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COLOURING VERTICES OF PLANE GRAPHS UNDER RESTRICTIONS GIVEN BY FACES

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Abstract

We consider a vertex colouring of a connected plane graph G. A colour c is used k times by a face α of G if it appears k times along the facial walk of α. We prove that every connected plane graph with minimum face degree at least 3 has a vertex colouring with four colours such that every face uses some colour an odd number of times. We conjecture that such a colouring can be done using three colours. We prove that this conjecture is true for 2-connected cubic plane graphs. Next we consider other three kinds of colourings that require stronger restrictions.

Keywords: vertex colouring, plane graph, weak parity vertex colouring, strong parity vertex colouring, proper colouring, Lebesgue theorem.

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1. Introduction

We use the standard terminology according to [4] except for few notations defined throughout. Graphs considered can have multiple edges but loops are not allowed. We will adapt the convention that a graph is *planar* if it can be embedded in the plane without edges crossing, and *plane* if it is already embedded in the plane.

If a planar graph G is embedded in the plane $\mathcal{M}$, then the maximal connected regions of $\mathcal{M} - G$ are called the *faces*. The *facial walk* of a face α

of a connected plane graph G is the shortest closed walk traversing all edges incident with α. The *degree of a face* α is the length of its facial walk. Let a *k-face* be a face of degree k.

The *degree of a vertex* v of a connected plane graph G is the number of edges incident with v. Let a *k-vertex* be a vertex of degree k.

Let the set of vertices, edges, and faces of a connected plane graph G be denoted by $V(G)$, $E(G)$ and $F(G)$, respectively, or by V, E and F if G is known from the context.

Let φ be a vertex colouring of a connected plane graph G. We say that a face α of G uses a colour c under the colouring φ k times if this colour appears k times along the facial walk of α. (The first and the last vertex of the facial walk is counted as one appearance only.)

Colouring vertices of plane graphs under restrictions given by faces has recently attracted much attention, see e.g. [1, 5, 6, 7, 8, 9, 11, 14, 15, 16] and references there. Two natural problems of this kind are the following.

Problem 1. A vertex colouring φ is a *weak parity vertex colouring* of a connected plane graph G if each face of G uses at least one colour an odd number of times. The problem is to determine the minimum number $\chi_w(G)$ of colours used in a weak parity vertex colouring of G. Then the number $\chi_w(G)$ is called the *weak parity chromatic number*.

Problem 2. A vertex colouring φ is a *strong parity vertex colouring* of a 2-connected plane graph G if for each face α and each colour c the face α uses the colour c an odd number of times or does not use it at all. The problem is to find the minimum number $\chi_s(G)$ of colours used in a strong parity vertex colouring of G. Then the number $\chi_s(G)$ is called the *strong parity chromatic number*.

Our research has been motivated by a paper [3] which deals with parity edge colourings and strong parity edge colourings in graphs. Recall that a parity edge colouring is such a colouring in which each path uses some colour an odd number of times. A vertex variation of this problem, a parity vertex colouring with respect to paths of general graphs is introduced in the paper [2].

The rest of the paper is organized as follows. In Section 2 we provide a new proof of a theorem of Lebesgue [10]. This theorem is applied later in Section 3. In this Section we study weak parity vertex colouring. We prove that $\chi_w(G) \leq 4$ for every connected plane graph with minimum face degree at least 3. In Section 4 we conjecture that $\chi_w(G) \leq 3$ for all plane graphs of minimum face degree at least 3 and prove this conjecture for 2-connected cubic plane graphs. Section 5 of this paper is devoted to study the strong parity vertex colouring. Section 6 deals with the cyclic chromatic number which is a natural bound on strong parity chromatic number. In Section 7 we discuss two similar problems and formulate open questions.

2. Lebesgue Theorem

In this section we state one of the basic results concerning the structure of plane graphs, the Lebesgue theorem [10] proved in 1940. For the sake of completeness we give here a new simple proof of this theorem. To be able to state it we need two new notations.

A k-vertex v, $k \geq 3$, is said to be the $(a_1, a_2, \ldots, a_k)$-*vertex* if faces $\alpha_1, \alpha_2, \ldots, \alpha_k$, in order incident with v have degrees $a_1, a_2, \ldots, a_k$.

A connected plane graph with minimum vertex degree at least 3 and minimum face degree at least 3 is called *normal map*.

Theorem 2.1 (Lebesgue). *Every normal map contains at least one of the following vertices:*

1. *an (a, b, c)-vertex for*

$a = 3$ *and* $3 \leq b \leq 6$ *and* $c \geq 3$, *or*	$a = 4$ *and* $b = 4$ *and* $c \geq 4$, *or*
$a = 3$ *and* $b = 7$ *and* $7 \leq c \leq 41$, *or*	$a = 4$ *and* $b = 5$ *and* $5 \leq c \leq 19$, *or*
$a = 3$ *and* $b = 8$ *and* $8 \leq c \leq 23$, *or*	$a = 4$ *and* $b = 6$ *and* $6 \leq c \leq 11$, *or*
$a = 3$ *and* $b = 9$ *and* $9 \leq c \leq 17$, *or*	$a = 4$ *and* $b = 7$ *and* $7 \leq c \leq 9$, *or*
$a = 3$ *and* $b = 10$ *and* $10 \leq c \leq 14$, *or*	$a = 5$ *and* $b = 5$ *and* $5 \leq c \leq 9$, *or*
$a = 3$ *and* $b = 11$ *and* $11 \leq c \leq 13$, *or*	$a = 5$ *and* $b = 6$ *and* $6 \leq c \leq 7$, *or*

2. *a $(3, b, c, d)$-vertex for*

$b = 3$ *and* $c = 3$ *and* $d \geq 3$, *or*	$b = 4$ *and* $c = 4$ *and* $4 \leq d \leq 5$, *or*
$b = 3$ *and* $c = 4$ *and* $4 \leq d \leq 11$, *or*	$b = 4$ *and* $c = 5$ *and* $d = 4$, *or*
$b = 3$ *and* $c = 5$ *and* $5 \leq d \leq 7$, *or*	$b = 5$ *and* $c = 3$ *and* $5 \leq d \leq 7$, *or*
$b = 4$ *and* $c = 3$ *and* $4 \leq d \leq 11$, *or*	

3. *a $(3, 3, 3, 3, e)$-vertex for*
   $3 \leq e \leq 5$.

