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THE PERIPHERY GRAPH OF A MEDIAN GRAPH*

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Abstract

The periphery graph of a median graph is the intersection graph of its peripheral subgraphs. We show that every graph without a universal vertex can be realized as the periphery graph of a median graph. We characterize those median graphs whose periphery graph is the join

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of two graphs and show that they are precisely Cartesian products of median graphs. Path-like median graphs are introduced as the graphs whose periphery graph has independence number 2, and it is proved that there are path-like median graphs with arbitrarily large geodetic number. Peripheral expansion with respect to periphery graph is also considered, and connections with the concept of crossing graph are established.

Keywords: median graph, Cartesian product, geodesic, periphery, peripheral expansion.

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1. Introduction

Median graphs are one of the most studied classes of graphs, cf. a survey containing over 50 characterization [17] and the references therein. Several applications of median graphs have been found, in particular in recent studies in phylogenetics [5, 13]. A strong connection between median graphs and triangle free-graphs was established in [15] by which both classes have roughly the same recognition complexity. Another very useful result is a characterization of finite median graphs via a (finite) sequence of contractions that ends with the one-vertex graph, where at each step a so-called peripheral subgraph is contracted or deleted [19]. In terms of convexity theory peripheral subgraphs in median graphs can be described as minimal half-spaces, and they play a similar role also in some generalizations of median graphs, where a more strict type of convexity applies, cf. [2, 6, 11, 21].

It was shown in [10] that given a geodetic set S of a median graph, every periphery contains a vertex from S; this yields the concept of the *periphery transversal number* as the smallest number of vertices that meet all peripheral subgraphs. Moreover, median graphs with geodetic number 2 and those with periphery transversal number 2 coincide [1]. Hence it is natural to ask whether there is a general connection between the geodetic number of a median graph and the structure that is derived from intersecting peripheral subgraphs. For this purpose we introduce the *periphery graph* $P(G)$ of a median graph G as the graph whose vertices are peripheral subgraphs in G and two vertices are adjacent in $P(G)$ if and only if the peripheral subgraphs intersect.

Several intersection concepts on median graphs and partial cubes have been studied so far [2, 8, 16, 18, 22]. It turns out that the periphery graph is closely related to the crossing graph that was introduced by Bandelt and Chepoi [2] and independently by Klavžar and Mulder [18]. In fact, as we show in Section 3, unless a median graph G is a prism, the periphery graph of G is an induced subgraph of the crossing graph of G. In this paper we answer the following questions: *which graphs can be realized as the periphery graph of some median graph*, *what are the periphery graphs of Cartesian products*, and *how is the periphery graph related to the peripheral contraction and expansion*. We also consider its relation with geodetic number, and show that there are median graphs whose periphery graph has independence number 2, and have arbitrarily large geodetic number.

The paper is organized as follows. In the next section we fix the notation, and state some basic observations. In Section 3 we establish a useful connection between the crossing graph and the periphery graph of a median graph. Then in Section 4 we show that the periphery graph of a median graph G is the join of two graphs S and T if and only if $G = H \Box K$ and $P(H) = S$, $P(K) = T$. In Section 5 it is described how the peripheral expansion affects the periphery graph of a median graph.

2. Notation and Basic Observations

All graphs considered in this paper are undirected, simple and finite. The *distance* d (or d_G when G is not clear from the context) is the usual shortest path distance. A shortest path between vertices u and v will be called a *u,v-geodesic*. The set of vertices on all u,v-geodesics is called the *interval* between u and v, denoted $I(u,v)$. A subset S of vertices in a graph G is *convex* in G if $I(u,v) \subseteq S$ for any $u,v \in S$. It is well-known that convex sets in median graphs enjoy the *Helly property*, that is, any family of pairwise non-disjoint convex sets has a common intersection.

The Cartesian product $G \Box H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ where vertices (g,h) and (g',h') are adjacent if $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. It is well-known that for connected graphs G and H, $d_{G \Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$. Cartesian products of the form $H \Box K_2$ are called *prisms*.
For a connected graph and an edge xy of G we denote

$$W_{xy} = \{w \in V(G) \mid d(x,w) < d(y,w)\}.$$

Note that if G is a bipartite graph then $V(G) = W_{ab} \cup W_{ba}$ holds for any edge ab. Next, for an edge xy of G let U_{xy} denote the set of vertices u that are in W_{xy} and have a neighbor in W_{yx}. Sets in a graph that are U_{xy} for some edge xy will be called *U-sets*. Similarly we define *W-sets*. To simplify the notation, W_{xy} and U_{xy} will also denote the subgraph induced by these sets. It should be clear from the context what is meant in each case.

