_2\}$ and $\deg(f_1) \leq \deg(f_2)$, then the pair $(\deg(f_1), \deg(f_2))$ is called the *type* of e. A cycle in G is *facial* if its vertex set is equal to $V(f)$ for some $f \in F$.

A vertex x_1 is *cyclically adjacent* to a vertex $x_2 \neq x_1$ if there is a face f with $x_1, x_2 \in V(f)$. The *cyclic neighbourhood* $N_c(x)$ of a vertex x is the set of all vertices that are cyclically adjacent to x and the *closed cyclic* *neighbourhood* of x is $\bar{N}_{\mathrm{c}}(x) = N_{\mathrm{c}}(x) \cup \{x\}$. (The usual neighbourhood of x is denoted by $N(x)$.) The *cyclic degree* of x is $\mathrm{cd}(x) = |N_{\mathrm{c}}(x)|$. A *cyclic colouring* of G is a mapping $\varphi : V \to C$ in which $\varphi(x_1) \neq \varphi(x_2)$ whenever x_1 is cyclically adjacent to x_2 (elements of C are *colours* of φ). The cyclic chromatic number $\chi_{\mathrm{c}}(G)$ of the graph G is the minimum number of colours in a cyclic colouring of G.

For $p, q \in \mathbb{Z}$ let $[p, q] = \{z \in \mathbb{Z} : p \leq z \leq q\}$ and $[p, \infty) = \{z \in \mathbb{Z} : p \leq z\}$.

Let G be an embedding of a 2-connected graph and let v be its vertex of degree n. Consider a sequence $(f_1, \ldots, f_n)$ of faces incident to v in a cyclic order around v (there are altogether $2n$ such sequences) and the sequence $D = (d_1, \ldots, d_n)$ in which $d_i = \deg(f_i)$ for $i \in [1, n]$. The sequence D is called the *type* of the vertex v provided it is the lexicographical minimum of the set of all such sequences corresponding to v.

It is easy to see that $\mathrm{cd}(v) = \sum_{i=1}^{n}(d_i - 2)$. A *contraction* of an edge $xy \in E(G)$ consists in a continuous identification of the vertices x and y forming a new vertex $x \leftrightarrow y$ and the removal of the created loop together with all possibly created multiedges; let G/xy be the result of such a contraction. An edge xy of a 3-connected plane graph G is *contractible* if G/xy is again 3-connected.

If the graph G is 2-connected, any face f of G is incident to $\deg(f)$ vertices. In such a case $\chi_{\mathrm{c}}(G)$ is naturally lower bounded by $\Delta^*(G)$, the *maximum face degree* of G.

By a classical result of Whitney [9] all plane embeddings of a 3-connected planar graph are essentially the same. This means that $\chi_{\mathrm{c}}(G_1) = \chi_{\mathrm{c}}(G_2)$ if G_1, G_2 are plane embeddings of a fixed 3-connected planar graph G; thus, we can speak simply about the cyclic chromatic number of G. Plummer and Toft in [8] conjectured that if G is a 3-connected plane graph, then $\chi_{\mathrm{c}}(G) \leq \Delta^*(G) + 2$. They showed a weaker inequality $\chi_{\mathrm{c}}(G) \leq \Delta^*(G) + 9$. Let $\mathrm{PTC}(d)$ denote the conjecture by Plummer and Toft restricted to graphs with $\Delta^*(G) = d$. By the Four Colour Theorem, for a triangulation G we have $\chi_{\mathrm{c}}(G) \leq 4 = \Delta^*(G) + 1$. PTC(4) is known to be true by the work of Borodin [2]. Horňák and Jendrol' [5] proved $\mathrm{PTC}(d)$ for any $d \geq 24$. The bound was improved to 22 by Morita [7], but to the best of our knowledge, the proof was never published. Horňák and Zlámalová [6] proved $\mathrm{PTC}(d)$ for any $d \geq 18$. Enomoto *et al.* [4] obtained for $\Delta^*(G) \geq 60$ even a stronger result, namely that $\chi_{\mathrm{c}}(G) \leq \Delta^*(G) + 1$. The example of the (graph of) d-sided prism with maximum face degree d and cyclic chromatic number $d + 1$

shows that the bound is best possible. The best known general result (with no restriction on $\Delta^*(G)$) is the inequality $\chi_c(G) \leq \Delta^*(G) + 5$ of Enomoto and Horňák [3].