Proof. We proceed by contradiction. Suppose there is a normal map $G = (V, E, F)$ on a set V of n vertices which contains none of the vertices mentioned above. Let e be the number of edges and let f be a number of faces of G. From the Euler polyhedral formula $n - e + f = 2$ we can easily derive

$$\sum_{\alpha \in F} (2deg(\alpha) - 6) + \sum_{v \in V} (deg(v) - 6) = -12.$$

Consider an initial charge function $\varphi : V \cup F \to \mathbb{Q}$ such that $\varphi(\alpha) = 2deg(\alpha) - 6$ for $\alpha \in F$ and $\varphi(v) = deg(v) - 6$ for $v \in V$. Initially,

$$\sum_{\alpha \in F} \varphi(\alpha) + \sum_{v \in V} \varphi(v) = -12.$$

We use the following rule to transform φ into a new charge function $\psi : V \cup F \to \mathbb{Q}$ by redistributing charges locally so that $\sum_{\alpha \in F} \varphi(\alpha) + \sum_{v \in V} \varphi(v) = \sum_{\alpha \in F} \psi(\alpha) + \sum_{v \in V} \psi(v)$.

Rule. Each face α transfers the charge $\frac{2deg(\alpha)-6}{deg(\alpha)}$ to each vertex v incident with α. Therefore $\psi(\alpha) = 0$ for any $\alpha \in F$. Hence

$$\sum_{v \in V} \psi(v) = -12. \tag{1}$$

We are going to show that $\psi(v) \geq 0$ for every $v \in V$ which will trivially be a contradiction with (1). Let $v \in V$ be an $(a_1, a_2, \ldots, a_k)$-vertex. Then

$$\psi(v) = \varphi(v) + \sum_{\substack{\alpha \in F \\ v \in \alpha}} \frac{2deg(\alpha) - 6}{deg(\alpha)} = k - 6 + \sum_{i=1}^{k} \frac{2a_i - 6}{a_i} = 3k - 6 - 6 \sum_{i=1}^{k} \frac{1}{a_i}. \tag{2}$$

It is easy to see that if $k \geq 6$ then $\psi(v) \geq 0$. Recall that graph G does not contain any vertex from the list.

If $k = 3$ all possibilities for degrees of faces incident with the vertex v (here $m+$ denotes any integer $\geq m$) are listed in Table 1. It is a routine matter to verify that $\psi(v) = 3 - 6\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \geq 0$ for every triple (a_1, a_2, a_3) from the list of Table 1.

In Table 2 and Table 3 there are listed all possibilities for degrees of faces incident with the vertex v in the case if $k = 4$ and $k = 5$, respectively. Using (2) we can easily verify that for any possibility there is $\psi(v) \geq 0$.

Table 1. Degrees of faces incident with a 3-vertex.

a_1	3	3	3	3	3	3	4	4	4	4	5	5	5	6+
a_2	7	8	9	10	11	12+	5	6	7	8+	5	6	7+	6+
a_3	42+	24+	18+	15+	14+	12+	20+	12+	10+	8+	10+	8+	7+	6+

Table 2. Degrees of faces incident with a 4-vertex.

a_1	3	3	3	3	3	3	3	3	3	3	3	4+
a_2	3	4	3	5	3	6+	4	4	4	5+	5+	4+
a_3	4	3	5	3	6+	3	4	6+	5+	4	5+	4+
a_4	12+	12+	8+	8+	6+	6+	6+	4	5+	5+	5+	4+

Table 3. Degrees of faces incident with a 5-vertex.

a_1	3	3	3	3	3	3	4+
a_2	3	3	3	3	4+	4+	4+
a_3	3	3	4+	4+	3	4+	4+
a_4	3	4+	3	4+	4+	4+	4+
a_5	6+	4+	4+	4+	4+	4+	4+

■

3. Weak Parity Vertex Colourings

Using distinct colours on all vertices of a connected plane graph G produces a weak parity vertex colouring of G. Hence $\chi_w(G)$ is well defined for every connected plane graph G. It is easy to see that $\chi_w(G) = 1$ if and only if all faces of G have odd degree.

Next we determine the parameter $\chi_w(D_r)$ for the graphs of r-sided prisms. The r-sided prism, $r \geq 3$, is a plane graph consisting of two r-faces; an internal r-face $\alpha = (u_1, u_2, \ldots, u_r)$ and an external r-face $\beta = (v_1, v_2, \ldots, v_r)$ and r 4-faces $\alpha_i = (u_i, u_{i+1}, v_{i+1}, v_i)$ for all $i = 1, \ldots, r$; indices modulo r.

Theorem 3.1. *Let D_r be an r-sided prism, $r \geq 3$. Then*

$$\chi_w(D_r) = \begin{cases} 2 & \text{if} \quad r \equiv 0 \pmod 4, \\ 3 & \text{if} \quad r \not\equiv 0 \pmod 4. \end{cases}$$

Proof. It is easy to see that $\chi_w(D_r) \geq 2$. To prove the theorem we distinguish two cases according to $r \equiv k \pmod 4$.

Case 1. If $r \equiv 0 \pmod 4$ then we define a suitable weak parity vertex 2-colouring φ as follows: $\varphi(u_1) = \varphi(v_3) = 2$, $\varphi(v_1) = \varphi(v_2) = \varphi(v_i) = 1$ for all $i \geq 4$; $\varphi(u_3) = 1$, $\varphi(u_j) = 1$ for all $j \geq 2$, $j \equiv 0 \pmod 2$ and $\varphi(u_j) = 2$ for $j \geq 5$, $j \equiv 1 \pmod 2$. It is a routine matter to check that φ fulfills our requirements.

Case 2. Let $r \not\equiv 0 \pmod 4$. Suppose that there is a weak parity vertex 2-colouring φ of D_r. An edge xy is *monochromatic* if $\varphi(x) = \varphi(y)$; otherwise it is *heterochromatic*. Every 4-face α_i uses any colour an odd number of times. Therefore no consecutive edges u_iv_i and $u_{i+1}v_{i+1}$, $i = 1, \ldots, r$ are simultaneously monochromatic or heterochromatic.

If $r \equiv 1 \pmod 2$ we immediately have a contradiction that φ is a weak parity vertex 2-colouring.

Let $r \equiv 2 \pmod 4$. As both α and β have even degree the colour 1 (and also the colour 2) is used on α and on β an odd number of times. Altogether the colour 1 is used on the vertices of D_r an even number of times.