Note that U-sets and W-sets are convex in median graphs. Moreover, a bipartite graph G is a *median graph* if and only if all its U-sets are convex [3]. If for some edge xy in a median graph, $W_{xy} = U_{xy}$, we call the set (the subgraph) U_{xy} *peripheral set* (*subgraph*) or *periphery*.

Edges $e = xy$ and $f = uv$ of a graph G are in the Djoković-Winkler relation Θ [12, 23] if $d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)$. Relation Θ is reflexive and symmetric. If G is bipartite, then Θ can be defined as follows: $e = xy$ and $f = uv$ are in relation Θ if $d(x,u) = d(y,v)$ and $d(x,v) = d(y,u)$. It is well-known that the relation Θ is an equivalence relation on the edge set of every median graph, and the classes of the corresponding partition will be called Θ-classes. For a Θ-class we will use the notation

$$F_{ab} = \{f | f \in E(G), e\Theta f\}$$

where e has endvertices a and b. We say that a Θ-class F_{ab} is *peripheral* (or that Θ-class induces a periphery) if $W_{ab} = U_{ab}$ or $W_{ba} = U_{ba}$.

Lemma 1 [19]**.** *Let G be a median graph. Then for any edge ab of G the set W_{ab} contains at least one periphery.*

Let G be a median graph. The *intersection graph of peripheries* (or *periphery graph* for short) $P(G)$ is the graph whose vertices are peripheries in G and two vertices in $P(G)$ are adjacent if the corresponding peripheries share a common vertex.

Given an arbitrary graph G, a set S of its vertices is called *geodetic set* of G if for every vertex $x \in V(G)$ there exist $u, v \in S$ such that $x \in I(u,v)$. The *geodetic number* $g(G)$ of a graph G is the least size of a geodetic set in G.

Let G be a median graph. We say that a set S is a *periphery transversal* if every peripheral subgraph of G contains a vertex of S. We denote by $pt(G)$ the size of a minimum periphery transversal in a median graph G. Since every geodetic set is periphery transversal [10],

$$pt(G) \leq g(G).$$

Because every set of pairwise disjoint peripheries in G corresponds to an independent set in $P(G)$, we derive that

$$\alpha(P(G)) \leq pt(G)$$

where $\alpha(G)$ is the independence number of G.

A *clique-cover* of size k of a graph G is a partition of the vertex set $V(G)$ into $C_1, C_2, \ldots, C_k$, such that each C_i, $1 \leq i \leq k$, induces a complete subgraph. The *clique-covering number* $\kappa(G)$ is the cardinality of a minimum clique-cover in G.

Proposition 2. *Let G be a median graph. Then $pt(G) = \kappa(P(G))$.*

Proof. Let P be a periphery transversal of G and $k = pt(G)$. Then, for $x_i \in P$ let $P_i = \{U_1, U_2, \ldots, U_{k_i}\}$ be the set of peripheral sets in G that contain x_i. In $P(G)$ vertices of P_i form a clique (since they all intersect in x_i). Therefore $P_1, P_2, \ldots, P_k$ yields a clique cover of $P(G)$. We infer $pt(G) \geq \kappa(P(G))$.

Let $C_1, C_2, \ldots, C_l$ be a clique cover of $P(G)$. Each C_i consists of vertices that correspond to peripheries in G that pairwise intersect. These are convex sets, thus their common intersection is nonempty by the Helly property. Hence $pt(G) \leq \kappa(P(G))$. ■

A median graph is called *path-like* if and only if $\alpha(P(G)) = 2$ (that is, no three peripheries are pairwise disjoint in G). Clearly path-like trees are precisely paths. Using Lemma 1, we derive a simple characterization of path-like median graphs.