Conjecture by Plummer and Toft is still open. This means that we do not know any G with $\chi_c(G) - \Delta^*(G) \geq 3$. On the other hand, all G's with $\chi_c(G) - \Delta^*(G) = 2$ we are aware of satisfy $\Delta^*(G) = 4$. Therefore, the conjecture could even be strengthened: If G is a 3-connected plane graph G with $\Delta^*(G) \neq 4$, then $\chi_c(G) \leq \Delta^*(G) + 1$.

In this paper we show that PTC(d) is true for 3-connected plane graphs of minimum degree 5 or of minimum degree 4 and maximum face degree at least 6.

2. Auxiliary Results

In the proof of the result of this paper we shall need a special information on the structure of 3-connected plane graphs contained in Lemma 1 that follows by results of Ando *et al.* [1].

Lemma 1. *If a vertex of degree at least four of a* 3*-connected plane graph* G *with* $|V(G)| \geq 5$ *is not incident to a contractible edge, then it is adjacent to three* 3*-vertices.* ■

Let $d \in [5, \infty)$. A 3-connected plane graph G is said to be *d-minimal* if $\Delta^*(G) \leq d$ and $\chi_c(G) > d + 2$, but $\Delta^*(H) \leq d$ implies $\chi_c(H) \leq d + 2$ for any 3-connected plane graph H such that the pair $(|V(H)|, |E(H)|)$ is lexicographically smaller then the pair $(|V(G)|, |E(G)|)$.

The next lemma shows that a d-minimal graph cannot contain some configurations.

Lemma 2. *Let* $d \in [5, \infty)$ *and let* G *be a* d*-minimal graph. Then* G *does not contain any of the following configurations:*

1. *a vertex* x *with* $\deg(x) \geq 4$ *and* $\mathrm{cd}(x) \leq d + 1$ *that is incident to a contractible edge;*
2. *an edge of type* $(3, d_2)$ *with* $d_2 \in [3, 4]$*;*
3. *the configuration* $\mathcal{C}_i$ *of Figure* i, $i \in [1, 2]$, *where* $d = 6$ *and the configuration* $\mathcal{C}_3$ *of Figure* 3, *where* $d = 7$ *and where encircled numbers represent degrees of corresponding vertices and vertices without degree specification are of an arbitrary degree.*

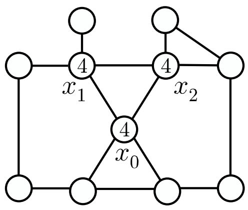

Figure 1. $\mathrm{cd}(x_1) \le 10$

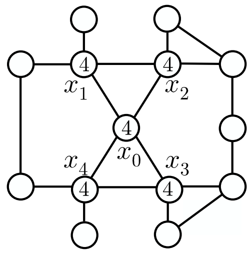

Figure 2. $\mathrm{cd}(x_1) \le 9$, $\mathrm{cd}(x_4) \le 9$

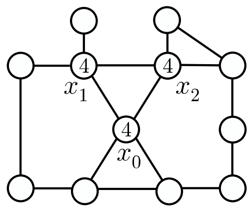

Figure 3. $\mathrm{cd}(x_1) \le 10$

Proof. 1. The statement has already been proved in [5] (Lemma 3.1(e)).