On the other hand there are exactly $\frac{r}{2}$ heterochromatic edges in the set $S = \{u_iv_i; i = 1, \ldots, r\}$. Hence, on the heterochromatic edges from S, there is an odd number of vertices coloured with colour 1. Because on the monochromatic edges of S there is an even number of vertices of colour 1 we have together an odd number of vertices in D_r having the colour 1. This contradicts to the above derived fact, that the colour 1 was used an even number of times.

This yields $\chi_w(D_r) \geq 3$ if $r \not\equiv 0 \pmod 4$. The opposite inequality can be proved by a construction of a suitable weak parity vertex 3-colouring. It is left for the reader. ■

For plane graphs we are able to prove the following.

Theorem 3.2. *Let G be a connected plane graph with minimum face degree at least* 3. *Then*

$$\chi_w(G) \leq 4.$$

Proof. Suppose that the theorem is not true. Let $G = (V, E, F)$ be the minimal counterexample (with respect to the number of vertices). First, we investigate properties of G.

Claim 1. There is no 1-vertex in G.

Proof. Assume that such a vertex u exists in G. Then there is a neighbour v of u in G such that the edge uv lies on a facial walk of the face α. Clearly, the vertex u appears exactly once on the facial walk of α while v appears d times, $d \geq 2$. There are the following two possibilities.

Case 1. The vertex u is not incident with a 4-face. Let $\widetilde{G} = G - u$ be a plane graph obtained from G by deleting the vertex u and let $\widetilde{\alpha}$ be a face of $\widetilde{G}$ corresponding to the face α. Clearly $\widetilde{G}$ has a weak parity vertex 4-colouring $\widetilde{\varphi}$. Then a colouring φ of G can be defined as follows: $\varphi(x) = \widetilde{\varphi}(x)$ if $x \neq u$ and $\varphi(u) = \varphi(v)$. To see that φ is a weak parity vertex 4-colouring observe that the vertex v appears $d - 1$ times on the facial walk of $\widetilde{\alpha}$ and d times on the facial walk of α, which means that the colour $c = \varphi(u) = \varphi(v)$ appears on α two times more than on $\widetilde{\alpha}$.

Hence, α uses a colour an odd number of times if and only if $\widetilde{\alpha}$ does.

Case 2. The vertex u is incident with a 4-face $\alpha = (v, u, v, w)$. This means, that there is a multiedge vw. Let $\widetilde{G} = G - u - vw$ be a plane graph obtained from G by deleting the vertex u and deleting one edge vw. Clearly $\widetilde{G}$ is connected and has a weak parity vertex 4-colouring $\widetilde{\varphi}$. Using the 4-colouring $\widetilde{\varphi}$ we define a colouring φ of G as follows: $\varphi(x) = \widetilde{\varphi}(x)$ for every $x \neq u$ and $\varphi(u) \in \{1, 2, 3, 4\} - \{\varphi(v), \varphi(w)\}$. It is easy to check that φ is a weak parity vertex 4-colouring of G.

So we have a contradiction in both cases. ∎

Claim 2. There is no 2-vertex in G.

Proof. Assume that such a vertex u exists in G. We distinguish the following three cases.

Case 1. The vertex u is incident with two 3-faces. Let v and w be its two neighbours. Let $\widetilde{G} = G - u - vw$ be a graph obtained from G by deleting the vertex u and one of the edges vw. Clearly, $\widetilde{G}$ has a weak parity vertex 4-colouring $\widetilde{\varphi}$. Then a desired weak parity vertex 4-colouring φ of G can be defined as follows: $\varphi(x) = \widetilde{\varphi}(x)$ if $x \neq u$ and $\varphi(u) \in \{1, 2, 3, 4\} - \{\varphi(v), \varphi(w)\}$.

Case 2. The vertex u is incident with a 3-face α, but not with two 3-faces. Then the second face incident with u is a k-face β, $k \geq 4$. Let $\widetilde{G} = G - u$ be a graph obtained from G by deleting the vertex u. The graph $\widetilde{G}$ is connected, thus it has a weak parity vertex 4-colouring $\widetilde{\varphi}$. Let c be a colour used by $\widetilde{\beta}$ an odd number of times, where $\widetilde{\beta}$ is a face corresponding to the face β of G. We extended this colouring $\widetilde{\varphi}$ of $\widetilde{G}$ to a 4-colouring φ of G as follows: $\varphi(x) = \widetilde{\varphi}(x)$ for every $x \in V(G)$, $x \neq u$ and we put $\varphi(u) \in \{1, 2, 3, 4\} - \{c\}$. Clearly, the face β uses the colour c also an odd number of times. Since α is a 3-face it uses some colour an odd number of times.

Case 3. The vertex u is not incident with any 3-face. Let u be adjacent with the vertices v and w. Let $\widetilde{G} = G - u + vw$ be a plane graph obtained from G by deleting the vertex u and inserting the edge vw. Let α and β be the faces of G incident with the vertex u and let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be the faces of $\widetilde{G}$ corresponding to α and β, respectively. Clearly, $\widetilde{G}$ has a weak parity vertex 4-colouring $\widetilde{\varphi}$. Let us colour the vertices of G as follows: $\varphi(x) = \widetilde{\varphi}(x)$ for every $x \neq u$. Let c_1 be a colour used by $\widetilde{\alpha}$ an odd number of times and let c_2 be a colour used by $\widetilde{\beta}$ an odd number of times. Then we put $\varphi(u) = c \in \{1, 2, 3, 4\} - \{c_1, c_2\}$. The face α (β) also uses the colour c_1 (c_2) an odd number of times. Hence, φ is a required weak parity vertex 4-colouring of G.

Thus we have a contradiction in all three cases. ■

By Claims 1 and 2 the graph G has minimum vertex degree $\delta(G) \geq 3$. For the rest of the proof we need some new notations. For a k-vertex v let $v_1, v_2, \ldots, v_k$ be neighbours of v in a clockwise order. Now we define the plane graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{F})$ as follows: $\widetilde{V} = V - v$, $\widetilde{E} = E(\langle V - v\rangle_G) \cup \bigcup_{i=1}^{k}\{v_i v_{i+1}\}$. The set $\widetilde{F}$ is defined in the following way. Let $\alpha_i \in F$ be a face of G which is incident with edges vv_i and vv_{i+1} (indices modulo k). Let $\widetilde{\alpha} \in \widetilde{F}$ be a face of $\widetilde{G}$ determined by the vertices $v_1, v_2, \ldots, v_k$. Let $\widetilde{\alpha_i} \in \widetilde{F}$ be the face different from $\widetilde{\alpha}$ and incident with the edge $v_i v_{i+1}$ if α_i is not a

3-face. If α_i is a 3-face then $\widetilde{\alpha_i}$ is not defined (it does not exist). If $\beta \in F$ and $\beta \neq \alpha_i$ for every $i \in \{1, \ldots, k\}$ then $\beta \in \widetilde{F}$.