Proposition 3. *A median graph is path-like if and only if no three W-sets in G are pairwise disjoint.*

Proof. Suppose G is a path-like median graph. Then no three peripheral sets are pairwise disjoint. Suppose to the contrary that there are three W-sets that are pairwise disjoint. Then by Lemma 1 there exist three peripheries in these three W-sets, respectively, thus these three peripheries are also pairwise disjoint, a contradiction. The converse is clear since every peripheral set is a W-set. ■

Using the above observations and propositions, we have the following straightforward

Remark 4. Every median graph with geodetic number 2 is path-like.

3. Periphery Graph and Crossing Graph of a Median Graph

For a median graph G the *crossing graph* $G^{\#}$ is the graph whose vertices are the Θ-classes of G and two vertices are adjacent if there exists a C_4 that contains edges of both Θ-classes (we say that these two Θ-classes *cross*). The concept of the crossing graph was introduced by Bandelt and Chepoi under the name *incompatibility graph* [2], and independently by Klavžar and Mulder who considered a more general case of partial cubes [18], see also [9] for a different generalization of the crossing graph concept.

Let G be a median graph and $ab \in E(G)$ such that $P = U_{ab} = W_{ab}$ is a periphery and $U_{ba} \neq W_{ba}$. Then P corresponds to a vertex in $P(G)$, and the Θ-class $F_{ab} = \{xy : x \in U_{ab}, y \in U_{ba}\}$ corresponds to a vertex in $G^{\#}$. Let uv be another edge such that $P' = U_{uv} = W_{uv}$. If $P \cap P' \neq \emptyset$ then it is clear that F_{ab} and F_{uv} cross on some C_4. Conversely, if F_{ab} and F_{uv} are adjacent in $G^{\#}$ then also P and P' intersect and there is an edge between them in $P(G)$. By identifying the peripheries $U_{ab} = W_{ab}$ for which $U_{ba} \neq W_{ba}$ with their corresponding Θ-classes F_{ab}, we derive that the subgraph of $P(G)$ induced by these peripheries is an induced subgraph of $G^{\#}$. Consequently, if in G there are no peripheries such that both $U_{ab} = W_{ab}$ and $U_{ba} = W_{ba}$ hold then $P(G)$ is an induced subgraph of $G^{\#}$. Note that this happens precisely when G cannot be realized as the prism $H \Box K_2$ for some subgraph H (that corresponds to some $W_{ab} = U_{ab}$). We extract these observations in the following proposition.

Proposition 5. *Let G be a median graph such that G is not isomorphic to $H \Box K_2$ for some (median) graph H. Then the periphery graph $P(G)$ is an induced subgraph of the crossing graph $G^{\#}$. If in addition all Θ-classes are peripheral, then $P(G) = G^{\#}$.*

Consider for instance the bipartite wheel BW_n that consists of the vertex u adjacent to n vertices $x_1, \ldots, x_n$, and there are another n vertices $y_1, \ldots, y_n$, where y_i is adjacent to x_i and x_{i+1} (modulo n) for $i = 1, \ldots, n$. There are n peripheries of BW_n that are isomorphic to the paths P_3, and there are exactly n Θ-classes that can be written as F_{ux_i}. Hence by Proposition 5 we infer that $P(BW_n) = BW_n^{\#}$ which is isomorphic to C_n, as observed also in [18]. As another example consider hypercubes $G = Q_n$. For every Θ-class

F_{ab} in a hypercube G, the graph obtained from G by deletion of edges from F_{ab} consists of two hypercubes, i.e., $W_{ab} = U_{ab}$ and $W_{ba} = U_{ba}$. Thus $G^{\#} = K_n$ and $P(G)$ is obtained from K_n by deleting edges of a perfect matching. Finally, there can be many Θ-classes in a median graph that do not induce any periphery. For instance in the square grid $P_n \Box P_m$, we only have 4 peripheries, but $n+m-2$ Θ-classes. Note that $(P_n \Box P_m)^{\#} = K_{n-1,m-1}$ and $P(P_n \Box P_m) = C_4$ being an induced subgraph of $(P_n \Box P_m)^{\#}$, as follows from Proposition 5.

Klavžar and Mulder proved that any graph can be realized as the crossing graph of some median graph [18]. The natural question appears: which graphs can be realized as the periphery graph of a median graph.

The vertex u in a graph G is called *universal* if it is adjacent to all other vertices in G.