2. The statement has already been proved in [6] (Lemma 3.6).

3. For the rest of the proof suppose that G contains a configuration C_i, $i \in [1,3]$, described in Lemma 2.3. Then 4-vertex x_0 of the configuration C_i, $i \in [1,3]$, is incident to a contractible edge (because of Lemma 1). The graph G' obtained by contracting of this edge is a 3-connected plane graph satisfying $\Delta^*(G') \le \Delta^*(G) \le d$ and $|V(G')| = |V(G)| - 1$, hence there is a cyclic colouring $\varphi : V(G') \to C$. This colouring will be used to find a cyclic colouring $\psi : V(G) \to C$ in order to obtain a contradiction with $\chi_c(G) > d+2$. If not stated explicitly otherwise, we put $\psi(u) = \varphi(u)$ for any $u \in V(G) - \{x_0\}$.

$i \in \{1,3\}$: First note that $\mathrm{cd}(x_0) = d+2$. If there is a colour $c \in C - \varphi(N(x_0))$, then we put $\psi(x_0) = c$, else, by assumptions, there is a colour c^* such that $c^* \notin \varphi(\bar{N}(x_1) \cup \bar{N}(x_2) - N(x_0))$. Therefore we can put $\psi(x_1) = c^*$ ($\psi(x_2) = c^*$) and $\psi(x_0) = \varphi(x_1)$ ($\psi(x_0) = \varphi(x_2)$).

$i = 2$: If there is a colour $c \in C - \varphi(N(x_0))$, then we put $\psi(x_0) = c$, else there is exactly one $j \in C$ such that $|\{\varphi(u) = j : u \in N(x_0)\}| = 2$. Without loss of generality we can suppose that $j \ne \varphi(x_2)$.

If $\varphi(x_1) \ne j$, then $C - \varphi(\bar{N}(x_1)) \ne \emptyset$, so we can put $\psi(x_0) = \varphi(x_1)$ and colour properly x_1.

Now let us suppose that $\varphi(x_1) = j$. If $\varphi(x_3) \ne j$, then $C - \varphi(\bar{N}(x_3)) \ne \emptyset$ and we can recolour x_3 and put $\psi(x_0) = \varphi(x_3)$.

If $\varphi(x_3) = j$, then we put $\psi(x_2) = \psi(x_4) = j$, $\psi(x_0) = \varphi(x_2)$, $\psi(x_3) = \varphi(x_4)$ and $\psi(x_1) = c$, where $c \in C - \varphi(\bar{N}(x_1))$. ■

The result of this paper will be proved by contradiction, using the Discharging Method. For any vertex v of 3-connected graph $G = (V, E, F)$ let

$$c_0(v) = 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{\deg(f)}$$

be the *initial charge* of vertex v. Then, using Euler's formula and the handshaking lemma, is easy to see that $\sum_{v \in V} c_0(v) = 2$.

In this section we shall establish (Lemma 2) that the structure of a d-minimal graph $G = (V, E, F)$ is restricted. In the next section we use the Discharging Method to distribute the initial charges of vertices of G such that every vertex $v \in V(G)$ will have a nonpositive new charge $c_1(v)$, but the sum of all charges will be the same. Then we will show that the restriction of structure of G is so strong that the existence of G is incompatible with $\sum_{v \in V} c_1(v) = 2$.

If a vertex v is of type $(d_1, \ldots, d_n)$, then

$$c_0(v) = \gamma(d_1, \ldots, d_n) = 1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{d_i}.$$

Clearly, if π is a permutation of the set $[1, n]$, then $\gamma(d_{\pi(1)}, \ldots, d_{\pi(n)}) = \gamma(d_1, \ldots, d_n)$. Let the *weight* of a sequence $D = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ be defined by $\mathrm{wt}(D) = \sum_{i=1}^{n} d_i$. For $n \in [2, \infty)$, $q \in [0, n-2]$, $(d_1, \ldots, d_{n-1}) \in [1, \infty)^{n-1}$ and $w \in [\sum_{i=1}^{n-1} d_i + 1, \infty)$ let $S_q(d_1, \ldots, d_{n-1}; w)$ be the set of all sequences $D = (d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n) \in \mathbb{Z}^n$ satisfying $d'_i \geq d_i$ for any $i \in [q+1, n-1]$ and $\mathrm{wt}(D) \geq w$. The following lemma has been proved as Lemma 4 in [6].