Clearly, $\widetilde{G}$ has a weak parity vertex 4-colouring $\widetilde{\varphi}$. For each $\beta \in \widetilde{F}$, let $c(\beta)$ denote the set of colours used by β an odd number of times. Let $c_i \in c(\widetilde{\alpha_i})$. Next we use the Lebesgue's Theorem 2.1. By this theorem G contains a k-vertex ($3 \leq k \leq 5$) of one of the types listed in this theorem. We distinguish the following three cases.

Case 1. Let $k = 3$. Then G contains a 3-vertex incident with faces α_1, α_2, and α_3. Consider $\widetilde{G}$ and its weak parity vertex 4-colouring $\widetilde{\varphi}$. For each $i \in \{1, 2, 3\}$ choose a colour c_i from $c(\widetilde{\alpha_i})$. Using the colouring $\widetilde{\varphi}$ we define a 4-colouring φ of G as follows: $\varphi(x) = \widetilde{\varphi}(x)$ for every $x \neq v$. The vertex v is coloured with a colour from $\{1, 2, 3, 4\} - \bigcup_{i=1}^{3}\{c_i\}$. It is easy to see that $c_i \in c(\alpha_i)$ for every $i \in \{1, 2, 3\}$. Hence, φ is a weak parity vertex 4-colouring of G. It is a contradiction.

Case 2. Let $k = 4$. Then G contains a $(3, a_2, a_3, a_4)$-vertex v incident with a 3-face α_1, an a_2-face α_2, an a_3-face α_3, and an a_4-face α_4. The graph $\widetilde{G}$ has a weak parity vertex 4-colouring $\widetilde{\varphi}$. Using this 4-colouring $\widetilde{\varphi}$ we define a weak parity vertex 4-colouring φ of G in the following way: $\varphi(x) = \widetilde{\varphi}(x)$ for every $x \neq v$. For $\varphi(v)$ we choose one colour from the set $\{1, 2, 3, 4\} - \bigcup_{i=2}^{4}\{c_i\}$ where $c_i \in c(\widetilde{\alpha_i})$. It is easy to see that the set $c(\beta)$ is nonempty for any face $\beta \in F - \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Furthermore, $c(\alpha_1) \neq \emptyset$ since α_1 is a 3-face. From the construction of φ it is easy to see that $c_i \in c(\alpha_i)$ for every $i \in \{2, 3, 4\}$. So we have a contradiction.

Case 3. Let $k = 5$. In this case G contains a $(3, 3, 3, 3, a)$-vertex, i.e. a vertex incident with four 3-faces and an a-face α. Clearly, $\widetilde{G}$ has a weak parity vertex 4-colouring $\widetilde{\varphi}$. Using this colouring $\widetilde{\varphi}$ we define a 4-colouring φ of G as follows: $\varphi(x) = \widetilde{\varphi}(x)$ if $x \neq v$. For $\varphi(v)$ we choose a colour from the set $\{1, 2, 3, 4\} - \{c\}$, where $c \in c(\widetilde{\alpha})$. It is easy to see that φ is a weak parity vertex 4-colouring of G, a contradiction. ∎

A vertex colouring is *proper* if adjacent vertices have different colours. When we use the four colour theorem for 2-connected plane graphs then we are able to prove a little stronger result.

Theorem 3.3. *Let G be a 2-connected plane graph. Then there is a proper weak parity vertex 4-colouring of G, such that each face of G uses some colour exactly once.*

Proof. Let α be a k-face with a facial walk $(v_1, v_2, \ldots, v_k, v_1)$, $k \geq 4$. We insert the diagonals $v_1v_3, v_1v_4, \ldots, v_1v_{k-1}$ into the face α. If we perform this operation for all faces of G of degree at least 4, we obtain a triangulation T such that $V(G) = V(T)$. Applying the four colour theorem we colour the vertices of T with four colours such that adjacent vertices receive distinct colours.

Let $c_T(x)$ be a colour of the vertex x of T under this colouring. Let $\varphi(x) = c_T(x)$ be a colouring of vertices of G. Then clearly $\varphi(v_1) \neq \varphi(v_i)$ for all $2 \leq i \leq k$ on the face α of G. Hence, the face α uses the colour $\varphi(v_1)$ only once. The same holds for any other face β of G. This means that the colouring φ is a proper weak parity vertex 4-colouring of G having the desired property. ∎

4. A Weak Parity Vertex 3-Colouring of Cubic Plane Graphs

When analysing the proof of our Theorem 3.2 we are not able to decrease the upper bound 4 to 3 only in two cases. So we strongly believe that the following holds.

Conjecture 4.1. Let G be a connected plane graph of minimum face degree at least 3. Then

$$\chi_w(G) \leq 3.$$

The requirement on the minimum face degree is substantial. To see this consider a graph H obtained from K_4 by replacing each of its edges with two parallel edges. There is a plane drawing D of H having four triangles and six digons. Clearly, $\chi_w(D) = 4$.

A set of vertices $S \subseteq V$ of a 2-connected plane graph $G = (V, E, F)$ is *face-independent* if no two vertices of S are incident with the same face.

A set $S \subseteq V$ is *maximal face-independent* if S is face-independent, but the set $S \cup \{v\}$ is not face-independent for every vertex $v \in V - S$.

The following main result of this section supports the above Conjecture 4.1.

Theorem 4.1. *Let G be a 2-connected cubic plane graph. Then*

$$\chi_w(G) \leq 3.$$

Moreover, the bound 3 is best possible.

Proof. Examples of prisms (see Theorem 3.1) show that the bound 3 is best possible. To prove that 3 is the upper bound we have to show that there is a weak parity vertex 3-colouring for every 2-connected cubic plane graph G.