Theorem 6. *Let H be a graph. Then there exists a median graph G such that $P(G) = H$ if and only if H has no universal vertices.*

Proof. First suppose that H has a universal vertex x, and assume that there is a median graph G such that $P(G) = H$. Then x corresponds to some periphery $W_{ab} = U_{ab}$. By Lemma 1 there is a periphery contained in W_{ba}, and let y be its corresponding vertex in $P(G)$. Clearly, x and y are not adjacent since W_{ab} and W_{ba} are disjoint, a contradiction. This direction is proved.

For the converse let H be a graph without universal vertices. We will use the construction of the simplex graph, introduced by Bandelt and van de Vel [4]. Given an arbitrary graph G the simplex graph $\sigma(G)$ of G has simplices of G as vertices (where simplices are sets of vertices that induce a complete subgraph of G, including the empty set), and two simplices are adjacent in $\sigma(G)$ if and only if they are comparable and differ in one vertex. It is well-known that the simplex graph of any graph is a median graph, and its Θ-classes are induced by edges between the vertex $\emptyset$ (of degree $|V(G)|$) and one-element simplices, each corresponding to a vertex of G. Note that two Θ-classes in $\sigma(G)$ cross if and only if the corresponding vertices are adjacent in G, hence $(\sigma(G))^{\#} = G$, see [18]. We claim that for a vertex $u \in V(G)$, the subset $W_{\{u\}\emptyset}$ in $\sigma(G)$ (of vertices closer to the simplex $\{u\}$ than to the simplex $\emptyset$) is peripheral. Indeed, if $S \in W_{\{u\}\emptyset}$ we infer that $u \in S$, and S is adjacent to $S \setminus \{u\}$ in $\sigma(G)$, and $S \setminus \{u\}$ is clearly in $W_{\emptyset\{u\}}$. Hence every Θ-class in $\sigma(G)$ induces a periphery.

Now, since H has no universal vertex we claim that $\sigma(H)$ is not a prism (Cartesian product with K_2). For, if $\sigma(H)$ would be a prism, $\sigma(H) = M \Box K_2$, then there would exist a Θ-class $F_{\{u\}\emptyset}$ that would cross with all other Θ-classes. Hence for any $x \in V(H)$ there would exist a simplex $\{u, x\} \in \sigma(H)$ which implies that u and x are adjacent in H. This in turn implies that u is a universal vertex, a contradiction with the assumption on H. Thus $\sigma(H)$ is not a prism, and so by Proposition 5, $P(\sigma(H)) = (\sigma(H))^{\#}$. As noted above $(\sigma(H))^{\#} = H$, and so $P(G) = H$ for $G = \sigma(H)$. The proof is complete. ■

By the well-known Mycielski construction [20] there are triangle-free graphs (graphs G with $\omega(G) = 2$), with arbitrarily large chromatic number. By duality of these notions, namely, $\alpha(\overline{G}) = \omega(G)$ and $\kappa(\overline{G}) = \chi(G)$, we infer that there are graphs with independence number equal to 2 that have arbitrarily large clique-covering number. Since Mycielski construction gives connected graphs (thus with no isolated vertices), their complements have no universal vertices. Such graphs can thus be realized as periphery graphs $P(G)$ of median graphs G, which are path-like median graphs. This fact was observed implicitly already in [2].

Corollary 7. *There are path-like median graphs with arbitrarily large periphery transversal number, and (by consequence) arbitrarily large geodetic number.*

4. Periphery Graphs of Cartesian Products

Let $A \oplus B$ denote the *join* of graphs A and B, i.e., the graph obtained from the disjoint union of A and B by joining every vertex of A with every vertex of B by an edge.

Brešar and Klavžar proved the following theorem for crossing graphs of median graphs.

Theorem 8 [7]**.** *Let G be a median graph. Then $G^{\#} = A \oplus B$ if and only if $G = H \Box K$, where $H^{\#} = A$ and $K^{\#} = B$.*

We prove in this section an analogous result for the periphery graph. We start with the easier direction.