Lemma 3. *The maximum of* $\gamma(d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n)$ *over all sequences* $(d_1, \ldots, d_q, d'_{q+1}, \ldots, d'_n) \in S_q(d_1, \ldots, d_{n-1}; w)$ *is equal to* $\gamma(d_1, \ldots, d_{n-1}, w - \sum_{i=1}^{n-1} d_i)$. ■

Claim 1. 1. If $c_0(v) > 0$ for a vertex v of a 3-connected graph $G = (V, E, F)$ with $\Delta^*(G) \geq 5$, then $\deg(v) \leq 4$.

2. If $c_0(v) > 0$ for a 4-vertex v of a 3-connected graph $G = (V, E, F)$, then the type of v is from the set $\{(3,5,3,5), (3,5,3,6), (3,5,3,7)\}$.

Proof. 1. Clearly, for vertices of degree at least 6 it holds

$$c_0(v) = 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{\deg(f)} \leq 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{3}$$

$$= 1 - \frac{\deg(v)}{2} + \frac{\deg(v)}{3} = 1 - \frac{\deg(v)}{6} \leq 0.$$

By Lemmas 2.2 and 3, for vertices of degree 5 it holds $c_0(v) \leq \gamma(3,5,3,5,5) \leq 0$.

2. The statement can be derived from Lemmas 2.2 and 3 and the following facts:
If a 4-vertex v is not adjacent to a 3-face, then $c_0(v) \leq \gamma(4,4,4,4) \leq 0$.
If a 4-vertex v is adjacent to exactly one 3-face, then $c_0(v) \leq \gamma(3,5,4,5) \leq 0$.
If a 4-vertex v is adjacent to exactly two 3-faces, but no 5-face, then $c_0(v) \leq \gamma(3,6,3,6) \leq 0$.
If a 4-vertex v is adjacent to exactly two 3-faces, 5-face and face of degree at least 8, then $c_0(v) \leq \gamma(3,5,3,8) \leq 0$. ■

A vertex $v \in V$ is *positive* if $c_0(v) > 0$, otherwise it is nonpositive. For a vertex $v \in V$ let $n(v)$ denote the number of all neighbours of v of positive initial charge.

3. Discharging

Theorem 4. *For every 3-connected plane graph G with $\delta(G) = 4$ and $\Delta^*(G) \geq 6$ or with $\delta(G) \geq 5$ it holds $\chi_c(G) \leq \Delta^*(G) + 2$.*

Proof. Let G be a Δ^*-minimal graph.

Case A. If $\delta(G) \geq 5$, then by the definition of the initial charge and Claim 1.1 we have $c_0(v) \leq 0$ for any $v \in V(G)$, contradicting Euler's formula. If $\delta(G) = 4$ and $\Delta^*(G) \geq 9$, then, by Lemmas 1 and 2.1, G does not contain positive 4-vertices. Thus, by the definition of initial charge and Claim 1.1, we have $c_0(v) \leq 0$ for every $v \in V(G)$, contradicting Euler's formula.

Case B. Let $\delta(G) = 4$ and $\Delta^*(G) \in [6,8]$. Let us state the only redistribution rule **R:** A vertex v with $c_0(v) < 0$ sends to its neighbour w with $c_0(w) > 0$ the amount $\frac{c_0(v)}{n(v)}$.

Now our aim is to show that $c_1(v) \leq 0$ for any $v \in V(G)$ (where $c_1(v)$ is the charge of v after using R).