Consider a maximal face-independent set R of vertices of $V(G)$ and colour each vertex of R with red colour; colour all other vertices white. Each face incident with a vertex from R is called a *red face*. Let $F(R)$ be the set of red faces. The face which belongs to the set $F(G) - F(R)$ is called a *white face*. An edge incident with two white faces is called a *white edge*. Let us call this colouring the initial one. Observe that in this initial colouring each vertex is incident with at least one red face.

A construction of our colouring will continue in three main steps. First we outline the idea of this construction.

In the first step we recolour the white faces that are adjacent with at least three other white faces into pink ones by colouring some other vertices with red colour and changing also some white faces into red ones. The first part of the construction terminates when in the obtained partial colouring there is no white face adjacent with at least three other white faces. In this moment all white faces will create chains of white faces such that each white face in it is adjacent to at most two other white faces.

In the second step some vertices of these chains will be coloured with blue colour and their faces became blue. The construction in the second step will terminate if there is no white face in the obtained colouring.

In the third step the pink faces will be changed into blue ones by colouring some of the remaining white vertices with blue colour.

Note that the construction is led in such a way that if a face is called red (or blue) then it uses a red (a blue) colour an odd number of times.

The general construction of the first step is the following.

Let $\alpha = (v_1, v_2, \ldots, v_k)$ be a white k-face, $k \geq 6$, that is adjacent with at least three other white faces. Let $\beta_1, \beta_2, \ldots, \beta_k$ be faces adjacent with α in order such that $v_i v_{i+1}$ is a common edge of α and β_i, $i \in \{1, 2, \ldots, k\}$; indices modulo k. Let, w.l.o.g., $v_1 v_2$ and $v_j v_{j+1}$ be white edges and let the edges $v_2 v_3, v_3 v_4, \ldots, v_{j-1} v_j$ be not white edges. Then clearly $\beta_2, \beta_3, \ldots, \beta_{j-1}$ are not white faces and the face β_k is a red one. Then we extend previous colouring in the following way. The vertices $v_2, v_3, \ldots, v_j$ are coloured with red colour. The vertices v_1 and v_{j+1} remain white. Faces β_1 and β_j have became red faces. The vertex v_1 is now incident with two red faces, namely with β_1 and β_k. After above changes the face α is called *pink*. Notice that the faces $\beta_2, \beta_3, \dots, \beta_{j-1}$ have two more vertices coloured with red colour. To continue in the construction in the first main step we choose another white face adjacent with at least three white faces. If such a white face does not exist we continue with the second main step.

Now each white face has at most two white neighbouring faces. Hence, the white faces are grouped into open or closed chains. Let $\alpha_1, \alpha_2, \dots, \alpha_t$ form an open (closed) chain of white faces with white edges $e_1, e_2, \dots, e_{t-1}$ ($e_1, e_2, \dots, e_t$, respectively) separating gradually these faces. Faces incident to two white edges have degree at least 4, otherwise there would be a vertex incident to only white faces.

If $\alpha_1, \alpha_2, \dots, \alpha_t$ is an open (closed) chain with even t then exactly one vertex of each of the white edges $e_1, e_3, e_5, \dots, e_{t-1}$ is coloured with blue colour and each white faces of this chain incident with a blue coloured vertex is said to be a *blue face*.

If $\alpha_1, \alpha_2, \dots, \alpha_t$ is an open chain with odd t then we do the same as above with the chain of the faces $\alpha_1, \alpha_2, \dots, \alpha_{t-1}$. The face α_t is incident with a vertex w which is incident with two not white faces (and neither with e_{t-1}). We colour this vertex with blue colour and α_t becomes blue.

If $\alpha_1, \alpha_2, \dots, \alpha_t$ is a closed chain with odd t then at least one of white faces, say α_t, has degree at least 5. (Otherwise, the graph would be isomorphic to a t-sided prism with both t-gonal faces red, what is impossible.) The face α_t then contains a white vertex w incident with two not white faces. We colour it with the blue colour and the face α_t becomes blue. The remaining faces $\alpha_1, \alpha_2, \dots, \alpha_{t-1}$ create an open chain of even number of faces and we proceed as above.

So in the second step all white faces are changed into blue ones. After finishing this second step every face of G is either red, or blue or pink. As we mentioned above, every red face uses red colour an odd number of times, every blue face uses the blue colour an odd number of times. So we need to check pink faces. This is done in the third step gradually.

Let α be a pink face. If α already uses the blue colour an odd number of times we rename it to a blue one. If α uses the blue colour an even number of times we colour the white vertex v_1 of α with the blue colour. This is always possible because the remaining two faces β_1 and β_k incident to v_1 are both red. If we have performed the third step on every pink face we are done. After the above constructed colouring the vertices are coloured either with red colour, or with blue colour or they are white. Every face is either red or blue which means that it uses one of colours red or blue an odd number of times.

This finishes the proof. ■

5. Strong Parity Vertex Colouring

A strong parity vertex colouring is well defined for every 2-connected plane graph G. To see this let us colour distinct vertices of G with different colours. Every colour on each face of G is in this case used exactly once. In general this does not hold for connected plane graphs that are not 2-connected, see the graph H on Figure 1.

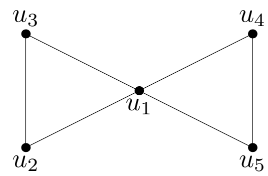

Figure 1. Graph H.

For a contrary, let H have a strong parity vertex colouring φ. Let $\varphi(u_1) = c$. Consider the outer face α which is a 6-face. The face α there uses at least two colours under φ and each of them an odd number of times. The vertex u_1 appears two times on the facial walk of α. Its colour c has to be used by α at least three times. If, w.l.o.g., the vertex u_2 uses the same colour c then also the vertex u_3 has to use c. Otherwise the 3-face (u_1, u_2, u_3) uses the colour c exactly two times which is not allowed. If colour c is used on vertices u_1, u_2 and u_3 then it is used by α four times. Consequently, the colour c must be used exactly on one of vertices u_4 or u_5. This means that the 3-face (u_1, u_4, u_5) uses exactly two colours which is a contradiction.

Let us provide a few examples of plane graph G for which the exact value of $\chi_s(G)$ is known. It is easy to see that $\chi_s(G) = 1$ if and only if all faces of G have odd degree. This means that if G contains a face with even degree then $\chi_s(G) \geq 2$. On Figure 2 there is a strong parity vertex 2-colouring of the cube Q.

Observe also that

$$\chi_s(G) \geq \chi_w(G)$$

for every 2-connected plane graph G.