Proposition 9. *Let H and K be median graphs. Then $P(H \Box K) = P(H) \oplus P(K)$.*

Proof. Using the distance formula in the Cartesian product $G = H\Box K$ we easily derive that the W-sets in G are precisely of the form $W_{x_1y_1}\Box K$ or $H\Box W_{x_2y_2}$, where x_1y_1 is an edge in H and x_2y_2 is an edge in K. Obviously, $W_{x_1y_1}\Box K$ (resp. $H\Box W_{x_2y_2}$) is a periphery in G if and only if $W_{x_1y_1}$ is a periphery in H (resp. $W_{x_2y_2}$ is a periphery in K). Thus $V(P(G)) = V(P(H)) \cup V(P(K))$.

Two peripheries of the form $W_{x_1y_1}\Box K$ ($H\Box W_{x_2y_2}$, respectively) in G intersect if and only if the corresponding peripheries in H (resp. K) intersect. Moreover, any periphery of the form $W_{x_1y_1}\Box K$ intersects with any periphery of the form $H\Box W_{x_2y_2}$, since $W_{x_1y_1}\Box W_{x_2y_2}$ is a subset of both $W_{x_1y_1}\Box K$ and $H\Box W_{x_2y_2}$. Thus in $P(G)$ any vertex corresponding to a vertex in $V(P(H))$ is adjacent to all vertices in $V(P(K))$. Hence $P(G) = P(H) \oplus P(K)$. ∎

From the proof of Proposition 9 we derive the following

Corollary 10. *A Θ-class in the Cartesian product of graphs is peripheral if and only if the corresponding Θ-class is peripheral in a factor.*

Let H be a subgraph of G. Then ∂H is the set of all edges xy of G with $x \in H$ and $y \notin H$. Imrich and Klavžar [14] proved that an induced connected subgraph H of a bipartite graph G is convex if and only if no edge of ∂H is in relation Θ with any edge of H. This result is known under the name *Convexity Lemma.*

Lemma 11. *Let G be a median graph. Then G is a prism if and only if G contains a Θ-class that crosses with every peripheral Θ-class.*

Proof. If $G = H\Box K_2$ then Θ-class that corresponds to K_2 obviously crosses with every other Θ-class of G.

Suppose that F_{ab} is a Θ-class that crosses with every peripheral Θ-class of G. We claim that $W_{ab} = U_{ab}$ and $W_{ba} = U_{ba}$. Suppose to the contrary and without loss of generality that there is a vertex $u \in W_{ab} \setminus U_{ab}$ and it is adjacent to a vertex $v \in U_{ab}$. By Convexity Lemma, uv is not in relation Θ to any of the edges of the subgraph induced by U_{ab}. Hence the set of edges F_{uv} is completely contained in the subgraph induced by W_{ab} and $W_{uv} \subset W_{ab}$.

Moreover, we claim that $W_{uv} \subset W_{ab} \setminus U_{ab}$. Suppose to the contrary that there is a vertex $z \in W_{uv} \cap U_{ab}$. Since z is closer to u than to v and G is bipartite, $d(u, z) = k - 1$ if $k = d(v, z)$. But this contradicts the fact that U_{ab} is convex. By Lemma 1, W_{uv} contains at least one periphery and the Θ-class that induces it cannot cross F_{ab} since $W_{uv} \subset W_{ab} \setminus U_{ab}$, a final contradiction. ■

Lemma 12. *Let G be a median graph, $P(G) = T \oplus S$ and $S_i \in S$ and $T_i \in T$ arbitrary peripheral subgraphs. Then for every Θ-class E there exists $e \in E$ that is contained in S_i or T_i.*

Proof. Let $S_i \in S$ and $T_i \in T$ be arbitrary peripheral subgraphs. Since $P(G) = T \oplus S$, S_i and T_i intersect in G. Suppose to the contrary that there is a Θ-class E with no edge in S_i nor in T_i. Let $e = ab \in E$. Note that S_i and T_i lie in the same W-set with respect to ab, say $S_i, T_i \in W_{ab}$. By Lemma 1, W_{ba} contains a peripheral subgraph that is disjoint with both S_i and T_i, a contradiction since $P(G)$ is a join. ■

Theorem 13. *Let G be a median graph. Then $P(G) = T \oplus S$ if and only if $G = H \Box K$ and $P(H) = S$, $P(K) = T$.*

Proof. By Proposition 9 one direction is proved: the periphery graph of the Cartesian product of median graphs is the join of the periphery graphs of the factors. To prove the converse we distinguish three cases.