(1) If $c_0(v) \leq 0$, then obviously $c_0(v) \leq c_1(v) \leq 0$.

(2) If $c_0(v) > 0$, then v is either of type $(3,5,3,6)$ with $c_0(v) = \frac{1}{30}$ or of type $(3,5,3,7)$ with $c_0(v) = \frac{1}{105}$ for the case $\Delta^*(G) \in \{7,8\}$ (because of

Lemmas 1 and 2.2 G does not contain vertices of type $(3,5,3,5)$) and v is either of type $(3,5,3,5)$ with $c_0(v) = \frac{1}{15}$ or of type $(3,5,3,6)$ with $c_0(v) = \frac{1}{30}$ for the case $\Delta^*(G) = 6$.

(21) If v is of type $(3,5,3,5)$, then:

(211) If there exist two distinct neighbours t_1, t_2 of vertex v such that $\deg(t_1), \deg(t_2) \geq 5$, then $c_1(v) \leq \frac{1}{15} + 2 \cdot \frac{1}{5} \cdot \gamma(3,5,3,5,5) \leq 0$.

(212) If at most one neighbour of vertex v is of degree at least 5, then, by absence of C_1 in G, $c_1(v) \leq \frac{1}{15} + 4 \cdot \gamma(3,5,4,5) = 0$.

(22) If v is either of type $(3,5,3,6)$ or of type $(3,5,3,7)$, then let t_2, t_3 be the neighbours of v incident with 5-face, let t_1, t_4 be the other two neighbours of v, where t_1 is a common neighbour of vertices v and t_2 and t_4 is a common neighbour of vertices v and t_3.

(221) If there exists $i \in [1,4]$ such that $\deg(t_i) \geq 5$, then $c_0(t_i) + \frac{1}{30}n(t_i) \leq 1 - \frac{7}{30}\deg(t_i) + \frac{1}{30}n(t_i) \leq 1 - \frac{7}{30}\deg(t_i) + \frac{1}{30}\deg(t_i) = 1 - \frac{1}{5}\deg(t_i) \leq 0$, and so $\frac{c_0(t_i)}{n(t_i)} \leq -\frac{1}{30}$. Therefore $c_1(v) \leq \frac{1}{30} - \frac{1}{30} = 0$.

(222) If $\deg(t_i) = 4$ for any $i \in [1,4]$, then let g_1 be another face incident with the edge t_1t_2 (and not incident with vertex v); similarly let g_2 be another face incident with the edge t_3t_4 (and not incident with vertex v). By Lemma 2.2 we have $\deg(g_i) \geq 5$, $i \in \{1,2\}$. Finally, let f_i be the fourth face incident with the vertex t_i (thus f_i is not incident with v and $f_i \notin \{g_1, g_2\}$).

(2221) If there exists $i \in [1,4]$ such that $\deg(f_i) \geq 5$, then $c_0(t_i) \leq \gamma(3,5,5,5) = -\frac{1}{15}$ and $n(t_i) \leq 2$. Therefore $c_1(v) \leq c_0(v) + \frac{1}{2} \cdot (-\frac{1}{15}) \leq 0$.

(2222) If there exists $i \in \{1,4\}$ such that $\deg(f_i) = 4$, then let $j \in \{1,2\}$ be such that face g_j is neighbour of face f_i.

(22221) If $\deg(g_j) \geq 6$, then $c_0(t_i) \leq \gamma(3,6,4,6) = -\frac{1}{12}$ and $n(t_i) \leq 2$. Thus $c_1(v) \leq c_0(v) + \frac{1}{2} \cdot (-\frac{1}{12}) \leq 0$.

(22222) If $\deg(g_j) = 5$, then $c_0(t_i) \leq \gamma(3,5,4,6) = -\frac{1}{20}$. Simultaneously $n(t_i) = 1$, else either G contains a vertex of type $(3,5,3,5)$ (for $\Delta^*(G) \in \{7,8\}$) or G contains a configuration C_1 (if $\Delta^*(G) = 6$). Then $c_1(v) \leq c_0(v) - \frac{1}{20} \leq 0$.