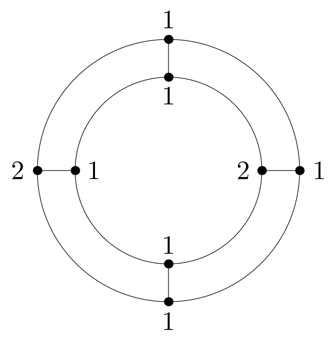

Figure 2. Strong parity vertex colouring of the cube Q.

A set of vertices $P \subseteq V$ of a 2-connected plane graph $G = (V, E, F)$ is *face-dominating* if every face of F is incident with a vertex of P.

Motivated by the strong parity vertex 2-colouring of the cube we are able to prove the following theorem.

Theorem 5.1. *Let G be a 2-connected plane graph all faces of which have even degree and G contains a set S of vertices which is face-independent and face-dominating. Then*
$$\chi_s(G) = 2.$$

Proof. It is enough to show a strong parity vertex 2-colouring. Let us colour the vertices of S with the colour 1 and the vertices of $V - S$ with the colour 2. Because of the properties of the set S every face of $F(G)$ contains exactly one vertex of S. Hence, each face uses the colour 1 exactly once. Then the colour 2 is on each face α used $deg(\alpha) - 1$ times. Because for every face α $deg(\alpha) \equiv 0 \pmod 2$ the proof is done. ■

The wheel W_n, $n \geq 4$, on n vertices is defined as a plane graph obtained by joining a vertex v_n (a central vertex) with every vertex of an $(n-1)$-vertex cycle C_{n-1}. It is an easy exercise to prove the following theorem.

Theorem 5.2. *Let W_n, $n \geq 4$, be an n-vertex wheel. Then*

$$\chi_s(W_n) = \begin{cases} 1 & \textit{if} \quad n \equiv 0 \pmod 2, \\ 3 & \textit{if} \quad n \equiv 3 \pmod 4, \\ 5 & \textit{if} \quad n \equiv 1 \pmod 4. \end{cases}$$

The following observation is very useful.

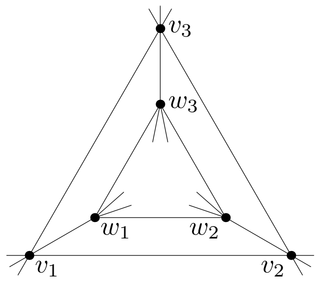

Figure 3. Configuration K.

Lemma 5.1. *If a 2-connected plane graph G contains a configuration K of Figure 3 then*

$$\chi_s(G) \geq 6.$$

Proof. Suppose that $\chi_s(G) = k$, $k \leq 5$. Then at least two vertices of K have the same colour under a strong parity vertex k-colouring φ. Let, w.l.o.g., these two vertices belong to the 4-face $\alpha = (w_1, w_2, v_2, v_1)$. Then on α there exists also a third vertex of the same colour. So let, w.l.o.g., $\varphi(w_1) = \varphi(w_2) = \varphi(v_2) = 1$ and let $\varphi(v_1) = 2$. Then $1 \in \{\varphi(v_3), \varphi(w_3)\}$ and consequently $\varphi(v_3) = \varphi(w_3) = 1$. We have a contradiction because all four vertices of the face (w_2, w_3, v_3, v_2) are coloured with the colour 1. ∎

Next we shall determine $\chi_s(D_r)$ for r-sided prisms, $r \geq 3$. Recall that an edge is monochromatic if both its ends are coloured with the same colour. Otherwise it is heterochromatic.

Theorem 5.3. *Let D_r be an r-sided prism, $r \geq 3$. Then*

$$\chi_s(D_r) = \begin{cases} 2 & \text{if } r \equiv 0 \pmod 4, \\ 4 & \text{if } r \not\equiv 0 \pmod 4 \text{ and } r \notin \{3, 7\}, \\ 5 & \text{if } r = 7, \\ 6 & \text{if } r = 3. \end{cases}$$

Proof. From Lemma 5.1 we have $\chi_s(D_3) \geq 6$. The opposite inequality follows from Figure 4.

From Figure 2 and from the fact that D_4 contains a 4-face we get $\chi_s(D_4) = 2$. Next we distinguish four cases.

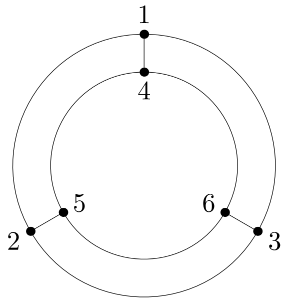

Figure 4. Strong parity vertex colouring of the 3-sided prism D_3.

Case 1. Let $r = 4k$, $k \geq 1$. Then starting with the graph of the cube of Figure 2 and using $(k-1)$ times the construction and colouring of Figure 5 we obtain a $4k$-sided prism with a strong parity vertex 2-colouring.

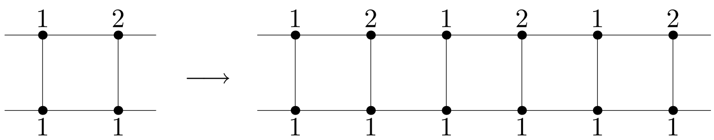

Figure 5. Extending the colouring of a prism.

Case 2. Let $r = 4k+2$, $k \geq 1$. Starting with the graph of Figure 6 and using $(k-1)$ times the construction and colouring of Figure 7 we get that $\chi_s(D_r) \leq 4$.

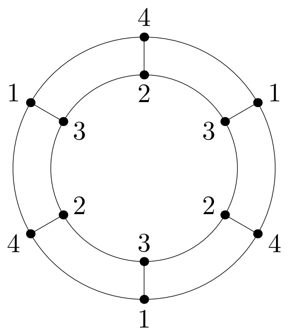

Figure 6. Strong parity vertex colouring of the graph D_6.

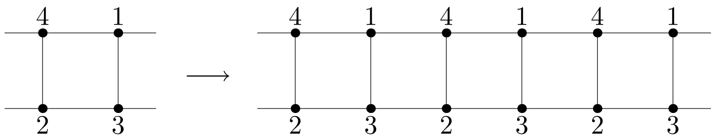

Figure 7. Extending the colouring of a prism.