First, suppose that G is a prism, $G = H \Box K_2$. Then $P(G) = P(H) \oplus P(K_2)$ by Proposition 9. Let $P(K_2) = \{A, B\}$. Note that in any representation of $P(G)$ as the join, say $P(G) = S \oplus T$, peripheries A and B have to be in the same partition, say S, since they are disjoint. If $S \oplus T$ is a different join representation of $P(G)$ as $P(H) \oplus P(K_2)$, there must be a vertex C in S. Hence $P(H) = S \setminus \{A, B\} \oplus T$ is also the join. By using induction (which is applicable since H is smaller than G), $H = H' \Box H''$ where $P(H') = S \setminus \{A, B\}$ and $P(H'') = T$. Thus $G = H' \Box K_2 \Box H''$ and $P(H' \Box K_2) = S$, $P(H'') = T$.

If G is not a prism and every Θ-class is peripheral, then by Proposition 5, $G^{\#} = P(G) = S \oplus T$. Hence by Theorem 8, $G = H \Box K$, where $H^{\#} = S$ and $K^{\#} = T$, and by Corollary 10, $P(H) = S$ and $P(K) = T$.

Finally, assume that $P(G) = S \oplus T$, G is not a prism and G contains non-peripheral Θ-classes. The first and main step in the proof is to show that then also $G^{\#}$ is a join.

By T_i let us denote the peripheral subgraph that is induced by a peripheral Θ-class T_i' (recall that if G is not a prism, every peripheral Θ-class induces exactly one peripheral subgraph). Also we denote by S' (T', resp.)

the set of Θ-class that induce peripheries from S (T resp.). Let P denote the set of non-peripheral Θ-classes that do not cross with some Θ-class from S' and let R denote the set of non-peripheral Θ-classes that do not cross with some Θ-class from T'. By Lemma 11, every non-peripheral Θ-class is either in P or in R, since P and R are disjoint (indeed, if there is a Θ-class E in $P \cap R$, then E does not cross with some $S'_i \in S'$ and E does not cross with some $T'_i \in T'$, a contradiction with Lemma 12). Now let F be an arbitrary Θ-class in P. As such it does not cross with some Θ-class in S'. Since S_i and T_i in Lemma 12 were arbitrarily chosen, there is an edge from F in every peripheral subgraph $T_i \in T$, hence F crosses with every Θ-class T'_i. Similarly, any Θ-class in R crosses with every Θ-class S'_i. To prove that $G^\# = (P \cup S') \oplus (R \cup T')$ we now only need to show that every vertex in P is adjacent to every vertex in R.

Suppose to the contrary that there are $P'_i \in P$ and $R'_i \in R$ that do not cross and choose $S'_i \in S'$ and $T'_i \in T'$ such that $P'_iS'_i \notin E(G^\#)$ and $R'_iT'_i \notin E(G^\#)$. Since P'_i and T'_i cross, there is an edge $ab \in P'_i$ that lies in the peripheral subgraph T_i. Similarly, there is an edge $cd \in R'_i$ in S_i. Since P'_i and R'_i do not cross, all edges from R'_i lie in the same W-set with respect to ab, say in W_{ab}, and hence also the whole S_i is contained in W_{ab} (since S'_i and P'_i do not cross). We may without loss of generality assume that $V(T_i) \subset W_{dc}$, since $R'_iT'_i \notin E(G^\#)$. Note that $W_{cd} \cap W_{ba} = \emptyset$ (otherwise P'_i and R'_i would cross). By Lemma 1, W_{cd} contains a peripheral subgraph A. Since $A \cap T_i = \emptyset$ and $P(G) = S \oplus T$, we derive that $A = T_j \in T$, $j \neq i$. Since P'_i is a non-peripheral Θ-class, $W_{ba} \neq U_{ba}$, and W_{ba} contains a peripheral subgraph B. From $B \cap S_i = \emptyset$ we derive that B intersects with every $T_i \in T$, hence $B = S_j \in S$, $j \neq i$. But now we have two peripheral subgraphs $A = T_j$ and $B = S_j$ that do not intersect, a contradiction to $P(G) = S \oplus T$. Hence $G^\# = (P \cup S') \oplus (R \cup T')$.