(223) Let now $\deg(f_1) = \deg(f_4) = 3$.

(2231) If $\Delta^*(G) \in \{7,8\}$, then:

(22311) If there exists $i \in \{2,3\}$ such that $\deg(f_i) = 3$, then, by C_3, v is of type $(3,5,3,7)$ and g_j adjacent to f_i, $j \in \{1,2\}$, is of degree at least 6, because G does not contain a vertex of type $(3,5,3,5)$. Then a vertex t_k, $k \in \{1,4\}$, which is a common neighbour of vertices v and t_i, has the initial charge $c_0(t_k) \leq \gamma(3,6,3,7) = -\frac{1}{42}$. Due to R, the vertex t_k sends at most $-\frac{1}{168}$ to the vertex v. If $\deg(f_{5-i}) = 3$, then also the vertex t_{5-k} sends at most $-\frac{1}{168}$ to the vertex v, else t_{5-i} sends at most $\frac{1}{2} \cdot (-\frac{1}{60})$ to v. Thus $c_1(v) \leq \max\{c_0(v) - 2 \cdot \frac{1}{168}, c_0(v) - \frac{1}{168} - \frac{1}{120}\} = c_0(v) - \frac{1}{84} \leq 0$.

(22312) Let now $\deg(f_2) = \deg(f_3) = 4$. Then $c_0(t_2), c_0(t_3) \leq \gamma(3,5,4,5) = -\frac{1}{60}$. Now if v is of type $(3,5,3,7)$, then $c_1(v) \leq c_0(v) - 2 \cdot \frac{1}{2} \cdot \frac{1}{60} \leq 0$, else, by C_3, $d = 7$, $\deg(g_1), \deg(g_2) \geq 6$ and so $c_1(v) \leq c_0(v) + 2\gamma(3,5,4,6) \leq 0$.

(2232) If $\Delta^*(G) = 6$, then due to absence of configuration C_2 in G, there exists $i \in \{2,3\}$ such that vertex t_i is of type $(3,5,4,6)$. Therefore $n(t_i) = 1$ and $c_1(v) \leq c_0(v) - \frac{1}{20} \leq 0$. ■

References

[1] K. Ando, H. Enomoto and A. Saito, *Contractible edges in* 3*-connected graphs*, J. Combin. Theory (B) **42** (1987) 87–93.

[2] O.V. Borodin, *Solution of Ringel's problem on vertex-face coloring of plane graphs and coloring of* 1*-planar graphs* (Russian), Met. Diskr. Anal. **41** (1984) 12–26.

[3] H. Enomoto and M. Horňák, *A general upper bound for the cyclic chromatic number of* 3*-connected plane graphs*, J. Graph Theory **62** (2009) 1–25.

[4] H. Enomoto, M. Horňák and S. Jendroľ, *Cyclic chromatic number of* 3*-connected plane graphs*, SIAM J. Discrete Math. **14** (2001) 121–137.

[5] M. Horňák and S. Jendroľ, *On a conjecture by Plummer and Toft*, J. Graph Theory **30** (1999) 177–189.

[6] M. Horňák and J. Zlámalová, *Another step towards proving a conjecture by Plummer and Toft*, Discrete Math. **310** (2010) 442–452.

[7] A. Morita, Cyclic chromatic number of 3-connected plane graphs (Japanese, M.S. Thesis), Keio University, Yokohama 1998.

[8] M.D. Plummer and B. Toft, *Cyclic coloration of* 3*-polytopes*, J. Graph Theory **11** (1987) 507–515.

[9] H. Whitney, *Congruent graphs and the connectivity of graphs*, Amer. J. Math. **54** (1932) 150–168.

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