To prove the opposite inequality, suppose that φ is a strong parity vertex 3-colouring of D_r using colours a, b, c. Then each 4-face contains one monochromatic edge. This means that there are exactly $2k+1$ monochromatic (and also heterochromatic) edges in the set $S = \{u_i v_i; i = 1, \ldots, r\}$ which cover all the vertices of D_r. Let the face α (face β) use the colour x, $x \in \{a, b, c\}$ x_α times (x_β times). Recall that x_α and x_β is always either an odd number or zero and that $\max\{x_\alpha, x_\beta\} \neq 0$ for every $x \in \{a, b, c\}$. Moreover,

$$a_\alpha + b_\alpha + c_\alpha = 4k + 2,$$

$$a_\beta + b_\beta + c_\beta = 4k + 2.$$

W.l.o.g., let $c_\beta = 0$. Then $a_\beta \neq 0$, $b_\beta \neq 0$, $c_\alpha \neq 0$, and consequently, $\min\{a_\alpha, b_\alpha\} = 0$. Let, w.l.o.g., $b_\alpha = 0$, consequently $a_\alpha \equiv 1 \pmod 2$ and $a_\beta \equiv 1 \pmod 2$. This means that in S there are no monochromatic edges with respect to colours b and c. Hence, each heterochromatic edge of S contains a vertex coloured with the colour a. Thus we have $b_\beta + c_\alpha = 2k+1$ which contradicts the proposition that both b_β and c_α are odd integers.

Case 3. Let $r = 4k+1$, $k \geq 1$ ($r = 4k+3$, $k \geq 2$). Because r is odd then in any strong parity vertex k-colouring there must be a 4-face without monochromatic edge. Therefore, $\chi_s(D_r) \geq 4$.

Starting with the graphs of Figure 8 and using the construction and the colouring of Figure 7 $(k-1)$ times ($(k-2)$ times) we get that

$$\chi_s(D_r) \leq 4 \text{ if } r = 4k+1, k \geq 1 \quad (\chi_s(D_r) \leq 4 \text{ if } r = 4k+3, k \geq 2).$$

Case 4. Let $r = 7$. Suppose there is a strong parity vertex 4-colouring φ of D_7 with four colours a, b, c, d. Let the colour $x \in \{a, b, c, d\}$ is used x_α times by the face α and x_β times by the face β. Clearly $x_\alpha \equiv 1 \pmod 2$ or $x_\alpha = 0$, analogously $x_\beta \equiv 1 \pmod 2$ or $x_\beta = 0$ for every $x \in \{a, b, c, d\}$. Since r is an odd integer, both faces α and β use exactly three colours. Let, w.l.o.g, the face α use the colours a, b, c and the face β use the colours a, b, d.

We say that a 4-face is monochromatic, if it is incident with a monochromatic edge; otherwise it is heterochromatic. It is easy to see that D_7 must have an odd number of heterochromatic 4-faces.

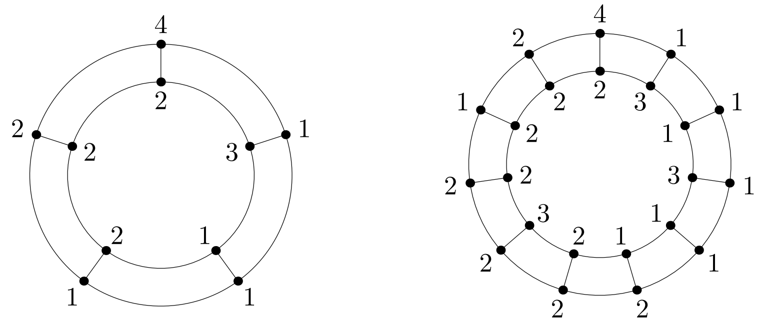

Figure 8. Strong parity vertex colouring of the graphs D_5 and D_{11}.

There are only two possible partitions of 7 into three odd summands, namely $7 = 3 + 3 + 1$ or $7 = 5 + 1 + 1$.

Next we show that $c_\alpha = 1$ and $d_\beta = 1$. Let, for a contrary, $c_\alpha \geq 3$. Clearly, if $\varphi(u_i) = c$ then $\varphi(u_{i-1}) \neq c$ and $\varphi(u_{i+1}) \neq c$, otherwise the face β uses the colour c which is impossible. Hence, w.l.o.g., $\varphi(u_1) = \varphi(u_3) = \varphi(u_5) = c$, $b_\alpha = 3$ and $a_\alpha = 1$. There are two possibilities: $\{\varphi(u_6), \varphi(u_7)\} = \{a, b\}$ or $\{\varphi(u_6), \varphi(u_7)\} = \{b, b\}$. In both cases we get a contradiction. This means that $c_\alpha = 1$. Analogously we can prove that $d_\beta = 1$.

If there is an edge u_iv_i with ends coloured with the colour c and d, then there are two heterochromatic 4-faces, but the third is impossible.

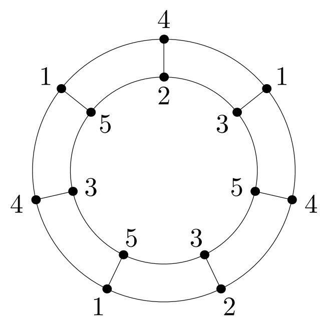

Figure 9. Strong parity vertex colouring of the 7-sided prism D_7.

If there is no edge u_iv_i having ends coloured with c and d, then there is exactly one heterochromatic 4-face. This enforces six monochromatic 4-faces whose vertices are coloured only with colours a and b. This leads to a contradiction with requirements on colouring of faces α and β.

So we have $\chi_s(D_7) \geq 5$. The opposite inequality we obtain from Figure 9. ■

6. The Cyclic Chromatic Number and the Strong Parity Vertex Colouring

The *cyclic chromatic number* $\chi_c(G)$ of a 2-connected plane graph G is the minimum number of colours in an assignment of colours to the vertices of G such that whenever two vertices are incident with the same face they have different colours. Obviously $\Delta^*(G) \leq \chi_c(G)$, where $\Delta^*(G)$ is the degree of a largest face. The cyclic chromatic number was introduced by Ore and Plummer [12] in 1969. There is a quite rich literature devoted to study this parameter. For recent results in this area see [5, 6, 7, 16] and the references therein.