Now, by Theorem 8, $G = H \Box K$ where $H^\# = P \cup S'$ and $K^\# = R \cup T'$. By Corollary 10, S (T, respectively) corresponds to the set of peripheral Θ-classes of H (K, respectively), thus $P(H) = S$ and $P(K) = R$, which completes the proof. ∎

Corollary 14. *The Cartesian product $G \Box H$ of two median graphs is path-like if and only if both G and H are path-like.*

It is clear from Proposition 9 that a clique in $P(G)$ with a clique in $P(H)$ forms a clique in $P(G \Box H)$. Thus the minimum number of cliques needed to cover the vertices of $P(G\Box H)$ will be the maximum of the clique-cover numbers of $P(G)$ and $P(H)$.

Proposition 15. *For median graphs G and H, $\kappa(P(G\Box H)) = \max\{\kappa(P(G)), \kappa(P(H))\}$.*

Combined with Proposition 2 we obtain also the following result.

Corollary 16. *Let G and H be median graphs. Then $pt(G\Box H) = \max\{pt(G), pt(H)\}$.*

As an application of Theorem 13 we can characterize graphs G for which $P(G)$ is the complete bipartite graph $K_{m,n}$. Since $K_{m,n}$ is of the form $P(H) \oplus P(K)$, peripheries in H and K are pairwise disjoint and converse is also true. We state this observation as a remark.

Remark 17. The periphery graph $P(G)$ is the complete bipartite graph if and only if $G = H\Box K$ where H and K are median graphs having no pairwise intersecting peripheries.

As another application of Theorem 13 we note that $P(G) = C_4$ implies that $G = H\Box K$, where H and K are median graphs having exactly two peripheries. By Lemma 1 it is easy to derive that median graphs with only two peripheries are just paths. Hence $P(G) = C_4$ if and only if G is the grid graph, $G = P_n\Box P_m$ for arbitrary $n, m \geq 2$.

In view of Theorem 6 and Theorem 13 it would be interesting to characterize some natural classes of graphs as periphery graphs of some median graphs. For instance, for which median graphs, their periphery graph is connected (2-connected, tree, chordal, ...). In particular, is there a nice structural characterization of path-like median graphs?

5. Peripheral Expansion

In the main result of this section we describe, what happens with the periphery graph of a median graph after we perform one step of peripheral expansion (the construction that we mentioned in the introduction, and is defined formally below).

Let H be a connected graph and P its *convex subgraph*, meaning the subgraph, induced by a convex subset $V(P)$ of $V(H)$. Then the *peripheral* *expansion* of H along P is the graph G obtained as follows. Take the disjoint union of a copy of H and a copy of P. Join each vertex u in the copy of P with the vertex that corresponds to u in the copy of H (actually in the subgraph P of H). We say that the resulting graph G is *obtained by a (peripheral) expansion from H along* P, and denote this operation in symbols by $G = pe(H, P)$. We also say that we *expand* P in H to obtain G. We will denote by H also the subgraph of G that corresponds (and is isomorphic to) H, and by P' the subgraph of G induced by $V(G) \setminus V(H)$.

As we noted in the introduction, a graph G is a median graph if and only if it can be obtained from K_1 by a sequence of peripheral expansions, a result due to Mulder [19].

Lemma 18. *Let $G = pe(H, P)$ and P' the subgraph of G induced by $V(G) \setminus V(H)$. Let S be a peripheral subgraph of H such that $P \not\subseteq S$ and $X = P \cap S \neq \emptyset$. Let X' be the subgraph of G induced by vertices in P' that are adjacent to a vertex from X. Then the subgraph $S' = S \cup X'$ is a peripheral subgraph in G.*

Proof. We will use the notation W_{ab}^H for a W-set in H and W_{ab}^G for the corresponding W-set in G. Since S is a peripheral subgraph of H there is an edge $ab \in E(H)$ such that $V(S) = W_{ab}^H = U_{ab}^H$. We claim that $V(S') = W_{ab}^G = U_{ab}^G$.

Let a' be an arbitrary vertex in X. Since $X \subset S$, there is a vertex $b' \in U_{ba}^H$, adjacent to a'. We note that $b' \in P$ for otherwise we obtain a contradiction to convexity of P (since b' lies on a shortest path from a' to any vertex of $P \cap (H \setminus S)$).