It is easy to see that for every 2-connected plane graph G there is

$$\chi_s(G) \leq \chi_c(G).$$

Using this observation and the results of [5, 6, 7, 16] we immediately have

Theorem 6.1. *Let G be a 2-connected plane graph and let Δ^* be the degree of a largest face of G. Then*

(i)

$$\chi_s(G) \leq \left\lceil \frac{5\Delta^*}{3} \right\rceil.$$

(ii) *If G is 3-connected, then*

$$\chi_s(G) \leq \Delta^* + 1 \quad \textit{for} \quad \Delta^* \geq 60,$$

$$\chi_s(G) \leq \Delta^* + 2 \quad \textit{for} \quad \Delta^* \geq 18.$$

7. Concluding Remarks

Let $\chi_{pw}(G)$ be the minimum number of colours in a proper weak parity vertex colouring of a connected plane graph G. It is easy to see that $\chi_w(K_4) = 1$ and $\chi_{pw}(K_4) = 4$ for the tetrahedral graph K_4. This together with Theorem 3.3 gives the following theorem.

Theorem 7.1. *Let G be a 2-connected plane graph. Then*

$$\chi_{pw}(G) \leq 4.$$

Moreover, the bound is tight.

It is easy to see that $\chi_{pw}(D_r) = 3$ for every $r \geq 3$.

Theorem 6.1 gives us the best known up to now upper bounds on the parameter $\chi_s(G)$ for any 2-connected plane graph G. We strongly believe that the following holds.

Conjecture 7.1. There is a constant K such that for every 2-connected plane graph G

$$\chi_s(G) \leq K.$$

We do not know any 2-connected plane graph H with $\chi_s(H) \geq 7$. Hence, we believe that $K = 6$ in the above conjecture.

Let $\chi_{ps}(G)$ be the minimum number of colours in a proper strong parity vertex colouring of a 2-connected plane graph G.

Problem 3. Let G be a 2-connected plane graph. Determine $\chi_{ps}(G)$.

Note that for the wheel W_5 there is

$$\chi_w(W_5) = 2, \quad \chi_{pw}(W_5) = 4, \quad \chi_s(W_5) = 5, \quad \chi_{ps}(W_5) = 5.$$

Lemma 7.1. *If a 2-connected plane graph G contains at least one 4-face, then*

$$\chi_{ps}(G) \geq 4.$$

Proof. Let $\alpha = (u_1, u_2, u_3, u_4)$ be a 4-face and let φ be a proper strong parity vertex colouring. It is enough to show, that $\varphi(u_i) \neq \varphi(u_j)$ for every $i \neq j$.

Assume that $\varphi(u_i) = \varphi(u_j) = 1$ for some $i \neq j$. Then the face α uses the colour 1 three times because φ is a strong parity vertex colouring. Hence, there are two adjacent vertices on the face α which have the same colour. It is a contradiction with the fact that φ is a proper colouring. ∎

The following theorem completes our investigations of vertex parity parameters for prisms.

Theorem 7.2. *Let D_r be an r-sided prism, $r \geq 3$. Then*

$$\chi_{ps}(D_r) = \begin{cases} 4 & \textit{if} \quad r \equiv 0 \pmod 2, \\ 5 & \textit{if} \quad r \equiv 1 \pmod 2 \ \textit{and} \ r \neq 3, \\ 6 & \textit{if} \quad r = 3. \end{cases}$$

Proof. From Theorem 5.3 follows that $\chi_{ps}(D_3) = 6$. Next we distinguish two cases.

Case 1. Let $r = 2k$, $k \geq 2$. Using the Lemma 7.1 we obtain $\chi_{ps}(D_r) \geq 4$. Hence, it is enough to show a proper strong parity vertex 4-colouring. Starting with the graphs of Figure 10 and using $(k-1)$ times the construction and colouring of Figure 7 we obtain a proper strong parity vertex 4-colouring of a $2k$-sided prism.

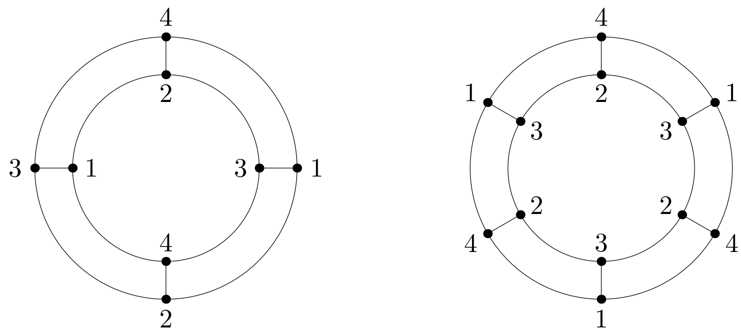

Figure 10. Proper strong parity vertex colouring of the graphs D_4 and D_6.

Case 2. Let $r = 2k+1$, $k \geq 2$. First we prove that $\chi_{ps}(D_r) \leq 5$. Starting with the graphs of Figure 11 and using $(k-1)$ times the construction and colouring of Figure 7 we obtain a proper strong parity vertex 5-colouring of a $(2k+1)$-sided prism.

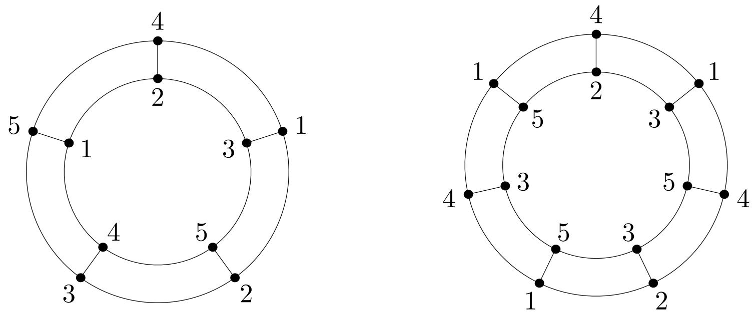

Figure 11. Proper strong parity vertex colouring of the graphs D_5 and D_7.

Now we show that 4 colours are not enough. Suppose that there is a proper strong parity vertex 4-colouring φ of D_r. From the proof of Lemma 7.1 follows that each 4-face uses exactly four colours. Therefore, $\{\varphi(u_i), \varphi(v_i)\} = \{\varphi(u_{i+2}), \varphi(v_{i+2})\}$ for every $i \in \{1, \ldots, r\}$ (indices modulo r). Hence, $\{\varphi(u_1), \varphi(v_1)\} = \{\varphi(u_r), \varphi(v_r)\}$. So we have a contradiction, because the face (u_r, v_r, v_1, u_1) uses the colours $\varphi(v_1)$ and $\varphi(u_1)$ exactly two times. ■

Acknowledgments

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