Let c' be a vertex in $V(X')$. Then the unique neighbor c of c' that lies in X, lies also in U_{ab}^G; c has a unique neighbor b_1 in U_{ab}^G which lies in P by the previous paragraph. Hence b_1' which is the neighbor of b_1 in P' is also the neighbor of c'. This implies that $c' \in U_{ab}^G$. Since $U_{ab}^H \subset U_{ab}^G$ we derive that $V(S') \subseteq U_{ab}^G$. We claim that $V(S')$ is equal to W_{ab}^G. First note that $V(H \setminus S)$ is a subset of W_{ba}^H and thus also of W_{ba}^G. Now, any $d' \in P' \setminus X'$ has a unique neighbor d in $P \setminus X$, and thus $d \in W_{ba}^G$. The complement of $(H \setminus S) \cup (P' \setminus X')$ in G is precisely S', thus $V(S') = W_{ab}^G$. Since $V(S') \subseteq U_{ab}^G$, we derive $W_{ab}^G = U_{ab}^G$ and so S' is a peripheral subgraph. ■

Proposition 19. *Let $G = pe(H, P)$ and P' the (peripheral) subgraph of G induced by $V(G) \setminus V(H)$. Then one of the following cases appear:*

(i) *If $P = H$, then $P(G) = P(H) \oplus P(K_2)$.*

(ii) *If P is equal to some periphery, then $P(G) = P(H)$.*

(iii) *Let P be a proper subgraph of some periphery in H and let $S_1, S_2, \ldots, S_k$ be peripheries in H such that $P \subset S_i$, $i \in \{1, \ldots, k\}$. Then $V(P(G)) = V(P(H)) \setminus \{S_1, \ldots, S_k\} \cup \{P'\}$, subgraph of $P(H)$, induced by vertices $V(P(H)) \setminus \{S_1, \ldots, S_k\}$, is an induced subgraph of $P(G)$, and for $T \in V(P(H)) \setminus \{S_1, \ldots, S_k\}$, $P'T \in E(P(G))$ if and only if P and T intersect in H.*

(iv) *If P is not a subgraph of any periphery (and is not equal to H), then $P(G)$ is obtained from $P(H)$ by adding a vertex that is adjacent to all those peripheries in $P(H)$ that have a nonempty intersection with P.*

Proof. Let $G = pe(H, P)$ and P' the subgraph of G induced by $V(G) \setminus V(H)$.

(i) If $P = H$, then $G = H \Box K_2$ and $P(G) = P(H) \oplus P(K_2)$ by Proposition 9.

(ii) Suppose that P is a peripheral subgraph in H. Peripheral subgraphs in H that have an empty intersection with P obviously stay peripheral also in G and they intersect in G if and only if they intersect in H. The subgraph P is clearly no longer peripheral in G but P' is (hence in $P(G)$ the vertex that corresponds to P in $P(H)$ is replaced by the vertex corresponding to P'). By Lemma 18, every peripheral subgraph S in H that has a nonempty intersection with P corresponds to a peripheral subgraph S' in G that has a nonempty intersection with P'. Clearly P' does not intersect with peripheries that have an empty intersection with P. Hence $P(G) = P(H)$.

(iii) It is clear that a peripheral subgraph S_i of H that contains P as a proper subgraph is not peripheral in G and that P' is a peripheral subgraph of G. Peripheral subgraphs in H that are disjoint with P clearly stay peripheral also in G, and peripheries in H that intersect with P (but do not contain P as a proper subgraph) only become larger in G after peripheral expansion by Lemma 18. Since we do not obtain any other new peripheral subgraph in G except P' after peripheral expansion, we derive $V(P(G)) = V(P(H)) \setminus \{S_1, \ldots, S_k\} \cup \{P'\}$ where $S_1, \ldots, S_k$ are peripheries in H that properly contain P. If two peripheries from $V(P(H)) \setminus \{S_1, \ldots, S_k\}$ are adjacent in $P(H)$ they are clearly also adjacent in $P(G)$, and any $T \in V(P(H)) \setminus \{S_1, \ldots, S_k\}$ is adjacent to P' in $P(G)$ if the corresponding periphery has a nonempty intersection with P in H by Lemma 18 (we may use it since P is not contained in T). If $T \cap P = \emptyset$ in H, then obviously $T \cap P' = \emptyset$ in G.

(iv) The result of the last case follows immediately by using Lemma 18. ■

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